The Aleksandrov-Bakelman type maximum principle for elliptic equations on a “book” type stratified set

Foma Mironenko, *SPbSU*

(joint work with Alexander Nazarov)

Used designations

$(x', x_n) = x \in \mathbb{R}^n$, where $x' \in \mathbb{R}^{n-1}$

$B^n(R)$ — n-dimensional ball of radius R in $\mathbb{R}^n$, $\quad B_0(R) := B^{n-1}(R) \times \{0\}$

$\mathbb{R}^n_\pm := \{x \in \mathbb{R}^n \mid x_n \gtrless 0\}$, $\quad B_\pm(R) := B^n(R) \cap \mathbb{R}^n_\pm$

$\mathbb{R}^n_{0,\pm} := \mathbb{R}^n_\pm \cup \{x_n = 0\}$, $\quad B_{0,\pm}(R) := B^n(R) \cap \mathbb{R}^n_{0,\pm} = B_0(R) \cup B_\pm(R)$

$\Phi_z : \mathbb{R}^n \to \mathbb{R}^n$ — for an upward convex function $z : \mathbb{R}^n \to \mathbb{R}$, a multi-valued mapping

$$\Phi_z(x_0) = \left\{ p \in \mathbb{R}^n \;\middle|\; \boldsymbol{\pi_p(x) := \langle p, x - x_0\rangle + z(x_0)} - \text{supporting plane to } z \right\}$$

$\mathrm{conv}(E) \subset \mathbb{R}^m$ — a convex hull of a set $E \subset \mathbb{R}^m$.

$\mathrm{conv}[u] : \mathrm{dom}(u) \to \mathbb{R}$ — a convex hull of a function $u : \mathrm{dom}(u) \to \mathbb{R}$. Which is the minimal **convex upwards** function majorizing u.

$u_+ := \max\{0, u\}$

$\mathcal{Z}_u := \{x \in \mathrm{dom}(u) \mid u(x) = \mathrm{conv}[u_+](x)\}$ — a contact set for function u

$\Omega_u := \{x \in \Omega \mid u(x) > 0\}$

"Book" type stratified set

Consider K n-dimensional hyperplanes embedded into $\mathbb{R}^{n+1}$ having $\mathbb{R}^{n-1} \times \{0\}^2$ as their joint intersection and denote each by $\mathbb{R}^{n,[k]}$. For the planes we choose first $n-1$ coordinates x' to be a basis of $\mathbb{R}^{n-1}$. And set the n-th coordinate $x_n^{[k]}$ orthogonal to x' in each plane.

We name the union of $\mathbb{R}^{n,[k]}$ a **stratified space**, denoting it $\mathbb{R}^{\circledast}$, and define one more entity:

$$\mathbb{R}^{\circledast}_{+} := \bigcup_{k=1}^{K} \mathbb{R}^{n,[k]}_{0,+}$$

which is a **"book" type stratified set**.
Finally, we can define the domain of our interest simply as **a ball of radius $\boldsymbol{R}$** in $\mathbb{R}^{\circledast}_{+}$:

$$\mathbb{B} := \left\{x \in \mathbb{R}^{\circledast} \;\middle|\; |x| < R\right\} = \bigcup_{k=1}^{K} B^{[k]}_{0,+}(R)$$

And we name it **a stratified ball**.

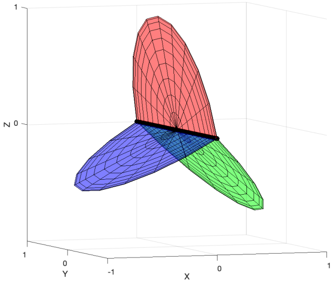

Figure: Stratified ball $\mathbb{B}$ with $K=3$, $n=2$

Functions on stratified set

Consider a **function $\boldsymbol{u} : \mathbb{B} \to \mathbb{R}$**. It can be represented as a set of it's components $u_1, u_2, \ldots, u_K$ defined over the halfball $B_{0,+}(R)$ and forced to be equal on $B_0(R)$: $u_i|_{B_0(R)} \equiv u_j|_{B_0(R)}$. Thus, function u could be rewritten as:

$$u\left(x^{[k]}\right) = u_k(x)$$

The key idea of Aleksandrov-Bakelman type estimates is **normal mapping** which requires a function's convex hull to be defined. For this purpose we introduce Lemma:

Lemma 1

Consider functions $v_1, v_2 : B_{0,+}(R) \to \mathbb{R}$, $v_1, v_2 \in \mathcal{C}\big(B_{0,+}(R)\big)$. An implication holds:

$$v_1|_{B_0(R)} \equiv v_2|_{B_0(R)} \quad \Longrightarrow \quad \textit{conv}[v_1]|_{B_0(R)} \equiv \textit{conv}[v_2]|_{B_0(R)}$$

In other words, function's convex hull restriction onto $B_0(R)$ does not depend on function's values out of $B_0(R)$.

In accordance with the Lemma, we define a **convex hull** for u component-wise:

$$\mathrm{conv}[u]\left(x^{[k]}\right) := \mathrm{conv}[u_k](x)$$

Problem setting

Define **operators over $\boldsymbol{B_+^{[k]}(R)}$**:

$$\mathcal{L}^{[k]}u \;:=\; \sum_{i,j=1}^{n} -a_{ij}^{[k]}(x)D_iD_ju \;+\; \sum_{i=1}^{n} b_i^{[k]}(x)D_iu$$

satisfying $a_{ij}^{[k]} \in L_\infty,\;\; b_i^{[k]} \in L_n,\;\; \big(a_{ij}^{[k]}\big)_{i,j=1}^{n} \geqslant \nu I_n$

Then define an **operator over $\boldsymbol{B_0(R)}$**:

$$\tilde{\mathcal{B}}u \;:=\; \sum_{l,m=1}^{n-1} -\alpha_{lm}(x')D_lD_mu \;+\; \sum_{l=1}^{n-1} \beta_l(x')D_lu \;+\; \mathcal{J}u$$

having $\alpha_{lm} \in L_\infty,\;\; \beta_l \in L_{n-1},\;\; \big(\alpha_{lm}\big)_{l,m=1}^{n-1} \geqslant \nu I_{n-1}$

Where $\mathcal{J}$ is a **"conjugation" operator**, defined for $u \in \bigcap_k \mathcal{C}^1\big(B_{0,+}^{[k]}(R)\big)$ with formula

$$\mathcal{J}u \;=\; \sum^{K} \beta_n^{[k]}(x') \cdot \lim_{x_n^{[k]}\to 0+} D_nu(x', x_n^{[k]})$$

Main result

Theorem (Maximum principle at stratified set)

Let $n \geqslant 2$. Consider $u : \mathbb{B} \to \mathbb{R}$, such that $u \in \bigcap_k \mathcal{C}^2\big(B_{0,+}^{[k]}(R)\big)$, $u \in \mathcal{C}\big(\overline{\mathbb{B}}\big)$, $u|_{\partial\mathbb{B}} < 0$ and let $\beta_n^{[k]} \leqslant 0$. The following estimate holds:

$$\max_{\overline{\mathbb{B}}} u_+ \leqslant N\left(n,\ \left(\frac{\left\| b^{[k]} \right\|_{n,\left(B_+^{[k]}(R)\right)_u}}{\nu}\right)_{k=1}^{K},\ \frac{\left\|\beta'\right\|_{n-1,\left(B_0(R)\right)_u}}{\nu}\right) \cdot \frac{R}{\nu} \cdot \left(\sum_{k=1}^{K} \left\|(\mathcal{L}^{[k]}u)_+\right\|_{n,\left(B_+^{[k]}(R)\right)_u} + \left\|(\tilde{\mathcal{B}}u)_+\right\|_{n-1,\left(B_0(R)\right)_u}\right)$$

Let $\boldsymbol{M = \max u_+} > 0$, $\boldsymbol{z} = \textbf{conv}[\boldsymbol{u_+}]$. Define a subset of $B_0(R)$ where the normal derivative of z is nonpositive in all k directions, which will come in for the estimates:

$$\widetilde{\mathcal{Z}} := \left\{ x \in \mathcal{Z}_u \cap B_0(R) \;\middle|\; \forall k:\ \ D_n^{[k]} z(x', 0) \leqslant 0 \right\}$$

Normal map on stratified set

The structure of the normal map is as follows:

$$\begin{cases} \Phi_z(x^{[k]}) = \big(\Phi_{z_k}(x)\big)^{[k]}, & x^{[k]} \in B_+^{[k]}(R) \\ \Phi_z(x) = \bigcup_{k=1}^{K} \big(\Phi_{z_k}(x)\big)^{[k]}, & x \in B_0(R) \end{cases}$$

Now we define a special set $\mathcal{D}$. Consider $\widehat{u}(x) := \max\{u_1(x), \dots, u_K(x)\}$ and convex hull $\widehat{z} :=$ conv$[\widehat{u}_+]$.

Let $\mathcal{P}_k$ be a set of $p \in B_{0,-}(M/2R)$ for which a supporting plane π_p of $\widehat{z}$ **touches the subgraph of $\boldsymbol{u_k}$**. All $\mathcal{P}_k$ are measurable and **their union gives the whole halfball $\boldsymbol{B_{0,-}(M/2R)}$**. Now we consider $\mathcal{D}_k := \mathcal{P}_k \setminus \bigcup_{j=1}^{k-1} \mathcal{P}_j$ and define the desired set:

$$\mathcal{D} := \bigcup_{k=1}^{K} \mathcal{D}_k^{[k]} \subseteq \mathbb{R}^{\circledast}$$

Lemma 2 (main property of $\mathcal{D}$)

$$\mathcal{D} \setminus \Phi_z\big(\mathbb{B} \setminus B_0(R)\big) \subseteq \Phi_z(\widetilde{\mathcal{Z}})$$

Proof of the Theorem (sketch)

The proof is based on the following two inequalities. First, for $g:\mathbb{R}^{\circledast}\to\mathbb{R}_{0,+}$

$$\int\limits_{\Phi_z\left(B_+^{[k]}(R)\right)} g(p)dp \leqslant \frac{1}{(\nu n)^n}\int\limits_{\mathcal{Z}_u\cap B_+^{[k]}(R)} g\big(Du(x)\big)\Big(\mathcal{L}^{[k]}u(x)-\big\langle b^{[k]}(x),Du(x)\big\rangle\Big)^n dx$$

Second, we denote $\widehat{g}(p):=\max_k g(p',p_n^{[k]}):\mathbb{R}^n\to\mathbb{R}_{0,+}$. Then

$$\int\limits_{\mathcal{D}\,\setminus\,\Phi_z(\mathbb{B}\setminus B_0(R))} g(p)dp \leqslant \frac{1}{(\nu(n-1))^{n-1}}\int\limits_{\widetilde{\mathcal{Z}}}\Big(\tilde{\mathcal{B}}u(x',0)-\big\langle\beta'(x'),D'u(x',0)\big\rangle\Big)^{n-1}\cdot$$

$$\cdot\int\limits_{-M/2R}^{0}\widehat{g}\Big(D'u\big(x',0\big),\ p_n\Big)dp_n\,dx'$$

The second inequality is a consequence of the main froperty of $\mathcal{D}$. We prove the Theorem by applying techniques first described by **A.D. Aleksandrov** and **I.Ya. Bakelman** and utilizing the two inequalities.