Guaranteed error estimates for approximate solutions of the problem with an obstacle for the p-Laplace operator

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FORMULATION OF THE PROBLEM

We consider an elliptic variational inequality, that arises in an obstacle problem for a nonlinear p-Laplace operator ($p > 1$). This free boundary problem reduces to minimizing the energy functional

$$J[v] = \int_\Omega \left(\frac{1}{p}|\nabla v|^p - fv\right) dx$$

on a closed convex set

$$\mathbb{K} = \{v \in W_0^{1,p}(\Omega) : v \geq \phi \text{ in } \Omega\}.$$

Here Ω is a bounded domain in the space $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$, the function f is given, $f \in L^q(\Omega)$ ($\frac{1}{p} + \frac{1}{q} = 1$), and the function ϕ is a sufficiently smooth obstacle function from the $C^{\max\{2,p\}}$ space.

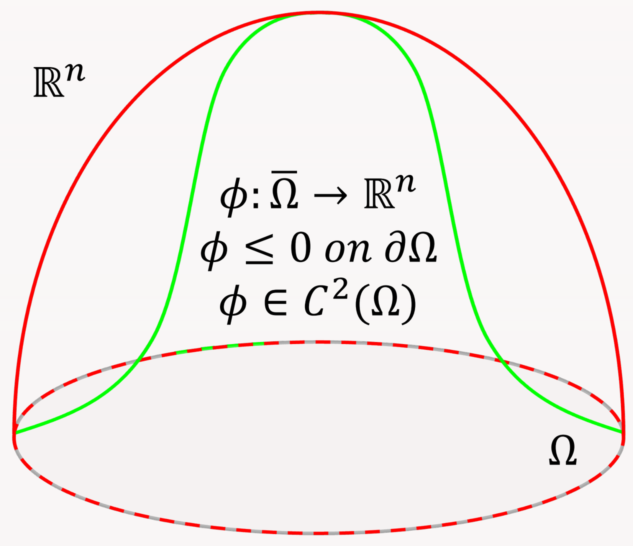

Fig.: A model problem with an obstacle, $p = 2$

A LITTLE BIT ABOUT P-LAPLACIAN (AND QR)

It is worth paying attention to the definition of the p-Laplace operator:

$$\Delta_p v := \operatorname{div}(|\nabla v|^{p-2}\nabla v),$$

$$|\nabla v|^{p-2} = \left[(v_{x_1})^2 + (v_{x_2})^2 + \cdots + (v_{x_n})^2\right]^{\frac{p-2}{2}},$$

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\psi \, dx = 0 \text{ for any } \psi \in C_0^\infty(\Omega).$$

GENERAL DUAL THEORY FOR CONVEX VARIATIONAL PROBLEMS

$$J[v] := G(\Lambda v) + F(v),$$

$$\Lambda: V \to Y, \qquad G: Y \to \mathbb{R}, \qquad F: V \to \mathbb{R}.$$

Theorem 1 ("Abstract error identity", see [1]): Let u, p^* be the exact solutions of the primal and dual problems, respectively. Then for any functions $v \in V$ and $y^* \in Y^*$ identity is true:

$$\mathcal{D}_G(\Lambda v, p^*) + \mathcal{D}_G(\Lambda u, y^*) + \mathcal{D}_F(v, -\Lambda^* p^*) + \mathcal{D}_F(u, -\Lambda^* y^*) = \mathcal{D}_G(\Lambda v, y^*) + \mathcal{D}_F(v, -\Lambda^* y^*),$$

where

$$\mathcal{D}_G(\Lambda u, y^*) := G(\Lambda u) + G^*(y^*) - (y^*, \Lambda u),$$

$$\mathcal{D}_F(v, -\Lambda^* p^*) := F(v) + F^*(-\Lambda^* p^*) + \langle \Lambda^* p^*, v\rangle. \quad (\mathcal{I})$$

Remark 1: This theorem is true for abstract G, F and Λ with general properties, as well as for abstract reflexive Banach spaces V, Y, V^* and Y^*.

Error estimates

[1] Repin, S.I. (2003). Two-sided estimates of deviation from exact solutions of uniformly elliptic equations. *Transl. of the Amer. Math. Soc.-Ser. 2, 209*, 143-172.

[2] Apushkinskaya, D.E., & Repin, S.I. (2020). Biharmonic obstacle problem: guaranteed and computable error bounds for approximate solutions. *Comput. Math. Math. Phys., 60*(11),1823–1838.

[3] Repin, S., & Frolov, M. (2002). A posteriori error estimates for approximate solutions of elliptic boundary value problems. *Comput. Math. Math. Phys, 42*(12), 1704–1716.

The regularity of solutions

[3] Andersson, J., Lindgren, E.A., & Shahgholian, H. (2014). Optimal regularity for the obstacle problem for the p-Laplacian. *J. Differential Equations, 259*, 2167-2179.

SPACES AND FUNCTIONALS OF OUR PROBLEM

Taking into account Remark 1, we need to define all the spaces, operators and the composite functionals, corresponding to our problem with p-laplacian.

Spaces and operators	Functionals
$V = W_0^{1,p}(\Omega)$, $Y = L^p(\Omega, \mathbb{R}^n)$, $\Lambda := \nabla$.	$G(y) = \frac{1}{p}\lVert y\rVert^p_{L^p}$, $F(v) = -\int_\Omega fv\,dx + \chi_{\mathbb{K}}(v)$, where $\chi_{\mathbb{K}}(v) = \begin{cases} 0, v \in \mathbb{K}, \\ +\infty, v \notin \mathbb{K}. \end{cases}$
$V^* = W_0^{-1,q}$, $Y^* = L^q(\Omega, \mathbb{R}^n)$, $\Lambda^*: Y^* \to V^*$, $\Lambda^* := \operatorname{div}$.	$G^*(y^*) = \frac{1}{q}\lVert y^*\rVert^q_{L^q}$, $F^*(-\Lambda^* y^*) = = \int_\Omega (f + \operatorname{div} y^*)\phi\,dx.$[2]

[2]) To define the functional $F^*(-\Lambda^* y^*)$ we need to introduce an intermediate space

$$H_q = H_q(\Omega, \operatorname{div}) := \{y \in Y^* : \operatorname{div} y^* \in L^q\}.$$

In the dual space, this functional is represented as a supremum, that takes finite values if and only if the condition $f + \operatorname{div} y^* \leq 0$ is satisfied. So, for any

$$y^* \in Q_{q,\leq 0} = \{y^* \in H_q : f + \operatorname{div} y^* \leq 0 \text{ a.e. in } \Omega\}$$

$$F^*(-\Lambda^* y^*) = \int_\Omega (f + \operatorname{div} y^*)\phi\,dx.$$

Composite functionals

$$\mathcal{D}_G(\Lambda v, y^*) = \int_\Omega \left(\frac{1}{p}|\nabla v|^p + \frac{1}{q}|y^*|^q - \nabla v \cdot y^*\right) dx,$$

at the same time, $\mathcal{D}_G(\Lambda u, y^*)$ and $\mathcal{D}_G(\Lambda v, p^*)$ are found by substituting u and p^* instead of v and y^*, respectively.

$$\mathcal{D}_F(v, -\Lambda^* y^*) = \int_\Omega (f + \operatorname{div} y^*)(\phi - v)\,dx,$$

and $\mathcal{D}_F(u, -\Lambda^* y^*)$ is obtained by replacing v with u. From the duality relation $J[u] = I^*[p^*]$ we get:

$$J[v] - J[u] = \mathcal{D}_G(\Lambda v, p^*) + \mathcal{D}_F(v, -\Lambda^* p^*).$$

From here at $v = u$ it follows that

$$\mathcal{D}_F(v, -\Lambda^* p^*) = \int_\Omega (f + \operatorname{div} p^*)(u - v)\,dx.$$

The left side of the abstract identity ($\mathcal{I}$)

Using the connection conditions of the primal and dual problems $p^* = \nabla u|\nabla u|^{p-2}$ and $|\nabla u|^p = |p^*|^q$, we obtain the expressions on the left side of the identity of Theorem 1:

$$\mu(v) := \int_\Omega \left(\frac{1}{p}|\nabla v|^p + \frac{1}{q}|\nabla u|^p - \nabla v \nabla u |\nabla u|^{p-2}\right) dx + \int_{\{u=\phi\}} (f + \Delta_p u)(u - v)\,dx,$$

$$\mu^*(y^*) := \int_\Omega \left(\frac{1}{p}|p^*|^q + \frac{1}{q}|y^*|^q - \gamma(x) p^* y^*\right) dx + \int_{\{u>\phi\}} (f + \operatorname{div} y^*)(\phi - u)\,dx,$$

$$\text{where } \gamma(x) = \begin{cases} 0, \text{ if } x \in \Omega \cap \{|\nabla u| = 0\}, \\ |p^*|^{\frac{2-p}{p-1}}, \quad \text{otherwise.} \end{cases}$$

The regularity of free boundary

[4] Figalli, A., Krummel, B., & Ros-Oton, X. (2017). On the regularity of the free boundary in the p-Laplacian obstacle problem. *J. Differential Equations, 263*(3), 1931-1945.

ERROR IDENTITY FOR P-LAPLACIAN

Theorem 2: For any functions $v \in \mathbb{K}$ and $y^* \in Q_{q,\leq 0}$, which are approximate solutions of the primal and dual problems, respectively, the following identity holds:

$$\mu(v) + \mu^*(y^*) = = \int_\Omega \left(\frac{1}{p}|\nabla v|^p + \frac{1}{q}|y^*|^q - \nabla v \cdot y^*\right) dx + \int_\Omega (f + \operatorname{div} y^*)(\phi - v)\,dx,$$

where the left side of the equality is defined by the formulas above.

Remark 2: The expressions $\mu(v)$ and $\mu^*(y^*)$ are non-negative quantities, that vanish when the exact and approximate solutions are equal in the primal and dual problems, respectively. It is also worth noting that the expression on the right side of the identity of this theorem is fully computable.

Remark 3: At this stage of obtaining an estimate, some "problem" arises, condition $f + \operatorname{div} y^* \leq 0$ is rather narrow and inconvenient for practical use. Let us show for the superquadratic case ($p \geq 2$) how to extend the admissible set for y^*.

ERROR ESTIMATE IN THE FORM OF INEQUALITY. SUPERQUADRATIC CASE ($p \geq 2$)

Lemma 1: For any function $z^* \in H_q$ the projection inequality holds:

$$\inf_{y^* \in Q_{q,\leq 0}} \lVert z^* - y^*\rVert^q_{L^q} \leq C_F^q \lVert (f + \operatorname{div} z^*)_+\rVert^q_{L^q},$$

where C_F is a constant from a Friedrichs - type inequality:

$$\lVert w\rVert_{L^p} \leq C_F \lVert \nabla w\rVert_{L^p},$$

and $(f + \operatorname{div} z^*)_+ = \max\{f + \operatorname{div} z^*, 0\}$.

Using the first Clarkson inequality for any functions $\nabla u, \nabla v \in L^p(\Omega)$, and, also representing the expression $\mu^*(y^*)$ in terms of the functions $z^* \in H_q$, we obtain a lower estimate for the left-hand side of the identity of the Theorem 2. Applying the general Young inequality and the Minkowski inequality for integrals to the right side of the identity in Theorem 2 leads us to an upper estimate for the left side of the identity. And further, using the projection inequality and the Cauchy inequality with sufficiently small ϵ, we obtain

Theorem 3: For any functions $v \in \mathbb{K}$, $z^* \in H_q$ and parameter $p \geq 2$ the total measure of the deviation of these functions from the exact solutions of the primal and dual problems satisfies the inequality:

$$\frac{(1 - 2^{p-1}\epsilon)}{2^{p-1}p}\lVert \nabla(u - v)\rVert^p_{L^p} + \int_{\{u>\phi\}} (f + \operatorname{div} z^*)(\phi - u)\,dx \leq \leq \mathfrak{M}(v, z^*, f, \phi, \epsilon),$$

where

$$\mathfrak{M}(v, z^*, f, \phi, \epsilon) := \frac{2}{p}\lVert \nabla v\rVert^p_{L^p} + \frac{2^q}{q}\lVert z^*\rVert^q_{L^q} + \int_\Omega (f + \operatorname{div} z^*)(\phi - v)\,dx + \frac{C_F^q(1 + 2^q\epsilon^{\frac{p}{q}})}{q\epsilon^{\frac{p}{q}}}\lVert (f + \operatorname{div} z^*)_+\rVert^q_{L^q}.$$