Construction of Approximate Solutions to an Elliptic Variational Problem with a Thin Obstacle

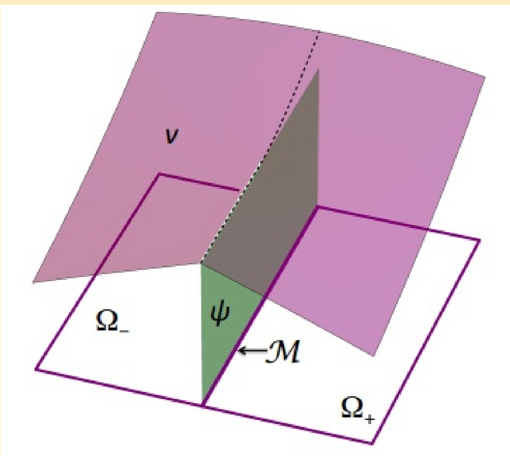
A.N. Potyomkina[1]

[1]) Peoples' Friendship University of Russia (RUDN University), Moscow; e-mail: alina.bodnya@yandex.ru

Presented results were obtained under supervision of Darya Apushkinskaya

Formulation of the problem

We consider elliptic variational inequalities generated by obstacle type problems with thin obstacles. This problem with a free boundary is reduced to minimizing the functional



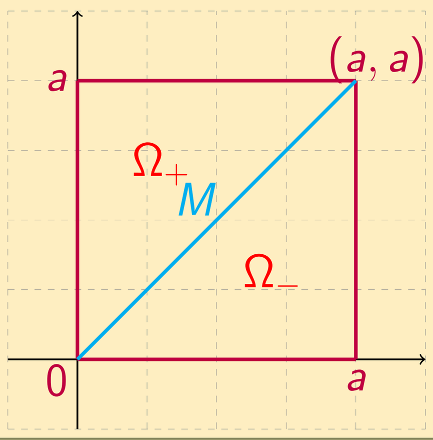
$$J[v] = \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx \qquad (1)$$

on a closed convex set $\mathbb{K} = \left\{ v \in H^1(\Omega) : \; v \geq \psi \text{ on } \mathcal{M} \cap \Omega, \; v = \varphi \text{ on } \partial\Omega \right\}$. Here Ω is an open, connected, and bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, $\mathcal{M}$ is smooth one – dimensional manifold in $\mathbb{R}^2$, which divides Ω into two Lipschitz subdomains Ω_- and Ω_+. The given functions $\psi : \mathcal{M} \to \mathbb{R}$ is a sufficiently smooth and $\varphi \in H^{1/2}(\partial\Omega)$. For this class of problems, we construct an approximate solution.

Construction of an approximate solution to the problem (1)

Let w_0 be a continuation of φ from $\partial\Omega$ to Ω and let $v = v_0 + w_0$. Obviously $v_0|_{\partial\Omega} = 0$, $w_0|_{\partial\Omega} = \varphi$. As a result, the original problem turns into a variational problem

$$\tilde{J}(v_0) = \frac{1}{2}\int_\Omega |\nabla v_0|^2 dx + \int_\Omega \nabla v_0 \nabla w_0 dx + \frac{1}{2}\int_\Omega |\nabla w_0|^2 dx \to min$$



Algorithm

$v = v_0 + w_0$, $v_0|_{\partial\Omega} = 0$, $w_0|_{\partial\Omega} = \varphi$

w_0 — continuation from the border to the region

Coordinate relaxation

Affine approximation

w_0

$\varphi \in H^{1/2}(\partial\Omega)$ — such a function allows continuations with $\partial\Omega$ inside Ω.
As a reference, we will take the exact solution and compare it with it. Therefore, when calculating our example, we take $\varphi = u|_{\partial\Omega}$, and then φ, continued inside, is u itself, where

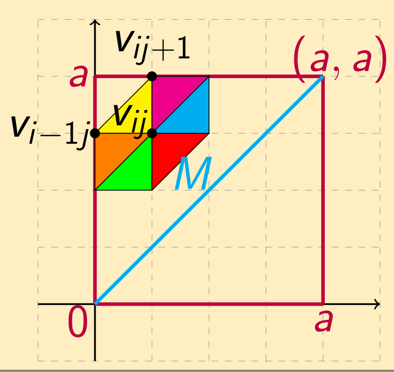
$$u(x, y) = \mathrm{Re}\left(\left[\frac{x+y}{\sqrt{2}} - a + i \left| \frac{y-x}{\sqrt{2}} \right| \right]^{\frac{3}{2}} \right).$$

We get that $w_0 = u$ in the entire domain.

Affine approximation

$$\tilde{J}(v_0) = \frac{1}{2}\int_\Omega |\nabla v_0|^2 dx + \int_\Omega \nabla v_0 \nabla w_0 dx$$

$$\begin{cases} v_{ij}(x_i, y_j) = Ax_i + By_j + C \\ v_{ij+1}(x_i, y_{j+1}) = Ax_i + By_{j+1} + C \\ v_{i-1j}(x_{i-1}, y_j) = Ax_{i-1} + By_j + C \end{cases}$$



Coordinate relaxation

If $i \neq j$:

$$v_{ij} \longrightarrow \tilde{J}(v_{ij})$$
$$v_{ij} + h \longrightarrow \tilde{J}(v_{ij} + h)$$
$$v_{ij} - h \longrightarrow \tilde{J}(v_{ij} - h)$$
$$v_{ij} = argmin(\tilde{J}(v_{ij}), \tilde{J}(v_{ij} + h), \tilde{J}(v_{ij} - h))$$

If $i = j$:

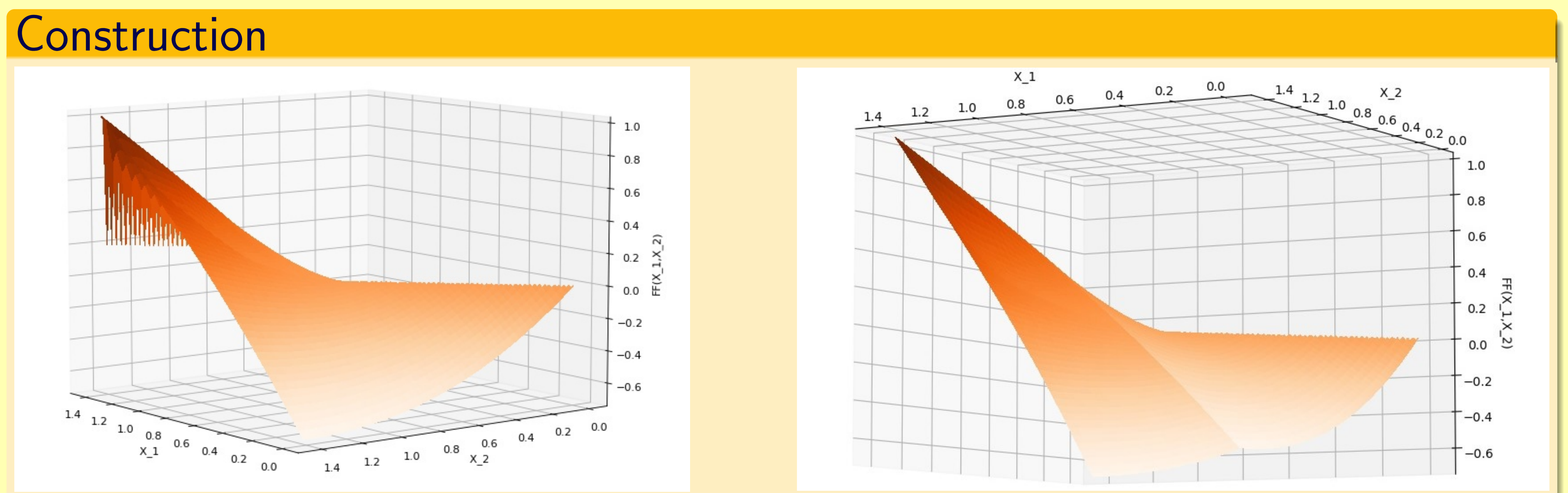
$$v_{ii} \longrightarrow \tilde{J}(v_{ii})$$
$$v_{ii} + h \longrightarrow \tilde{J}(v_{ii} + h)$$
$$v_{ii} - h \longrightarrow \tilde{J}(v_{ij} - h) \text{ and } v_{ii} - h \geq -w_{0ii}$$
$$v_{ii} = argmin(\tilde{J}(v_{ii}), \tilde{J}(v_{ii} + h), \tilde{J}(v_{ii} - h))$$

Numerical results

As an indicator of the accuracy of the approximate solution, we take the L^2-norm of the gradient of the difference between the exact and approximate solutions.

Number of iterations	$\|\nabla(u - v)\|_{2,\Omega}$
10	3.684996763005937
20	2.7119223135718897
30	2.0812470516475314
50	1.465314923616265
60	1.19669961812561
80	0.8244171157921146
90	0.7388018907419068
100	0.6719099635315782
200	0.40473180451218443

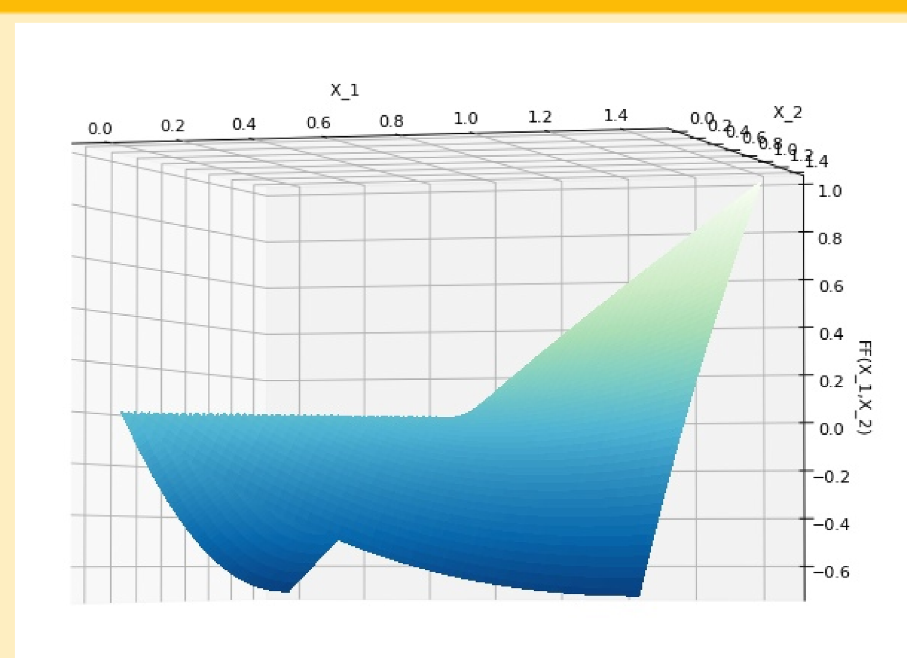
Construction



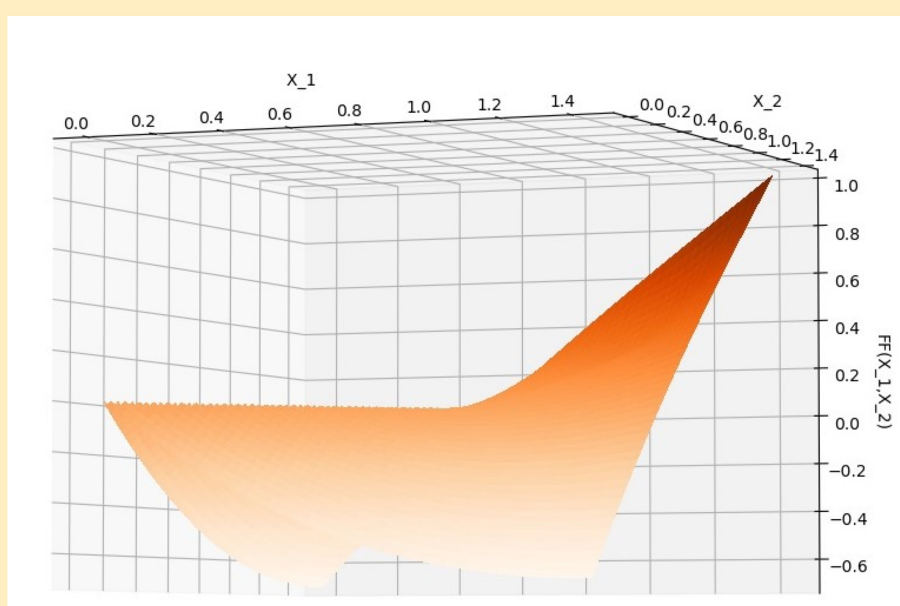
20 iterations　　200 iterations

Figure: Approximate solution

Comparison



Exact solution



Approximate solution

References

[1] D.E. Apushkinskaya, S.I. Repin, *Thin obstacle problem: Estimates of the distance to the exact solution.* Interfaces and Free Boundaries **20** (2018), 511-531.
[2] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems.* Springer, New York, 1984.
[3] Amosov A.A., Yu Dubinsky.A., Kopchenova N.V., or *Computational methods for engineers* **15** (1994), 487-490.
[4] In Berdyshev.I., Petrak L.B. or *Approximation of functions, compression of numerical information, applications.* Yekaterinburg: Ural Branch of the Russian Academy of Sciences, 1999.
[5] Kikuchi, N. & Oden, J. T., *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, vol. 8 *of SIAM Studies in Applied Mathematics.* Philadelphia, PA, 1988.