On the impact of dissipation ratio on vanishing viscosity solutions of Riemann problems for chemical flooding models

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1. Introduction

We are interested in solutions of the initial-value problem (1)–(2) arising in chemical flooding models for enhanced oil recovery (EOR). To distinguish physically meaningful weak solutions we use vanishing viscosity admissibility criterion. We demonstrate that when the flow function depends non-monotonically on the chemical agent concentration (which corresponds to the surfactant flooding), non-classical undercompressive shocks appear. They correspond to the saddle-saddle connections for the traveling wave dynamical system and are sensitive to precise form of the dissipation terms. The talk is based on preprint [1]. See a related work [2]. The analysis of a monotone case was done in [3], where it was proven that only Lax shocks have dissipative profiles.

2. Problem statement

Consider two-phase oil-water flow with dissolved chemical in water in porous media:

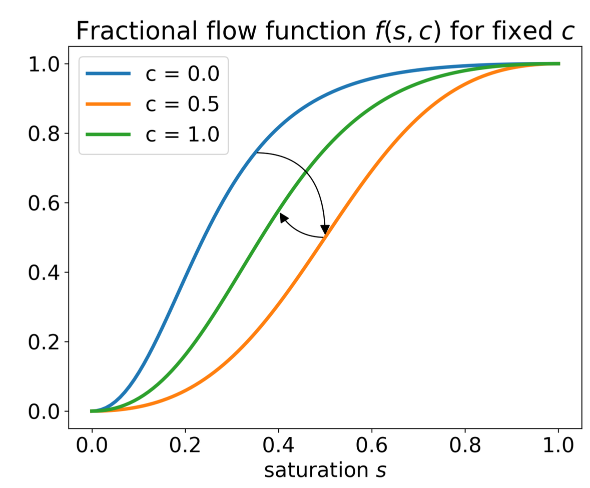
$$\begin{aligned} s_t + f(s,c)_x &= 0, \\ [c\,s + a(c)]_t + [c\,f(s,c)]_x &= 0, \end{aligned} \tag{1}$$

- $s \in [0,1]$ — water saturation
- $c \in [0,1]$ — chemical concentration
- $f(s,c)$ — fractional flow function
- $a(c)$ — adsorption function ($a(0)=0$, increasing, concave)

Find exact solution $s(x,t)$ and $c(x,t)$ to a Riemann problem:

$$(s,c)\Big|_{t=0} = \begin{cases} (1,1), & x < 0, \\ (0,0), & x \geq 0. \end{cases} \tag{2}$$

3. Properties of function f



(F1) $f \in C^2([0,1]^2)$; $f(0,c)=0$; $f(1,c)=1$;

(F2) $f_s(s,c) > 0$ for $0 < s < 1$, $0 \leq c \leq 1$; $f_s(0,c) = f_s(1,c) = 0$;

(F3) f is S-shaped in s: for each $c \in [0,1]$ function $f(\cdot,c)$ has a unique point of inflection $s^I = s^I(c) \in (0,1)$, such that $f_{ss}(s,c) > 0$ for $0 < s < s^I$ and $f_{ss}(s,c) < 0$ for $s^I < s < 1$.

(F4) f is non-monotone in c: $\forall s \in (0,1)\, \exists c^*(s) \in (0,1)$: $f_c(s,c) < 0$ for $0 < s < 1$, $0 < c < c^*(s)$; $f_c(s,c) > 0$ for $0 < s < 1$, $c^*(s) < c < 1$.

4. Vanishing viscosity formulation and travelling wave dynamical system

For model (1) we call a shock between states (s^-, c^-) and (s^+, c^+) (for $c^+ \neq c^-$) **admissible** if it could be obtained as a limit of smooth travelling wave solutions of (3) as $\varepsilon_{c,d} \to 0$

$$\begin{aligned} s_t + f(s,c)_x &= \varepsilon_c (A(s,c) s_x)_x, \\ (cs + \alpha a(c))_t + (cf(s,c))_x &= \varepsilon_c (cA(s,c)s_x)_x + \varepsilon_d (c_x)_x. \end{aligned} \tag{3}$$

Here $A(s,c)$ bdd from zero and infinity function. Denote $\kappa := \varepsilon_d/\varepsilon_c$. We are looking for a travelling wave solution of the form $s(x,t) = s(\varepsilon_c^{-1}(x - vt)), c(x,t) = c(\varepsilon_c^{-1}(x-vt))$, connecting (s^-,c^-) and (s^+,c^+), i.e. $s(\pm\infty) = s^\pm, c(\pm\infty) = c^\pm$. We get the following dynamical system:

$$\begin{cases} A(s,c)s_\xi = f(s,c) - v(s + d_1), \\ \kappa c_\xi = v(d_1 c - d_2 - a(c)), \end{cases} \tag{4}$$

Here d_1, d_2 are constants fully determined by $c^\pm, a(c^\pm)$. We focus on the existence of a saddle-to-saddle orbit for (4) depending on the parameters v, κ. This restriction comes from the compatibility of speeds condition in a sequence of waves for the solution to a Riemann problem (2).

5. Main theorem formulation

There exist $0 < v_{\min} < v_{\max} < \infty$, such that for all $\kappa = \varepsilon_d/\varepsilon_c \in (0, +\infty)$, there exists a unique

- points $s^-(\kappa) \in [0,1]$ and $s^+(\kappa) \in [0,1]$;
- velocity $v(\kappa) \in [v_{\min}, v_{\max}]$,

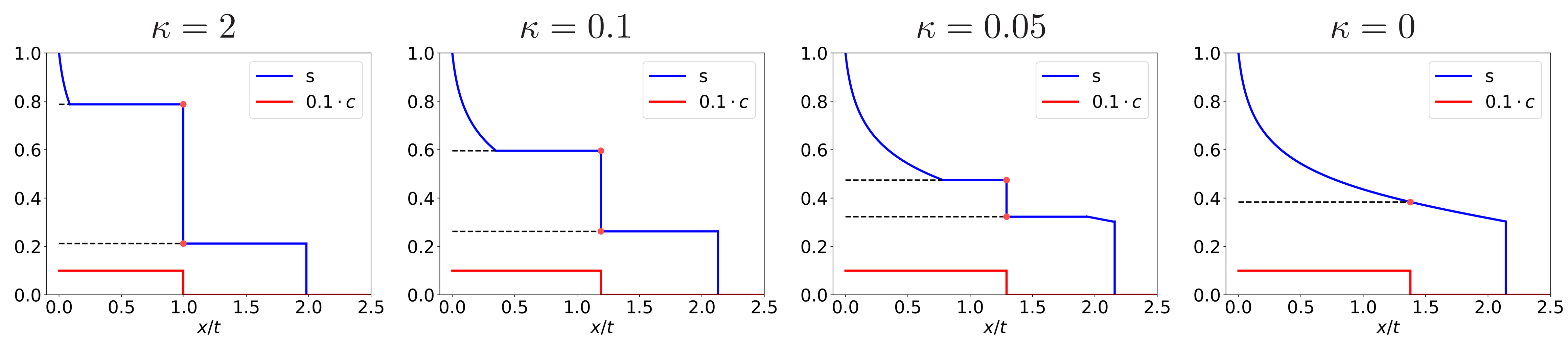
such that there exists a travelling wave solution of (3), moving with velocity $v(\kappa)$ and connecting two saddle points $u^-(\kappa) = (s^-(\kappa), c^-)$ and $u^+(\kappa) = (s^+(\kappa), c^+)$ of the corresponding dynamical system (4). Moreover, $v(\kappa)$ is monotone and continuous; $v(\kappa) \to v_{\min}$ as $\kappa \to \infty$; $v(\kappa) \to v_{\max}$ as $\kappa \to 0$.

6. Example of solution to a "boomerang" model

Consider the simplest model with non-monotone flow function (we call it the "boomerang" model): the fractional flow function f decreases in c from $c = 0$ up to some value $c^M \in (0,1)$, and then increases from c^M to $c = 1$ back to the same function, i.e. $f(s,1) = f(s,0)$.

$$f(s,c) = \frac{s^2}{s^2 + \mu(c)(1-s)^2}, \qquad \mu(c) = 1 + 4c(1-c), \qquad c^M = 0.5.$$

Depending on $\kappa = \varepsilon_d/\varepsilon_c$ the solution to a Riemann problem is different (see fig. below). Note that the shock wave that satisfies the classical Lax criterion gives the solution s that doesn't reflect any change of flow function (corresponds to the case $\kappa = 0$) and contradicts the physical intuition. This motivates us to use physically appropriate vanishing viscosity criterion.

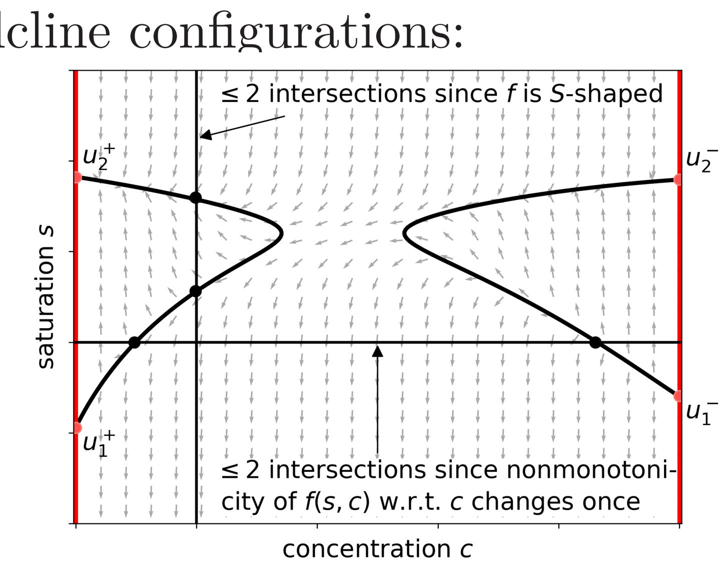


7. Sketch of proof

The Theorem can be divided into simpler statements:

- $\forall v \in [v_{\min}, v_{\max}]$ $\exists! \kappa(v)$: there is a saddle-to-saddle travelling wave with $\kappa(v)$.
- $\kappa(v)$ is continuous.
- $\nexists v_1 \neq v_2 : \kappa(v_1) = \kappa(v_2)$, thus $\kappa(v)$ is monotone.
- $\kappa(v) \to \kappa_{crit} \geqslant 0$ as $v \to v_{max}$.
- when $\kappa < \kappa_{crit}$ and $v = v_{\max}$ there is a saddle to saddle-node travelling wave

We analyze and classify all possible nullcline configurations:



9. References

[1] F. Bakharev, A. Enin, Yu. Petrova, and N. Rastegaev. Impact of dissipation ratio on vanishing viscosity solutions of the riemann problem for chemical flooding model. *arXiv preprint arXiv:2111.15001*, 2021.

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[3] T. Johansen and R. Winther. The solution of the riemann problem for a hyperbolic system of conservation laws modeling polymer flooding. *SIAM journal on mathematical analysis*, 19(3):541–566, 1988.