Singularities of solutions to the equations of an isentropic gas flow and singularities of solutions to the linear wave equation

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Solutions of the 1D hyperbolic quasilinear gas dynamics system have the same typical germs of catastrophes as the general solution of 1D linear wave equation. We have obtained the canonical form of the section of the hyperbolic umbilical catastrophe D_{4+}:

$$F = \frac{y_1^3 + y_2^3}{3} + k_3(k_1, k_2)y_1y_2 + k_2y_1 + k_1y_2.$$

We state that a similar "inheritance" should take place for typical singularities of solutions to systems of equations of an isentropic gas flow in spatially non-1D cases as well.

The classification of catastrophes of solutions to the more general quasilinear system

$$w_t + \lambda(w, z, x, t)w_x = g(w, z, x, t), \quad z_t + \mu(w, z, x, t)z_x = h(w, z, x, t), \lambda_w\lambda_z\mu_w\mu_z \neq 0$$

was made by A. Kh. Rakhimov [2]:

A_2	$x_1 + y_1^2 = 0, \quad x_2 + y_2 = 0$
A_3	$x_1 + y_1^3 + y_1y_2 = 0, x_2 + y_2 = 0$
$C_{2,2}$	$y_1^2 + x_1 + x_2y_2 = 0, \quad y_2^2 + x_2 + x_1y_1 = 0$

Knowing the catastrophes of solutions, we can understand the asymptotics in the neigbourhood of the gradient catastrophe point. It is the finite point where all first derivatives of the solution tend to infinity.

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Gas dynamics

One-dimensional gas dynamics equations

$$\begin{cases} u_t + uu_x + \alpha(\rho)\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1}$$

where $\alpha(\rho) = \frac{p_\rho}{\rho} > 0$ is a smooth function such that $\alpha(\rho) = 4 + \sum_{i\geq 1} \frac{\alpha_i}{i!}\Delta\rho^i$ and $\rho_*\alpha_1 + 12 \neq 0$ (otherwise it's not the general situation, it's the very special case). $\Delta\rho = \rho = \rho_*$, $\rho_* > 0$. $p(\rho)$ is an equation of the gas state. The flow is isentropic, so $\mathrm{d}S \equiv 0$.

Riemann invariants

$$r = u + \int_0^\rho \frac{c}{\rho}\mathrm{d}\rho, \quad l = u - \int_0^\rho \frac{c}{\rho}\mathrm{d}\rho, \quad c^2 = p_\rho = \rho\alpha(\rho) \tag{2}$$

(where $c > 0$ is the sonic speed, $r \neq l$, otherwise $\rho = 0$) diagonalize (1):

$$\begin{cases} r_t + (\frac{r+l}{2} + c)r_x = 0 \\ l_t + (\frac{r+l}{2} - c)l_x = 0. \end{cases} \tag{3}$$

The hodograph transformation

$$r_x = Jt_l, \quad r_t = -Jx_l, \quad l_x = -Jt_r, \quad l_t = Jx_r, \quad J = r_xl_t - r_tl_x, \quad j = x_rt_l - x_lt_r, \quad J = j^{-1},$$

linearises (3):

$$\begin{cases} x_l = t_l(\frac{r+l}{2} + c), \\ x_r = t_r(\frac{r+l}{2} - c), \end{cases}$$

so we have the hodograph transformation's jacobian

$$j = -2ct_rt_l. \tag{4}$$

Our goal is to study the catastrophe of solution to system (3) in case $j(r_*, l_*; t_*, x_*) = 0$, so the mapping $(t, x) \to (r, l)$ is no longer a diffeomorphism. All of derivatives of solutions to (3) approache infinity at the finite point $(r_*, l_*; t_*, x_*)$. The gradient catastrophe occurs.

The hodograph transformation and substitutions

$$t = B_u, \quad x = uB_u - B - \rho B_\rho \tag{5}$$

reduce (1) to the 2nd order hyperbolic equation $\alpha(\rho)B_{uu} = \rho B_{\rho\rho} + 2B_\rho$ or

$$8\alpha B_{rl} = \left(\alpha_\rho + 3\frac{\alpha}{\rho}\right)\sqrt{\frac{\rho}{\alpha}}(B_r - B_l), \tag{6}$$

where $B = \sum_{i+j\geq 0} b_{ij}\Delta r^i \Delta l^j$ is a formal power series defining the solutuion. The equation (6) is hyperbolic and it means that any formal solution corresponds to true solution.

$j(r_*, l_*; t_*, x_*) = -2ct_rt_l(r_*, l_*; t_*, x_*) = 0$ if $r \neq l$ (otherwise $\rho = 0$) means $b_{11} = -2b_{20}$ or $b_{11} = -2b_{02}$. We can impose no more than two restrictions on coefficients b_{ij} as governing parameters of the catastrophe perturbation can depend only on t, x. So we demand $b_{11} = -2b_{20} = -2b_{02} \Rightarrow b_{20} = b_{02}$. From the equation and (5) we have

$$b_{20} = -\frac{(b_{10}-b_{01})(\alpha_1\rho_*+12)}{128\sqrt{\rho_*}}, \quad b_{21} = -b_{12} = \frac{(b_{10}-b_{01})(2\rho_*^2\alpha_2-8\rho_*\alpha_1-\rho_*^2\alpha_1^2-48)}{1024\rho_*},$$
$$t_* = b_{10} + b_{01}, \quad x_* = \frac{r_*+l_*}{2}(b_{10} + b_{01}) - b_{00} - 2\sqrt{\rho_*}(b_{10} - b_{01}), \ldots$$

Relations (5) are equivalent to zeroes of derivatives of the function

$$F = \rho(ut - x - B) \tag{7}$$

of arguments u, ρ and parameters t, x and $j(r_*, l_*, t_*, x_*) = 0$ corresponds to $F_u(u_*, \rho_*, t_*, x_*) = 0$, $F_\rho(u_*, \rho_*, t_*, x_*) = 0$. There is no such "potential" function in the Rakhimov's study.

The description of typical degenerate critical points of locally smooth fuctions suchlike (7) is the typical problem of the catastrophe theory.

Considering all relations between b_{ij} we can write the expansion of F:

$$\begin{aligned} F = {} & 2\rho_*^{3/2}(b_{10} - b_{01}) + \rho_*z + \frac{\rho_*\Delta t}{2}(\Delta r + \Delta l) + \frac{\sqrt{\rho_*}}{4}z(\Delta r - \Delta l) + \\ & + \frac{\sqrt{\rho_*}}{8}\Delta t((\Delta r)^2 - (\Delta l)^2) + z(\Delta r - \Delta l)^2 + \frac{4-\alpha_1\rho_*}{512}\Delta t(\Delta r - \Delta l)^2\frac{\Delta r+\Delta l}{2} + \\ & + h_3z(\Delta r - \Delta l)^3 + A_+(\Delta r)^3 + A_-(\Delta l)^3 + \sum_{i+j\geq 4}(f_{ij}^0 + f_{ij}^1\Delta t + f_{ij}^2 z)(\Delta r)^i(\Delta l)^j, \end{aligned} \tag{8}$$

where

$$\Delta t = t - t_*, \quad z = \frac{(r_*+l_*)\Delta t}{2} - x + x_*, \quad h_2 = \frac{(4-\rho_*\alpha_1)}{256}, \quad h_3 = \frac{(\rho_*^{3/2}(\alpha_1^2-2\alpha_2)-4\sqrt{\rho_*}\alpha_1)}{6144}, \quad f_{ij}^0 = const,$$
$$A_+ = \frac{(b_{10}-b_{01})[8\rho_*\alpha_1+48-2\rho_*^2\alpha_2+\rho_*^2\alpha_1^2]}{3072} - \rho_*b_{30}, \quad A_- = -\frac{(b_{10}-b_{01})[8\rho_*\alpha_1+48-2\rho_*^2\alpha_2+\rho_*^2\alpha_1^2]}{3072} - \rho_*b_{03}.$$

According to the form of (8) function $F(r, l; t, x)$ is the 2-deformation of the function $F(r_*, l_*; t, x)$ and can be induced from the versal deformation defined by the three-parameter function family $G_{k_1,k_2,k_3}(y_1, y_2) = \frac{(y_1)^3+(y_2)^3}{3} - k_3y_1y_2 - k_2y_1 - k_1y_2$, which means $F(r, l; t, x)$ can be defined in the form

$$F(r, l; t, x) = \frac{(y_1)^3 + (y_2)^3}{3} - k_3y_1y_2 - k_2y_1 - k_1y_2 + \gamma, \tag{9}$$

where $k_j = k_j(t, x)$, $j = 1, 2, 3$ and $\gamma = \gamma(t, x)$ – smooth in the neigbourhood of the points $t = t_*$, $x = x_*$ functions; $y_i = y_i(r, l; t, x)$, $i = 1, 2$ is the locally smooth change of coordinates $(r, l, t, x) \to (y_1(r, l, t, x), y_2(r, l, t, x))$ which is the local diffeomorphism if t and x are fixed.

So we can define the canonical form of the umbilical catastrophe from the expansion (8) using not just formal transformations but smooth and bijective.

The critical points $F_{y_1} = 0$, $F_{y_2} = 0$ define the solutions to the gas dynamics system (1):

$$\begin{aligned} y_1^2 &= k_3(k_1, k_2)y_2 + k_2, \\ y_2^2 &= k_3(k_1, k_2)y_1 + k_1. \end{aligned} \tag{10}$$

Now we show that k_3 depends on k_1, k_2 and construct the transformation:

$$\begin{aligned} y_1(r, l; t, x) = {} & \sum_{n+m>0} a_{nm,00}(\Delta t)^n z^m + \Delta r[(3A_+)^{1/3} + \sum_{n+m>0} a_{nm,10}(\Delta t)^n z^m] + \\ & + \Delta l \sum_{n+m>0} a_{nm,01}(\Delta t)^n z^m + \sum_{i+j>1}\sum_{n+m=0}^{\infty} a_{nm,ij}(\Delta r)^i(\Delta l)^j(\Delta t)^n z^m, \\ y_2(r, l; t, x) = {} & \sum_{n+m>0} b_{nm,00}(\Delta t)^n z^m + \Delta l[(3A_-)^{1/3} + \sum_{n+m>0}^{\infty} b_{nm,01}(\Delta t)^n z^m] + \\ & + \Delta r \sum_{n+m>0} b_{nm,10}(\Delta t)^n z^m + \sum_{i+j>1}\sum_{n+m=0}^{\infty} b_{nm,ij}(\Delta r)^i(\Delta l)^j(\Delta t)^n z^m, \\ k_j(t, x) = {} & k_{j,10}\Delta t + k_{j,01}z + \textstyle\sum_{n+m>1} k_{j,nm}(\Delta t)^n z^m, \qquad j = 1, 2, 3. \end{aligned}$$

Here all coefficients are to be determined. We have the following relations:

$$a_{00,20} = \frac{f_{40}^0}{(3A_+)^{2/3}}, \quad a_{00,11} = \frac{f_{31}^0}{(3A_+)^{2/3}}, \quad b_{00,11} = \frac{f_{13}^0}{(3A_-)^{2/3}}, \quad b_{00,02} = \frac{f_{04}^0}{(3A_-)^{2/3}},$$
$$k_{2,10} = -\frac{\rho_*}{2(3A_+)^{1/3}}, k_{2,01} = -\frac{\sqrt{\rho_*}}{4(3A_+)^{1/3}}, \qquad k_{1,10} = -\frac{\rho_*}{2(3A_-)^{1/3}}, k_{1,01} = \frac{\sqrt{\rho_*}}{4(3A_-)^{1/3}},$$
$$k_{3,10} = -\frac{k_{2,10}a_{00,11}+k_{1,10}b_{00,11}}{(9A_+A_-)^{1/3}}, \qquad k_{3,01} = -\frac{k_{2,01}a_{00,11}+k_{1,01}b_{00,11}-2h_2}{(9A_+A_-)^{1/3}},$$
$$k_{10} = -\frac{1}{(3A_-)^{1/3}}\left[\frac{4h_2}{\rho_*} + \frac{f_{31}^0}{3A_+}\right], \quad k_{01} = \frac{1}{(3A_+)^{1/3}}\left[\frac{4h_2}{\rho_*} - \frac{f_{13}^0}{3A_-}\right], \ldots$$
$$k_3 = k_{10}k_1 + k_{01}k_2 + \textstyle\sum_{u+j\geq 2} k_{ij}k_1^ik_2^j.$$

Wave equation

The one-dimensional homogeneous linear wave equation with constant coefficients

$$u_{tt} = u_{xx} \tag{11}$$

is equivalent to the system $u_t = v_x, v_t = u_x$ which can be reduced from the linearization of (1). By using the hodograph transformation we have the system $t_v = x_u, t_u = x_v$ with the jacobian of the hodograph trasformation $j = (t_v)^2 - (t_u)^2$. Similar to (5) substitutions $t = B_u, x = B_v$ allow us to define the solutions of (11) by the general solution

$$B = f(u + v) + g(u - v) \tag{12}$$

of the wave equation $B_{uu} = B_{vv}$. Here $f(u+v)$ and $g(u-v)$ are infinitely differentiable functions with the expansions $f = f_0 + \sum_{j=1}^{\infty}\frac{f_j}{j!}(\Delta u + \Delta v)^j$ and $g = g_0 + \sum_{j=1}^{\infty}\frac{g_j}{j!}(\Delta u - \Delta v)^j$ respectively.

Now we consider the function

$$\Psi(u, v; t, x) = ut + vx - B(u, v) \tag{13}$$

analogous to the function F (7) such that relations $\Psi_u = 0$, $\Psi_v = 0$ are equivalent to $t = B_u$ and $x = B_v$.

Now we have relations

$$j(u_*, v_*; t_*, x_*) = f_2g_2 = 0, \quad t_* = f_1 + g_1, \quad x_* = f_1 - g_1.$$

Considering these let us write the expansion of (13):

$$\begin{aligned} \Psi = {} & (f_1 + g_1)u_* + (f_1 - g_1)v_* - (f_0 + g_0) + u_*\Delta t + v_*\Delta x + \\ & + \frac{(\Delta t+\Delta x)}{2}\widetilde{u} + \frac{(\Delta t-\Delta x)}{2}\widetilde{v} - \sum_{i=2}^{\infty}\frac{f_i}{i!}(\widetilde{u})^i - \sum_{j=2}^{\infty}\frac{g_j}{j!}(\widetilde{v})^j, \end{aligned}$$

where $\widetilde{u} = (\Delta u + \Delta v)$, $\widetilde{v} = (\Delta u - \Delta v)$. Using the well known general catastrophe theory technics we can describe the following catastrophe germs:

$$\begin{aligned} & g_2 \neq 0, , f_2 = 0, f_3 \neq 0 \Rightarrow \Psi_{germ} = -f_3\frac{(\widetilde{u})^3}{6}; \\ & g_2 \neq 0, f_2 = f_3 = 0, f_4 \neq 0 \Rightarrow \Psi_{germ} = -f_4\frac{(\widetilde{u})^4}{4!}; \\ & f_2 = g_2 = 0, f_3 \neq 0, g_3 \neq 0 \Rightarrow \Psi_{germ} = -f_3\frac{(\widetilde{u})^3}{6} - g_3\frac{(\widetilde{v})^3}{6}. \end{aligned} \tag{14}$$

All of them can be transformed to the corresponding germs of the soultions to the (1). That's why we can say about "inheritance" of catastrophes from the solutions to the wave equation.

Notes and goals

In the Becker-Stanyukovich case $p = c_1^2\rho^3$, $c_1 = const$ there is no k_3 term in the form (9).

If $\rho_*\alpha_1 = -12$ there is no terms $b_{ij}, i + j = 2$ in the series defining the solution to the equation (6).

"Inheritance" occurs in the elliptic case ($\alpha(\rho) < 0$) as well but we can't prove the conformity between the formal and true solution. This "inheritance" was noted in [1] but it was stated that $k_3 \equiv 0$ in the case $\alpha(\rho) = const$ (shallow water equations) which is not so.

There are so called "dropping" (means $\rho_* = 0$) cusp catastrophes that were described for both cases of $\alpha(\rho) > 0$ [3] and $\alpha(\rho) < 0$ [4] in (1). They are not "inherited" but similar "dropping" singularities should be typical for non-1D gas dynamics systems as well.

Right now we study 2D homogeneous linear wave equation with constant coefficients. The next object is the 3D homogeneous linear Laplace equation with constant coefficients.

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