Solvability of mixed control problems for the class of degenerate nonlinear equations with fractional derivatives

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In a reflexive Banach spaces $\mathcal{X}$, $\mathcal{Y}$ a continuous linear operator $L \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is given, $M \in \mathcal{C}l(\mathcal{X},\mathcal{Y})$ is a linear closed operator with domain D_M dense in $\mathcal{X}$, $\mathcal{U}$ is a Banach space, $B \in \mathcal{L}(\mathcal{U};\mathcal{Y})$, $N : \mathbb{R} \times \mathcal{X}^n \to \mathcal{Y}$ is a nonlinear operator, $\mathfrak{U} = L_q(t_0,T;\mathcal{U}) \times \mathcal{X}^m$ is a control space with the norm $\|(u,v)\|^2_{\mathfrak{U}} = \|u\|^2_{L_q(t_0,T;\mathcal{U})} + \|v\|^2_{\mathcal{X}^m}$.

$$LD_t^\alpha x(t) = Mx(t) + N(t, D_t^{\alpha_1}x(t), \dots, D_t^{\alpha_n}x(t)) + Bu(t), \quad t \in (t_0,T), \tag{1}$$

$$(Px)^{(k)}(t_0) = v_k, \;\; k = 0,1,\dots,m-1, \tag{2}$$

$$(u,v) = (u, v_0, v_1, \dots, v_{m-1}) \in \mathcal{U}_\partial, \tag{3}$$

$$J(x,u,v) \to \inf, \tag{4}$$

where $0 \le \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m-1 < \alpha \le m$, $m \in \mathbb{N}$, D_t^β is the Gerasimov — Caputo derivative, $k = 0,1,\dots,m-1$, $\mathcal{U}_\partial$ is a set of admissible controls, $\mathcal{U}_\partial \subset \mathfrak{U}$, J is the cost functional. We accept the following notation $g_\delta(t) := \Gamma(\delta)^{-1}t^{\delta-1}$ for $\delta > 0$, $t > 0$, $\tilde{g}_\delta(t) := \Gamma(\delta)^{-1}(t-t_0)^{\delta-1}$, $J_t^\delta h(t) := \int\limits_{t_0}^{t} g_\delta(t-s)h(s)ds$ for $t > t_0$. Let $m-1 < \alpha \le m \in \mathbb{N}$, D_t^m is an ordinary derivative, wich order is $m \in \mathbb{N}$, J_t^0 is an identity operator. The Gerasimov — Caputo derivative of fuction h defined as follows

$$D_t^\alpha h(t) = D_t^m J_t^{m-\alpha}\left(h(t) - \sum_{k=0}^{m-1} h^{(k)}(t_0)\tilde{g}_{k+1}(t)\right), \quad t \ge t_0.$$

Strong solution for a nonlinear fractional equation

Definition

Define L-resolvent set $\rho^L(M) := \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y};\mathcal{X})\}$ of an operator M and its L-spectrum $\sigma^L(M) := \mathbb{C}\setminus\rho^L(M)$, and denote $R^L_\mu(M) := (\mu L - M)^{-1}L$, $L^L_\mu(M) := L(\mu L - M)^{-1}$.

Definition

An operator M is called (L,σ)-bounded, if $\exists a > 0\ \forall \mu \in \mathbb{C}\ (|\mu| > a) \Rightarrow (\mu \in \rho^L(M))$.

Under the condition of (L,σ)-boundedness of operator M we have the projections

$$P = \frac{1}{2\pi i}\int\limits_\gamma R^L_\mu(M)\,d\mu \in \mathcal{L}(\mathcal{X}),\ \ Q = \frac{1}{2\pi i}\int\limits_\gamma L^L_\mu(M)\,d\mu \in \mathcal{L}(\mathcal{Y}),$$

where $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$. Put $\mathcal{X}^0 := \ker P$, $\mathcal{Y}^0 := \ker Q$; $\mathcal{X}^1 := \mathrm{im} P$, $\mathcal{Y}^1 := \mathrm{im} Q$. Denote by L_k (M_k) the restriction of the operator L (M) on $\mathcal{X}^k$ $(D_{M_k} = D_M \cap \mathcal{X}^k)$, $k = 0, 1$.

Theorem ((1) Sviridyuk G. A., Fedorov V. E.)

Let an operator M be (L,σ)-bounded. Then

(i) $M_1 \in \mathcal{L}(\mathcal{X}^1, \mathcal{Y}^1)$, $M_0 \in Cl(\mathcal{X}^0, \mathcal{Y}^0)$, $L_k \in \mathcal{L}(\mathcal{X}^k, \mathcal{Y}^k)$, $k = 0, 1$, *then* $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$;

(ii) *there exist operators* $M_0^{-1} \in \mathcal{L}(\mathcal{Y}^0, \mathcal{X}^0)$, $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1, \mathcal{X}^1)$.

Denote $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $G := M_0^{-1} L_0$. For $p \in \mathbb{N}_0$ operator M is called (L,p)-bounded, if it is (L,σ)-bounded, $G^p \neq 0$, $G^{p+1} = 0$.

$$LD_t^{\alpha} x(t) = Mx(t) + N(t, D_t^{\alpha_1} x(t), \dots, D_t^{\alpha_n} x(t)), \ t \in (t_0, T), \tag{5}$$

$$(Px)^{(k)}(t_0) = x_k, \quad k = 0, 1, \dots, m-1. \tag{6}$$

Let $\alpha_n \leq r \leq m-1$. A **strong solution** of problem (5), (6) is a function $x \in C^r([t_0,T];\mathcal{X}) \cap L_q(t_0,T;D_M)$, such that $Lx \in C^{m-1}([t_0,T];\mathcal{Y})$,

$$J_t^{m-\alpha}\left(Lx - \sum_{k=0}^{m-1} (Lx)^{(k)}(t_0)\tilde{g}_{k+1}\right) \in W_q^m(t_0,T;\mathcal{Y})$$

and equality (6) is valid and equality (5) is valid almost everywhere on (t_0, T).

Strong solution for a nonlinear fractional equation

Theorem (2)

Let $\alpha > 0$, $q > (\alpha - m + 1)^{-1}$, *an operator* M *be* $(L, 0)$*-bounded,* $N : [t_0, T] \times \mathcal{X}^n \to \mathcal{Y}$ *for all* $x_1, x_2, \dots, x_n \in \mathcal{X}$ *and almost all* $t \in (t_0, T)$ *satisfies the condition* $N(t, x_1, \dots, x_n) = N_1(t, Px_1, \dots, Px_n)$ *under some mapping* $N_1 : [t_0, T] \times (\mathcal{X}^1)^n \to \mathcal{Y}$, *thus* $QN_1 \in C^{m_n+1}([t_0, T] \times (\mathcal{X}^1)^n; \mathcal{Y})$ *uniformly Lipschitz in* $\bar{x} \in (\mathcal{X}^1)^n$, $(I - Q)N_1 \in C^m([t_0, T] \times (\mathcal{X}^1)^n; \mathcal{Y})$, $x_0, x_1, \dots, x_{m-1} \in \mathcal{X}^1$, *to solve the problem the equalities*

$$\text{если } \alpha_k < m_k, \text{ то } v^{(m_k+r)}(t_0) = 0, \ k = 1, 2, \dots, n, \ r = 0, \dots, \max\{m_n, m-1\},$$

$$D_t^k|_{t=t_0}[QN_1(t, D_t^{\alpha_1} v(t), D_t^{\alpha_2} v(t), \dots, D_t^{\alpha_n} v(t))] = 0, \quad k = 0, 1, \dots, m_n.$$

That problem (5), (6) *has a unique solution on* (t_0, T).

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*Baybulatova G.D., Plekhanova M.V. An Initial Problem for a Class of Weakly Degenerate Semilinear Equations with Lower Order Fractional Derivatives. The Bulletin of Irkutsk State University. Series Mathematics, 2021, vol. 35, pp. 34-48.

Mixed control

A strong solution of (1), (2) is a function in the space

$$\mathcal{Z}_{\alpha,q}(t_0,T;\mathcal{X}) := \{x \in L_q(t_0,T;D_M) \cap C^{m-1}([t_0,T];\mathcal{X}) :\\ J_t^{m-\alpha}\left(x - \sum_{k=0}^{m-1} x^{(k)}(t_0)\tilde{g}_{k+1}\right) \in W_q^m(t_0,T;\mathcal{X})\}.$$

Lemma (*)

$\mathcal{Z}_{\alpha,q}(t_0,T;\mathcal{X})$ is a Banach space with norm

$$\|z\|_{\mathcal{Z}_{\alpha,q}(t_0,T;\mathcal{Z})} = \|z\|_{C^{m-1}([t_0,T];\mathcal{Z})} + \|D_t^\alpha z\|_{L_q(t_0,T;\mathcal{Z})}.$$

Let the continuous operator $\gamma_0 : C([t_0,T];\mathcal{Z}) \to \mathcal{Z}$, $\gamma_0 x = x(t_0)$, $\gamma_0 \in \mathcal{L}(\mathcal{Q}_{\alpha,q}(t_0,T;\mathcal{Z});\mathcal{Z})$.

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* Plekhanova M.V. Degenerate distributed control systems with fractional time derivative / M.V. Plekhanova // Ural Mathematical Journal. – 2016. – C. 58–71.

Definition

The set of admissible elements $\mathfrak{W}$ *of problem is a set of such* $(x, u, \bar{v}) = (x, u, v_0, v_1, \ldots, v_{m-1})$, *that* $(u, \bar{v}) \in \mathcal{U}_\partial$, $x \in \mathcal{Z}_{\alpha,q}(t_0, T; \mathcal{X})$ *is a strong solution of* (1), (2) *and* $J(x, u, \bar{v}) < \infty$.

Definition

To solve problem (1)–(4) *means to find the set* $(\hat{x}, \hat{u}, \hat{v}_0, \hat{v}_1, \ldots, \hat{v}_{m-1}) \in \mathfrak{W}$, *which minimize the cost functional, i.e.,*

$$J(\hat{x}, \hat{u}, \hat{v}_0, \hat{v}_1, \ldots, \hat{v}_{m-1}) = \inf_{(x,u,\bar{v}) \in \mathfrak{W}} J(x, u, \bar{v}).$$

Definition

The functional is **coercive**, *if for all* R *the set* $\{(u, \bar{v}) \in \mathfrak{U} : J(x, u, \bar{v}) < R\}$ *is bounded.*

Proof of the main result

1. The set $\mathfrak{W}$ of admissible sets is nonempty.
2. Compactness condition.
3. Checking the coercivity of the functional.

Theorem (The main result)

Let $\alpha > 1$, $q > (\alpha - m + 1)^{-1}$, $q > (m_k - \alpha_k)^{-1}$ *for* $\alpha_k < m_k$, $k = 1, 2, \dots, n$, $\alpha_n \leq m - 2$, *an operator* M $(L, 0)$*-bounded,* $x_0, x_1, \dots, x_{m-1} \in \mathcal{X}^1$, $\mathcal{X}, \mathcal{X}_1$ *are reflexive Banach spaces,* $\mathcal{X}$ *compactly nested in space* $\mathcal{X}_1$, *the mapping* $N_1 : [t_0, T] \times \mathcal{X}_1^n \to \mathcal{Y}$ *for almost all* $t \in (t_0, T)$ *and for all* $x_1, x_2, \dots, x_n \in \mathcal{X}$ *satisfies the condition* $N(t, x_1, \dots, x_n) = N_1(t, Px_1, \dots, Px_n)$ *at some* $N_1 \in C^{m_n+1}([t_0, T] \times (\mathcal{X}^1)^n; \mathcal{Y})$*. Let* QN_1 *is uniformly Lipschitz in* $\bar{x}$ *mapping,* $(I - Q)N_1 \in C^m([t_0, T] \times (\mathcal{X}^1)^n; \mathcal{Y})$*. Assume that* $\mathcal{U}_\partial$ *is a non-empty closed convex subset in* $\mathfrak{U} = L_q(t_0, T; \mathcal{U}) \times \mathcal{X}^m$*, exists* $(u^0; v_k^0) \in \mathcal{U}_\partial$*, that* $QBu^0 \in C^{m_n+1}([t_0, T]; \mathcal{Y})$*,* $(I - Q)Bu^0 \in C^m([t_0, T]; \mathcal{Y})$*, for all solution* w *of problem*

$$D_t^\alpha w(t) = L_1^{-1} M_1 w(t) + L_1^{-1} Q N_1(t, D_t^{\alpha_1} w(t), \dots, D_t^{\alpha_n} w(t)) + L_1^{-1} Q B u^0(t),$$
$$w^{(k)}(t_0) = v_k^0, \ k = 0, 1, \dots, m-1,$$

if $\alpha_k < m_k$, then $w^{(m_k+r)}(t_0) = 0$, $k = 1, 2, \dots, n$, $r = 0, 1, \dots, m-1$, $D_t^k|_{t=t_0}[L_1^{-1} Q N_1(t, D_t^{\alpha_1} w(t), D_t^{\alpha_2} w(t), \dots, D_t^{\alpha_n} w(t)) + L_1^{-1} Q B u^0(t)] = 0$; $\mathcal{Z}_{\alpha,q}(t_0, T; \mathcal{X})$ *continuously embedded in a Banach space* $\mathbb{Y}$*, which continuously embedded in a Banach space* $W_q^{m-2}(t_0, T; \mathcal{X}_1)$*; the cost functional* J *is convex, bounded below and semi-continuous below on* $\mathbb{Y} \times L_q(t_0, T; \mathcal{U}) \times \mathcal{X}^m$*, the cost functional is coercive on* $\mathcal{Z}_{\alpha,q}(t_0, T; \mathcal{X}) \times L_q(t_0, T; \mathcal{U})$*. Then that problem* (1) – (4) *has a unique solution* $(\hat{x}, \hat{u}, \hat{v}) \in \mathcal{Z}_{\alpha,q}(t_0, T; \mathcal{X}) \times \mathcal{U}_\partial$*.*