A posteriori estimates for solutions to a problem with a thin obstacle for a second-order elliptic operator

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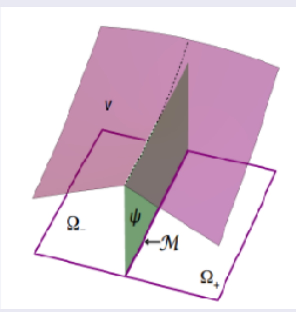
Objective

- Consider variational inequalities generated by problems with a thin obstacle for a second-order elliptic operator.
- Estimates of the distance between the exact solution and any approximate solution.
- The estimates obtained must be fully computable, i.e. depend only on known quantities.

First figure

problem with a thin obstacle



Formulation of the problem

Our aim is to minimize the functional

$$J(v) = \frac{1}{2}\int\limits_{\Omega} A(x)\nabla v \cdot \nabla v \, dx - \int\limits_{\Omega} fv \, dx \tag{1}$$

over the closed convex set

$$\mathrm{K} = \left\{ v \in H^1(\Omega): \ v \geq \Psi \text{ on } \mathcal{M} \cap \Omega, \ v = \varphi \text{ on } \partial\Omega \ \right\}.$$

Here Ω is an open, connected, and bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$, $\mathcal{M}$ is a smooth (n-1)-dimensional manifold in $\mathbb{R}^n$, which divides Ω into two Lipschitz subdomains Ω_+ and Ω_-. The functions Ψ is a sufficiently smooth obstacle function, $\varphi \in H^{1/2}$.

Applications

1. Theory of elasticity: equilibrium of an elastic membrane with a fixed edge over a thin object.
2. Temperature control.
3. The flow of fluid through a semi-permeable membrane (osmosis).

Theorem 1

For any $v \in \mathrm{K}$ holds the inequality assessment

$$\|\nabla(u-v)\|_A^2 \leq \|A\nabla v - y^*\|^2_{A^{-1}} + 2\int\limits_{\mathcal{M}} \lambda\,(v - \Psi)\, d\mu, \tag{2}$$

where

$$Q^*_{\lambda,\mathcal{M}} := \left\{ y^* \in L^2(\Omega, \mathbb{R}^n): \int\limits_{\Omega} y^* \cdot \nabla w dx = \int\limits_{\Omega} fw dx + \int\limits_{\mathcal{M}} \lambda w d\mu \quad \forall\, w \in H^1_0(\Omega) \right\}.$$

$$\Lambda := \{\lambda \in L^2(\mathcal{M}): \ \lambda(x) \geq 0 \text{ a.e. in } \mathcal{M}\},$$

u is the exact solution of the problem (1), λ and y^* are arbitrary functions from sets Λ and $Q^*_{\lambda,\mathcal{M}}$, respectively.

Lemma

For any function $q^* \in H(\Omega_\pm, \mathrm{div})$ and any function $\lambda \in \Lambda$ holds the inequality

$$\begin{aligned}\inf_{y^* \in Q^*_{\lambda,\mathcal{M}}} \|q^* - y^*\|_{A^{-1}} \leq & \mathrm{C}_1\|f + \mathrm{div}\, q^*\|_{\Omega_+} + \mathrm{C}_2\|f + \mathrm{div}\, q^*\|_{\Omega_-} \\ & + \mathrm{C}_3\|\lambda - [q^* \cdot n]\ ]\|_{\mathcal{M}}.\end{aligned} \tag{3}$$

Here $C_1 = \frac{C_{F\Omega_+}}{\sqrt{\nu}}$, $C_2 = \frac{C_{F\Omega_-}}{\sqrt{\nu}}$, $C_3 = \frac{C_{TrM}}{\sqrt{\nu}}$, $C_{F\Omega_\pm}$ and $C_{Tr\mathcal{M}}$ are the constants, defined by the Friedrichs-type inequalities.

$$\|w\|_{\Omega_\pm} \leq \ C_{F\Omega_\pm}\|\nabla w\|_{\Omega_\pm} \tag{4}$$

$$\|w\|_{\mathcal{M}} \leq \ C_{TrM}(\Omega_\pm)\|\nabla w\|_{\Omega_\pm}. \tag{5}$$

$$C_{TrM} := \max\left\{C_{TrM}(\Omega_+), C_{TrM}(\Omega_-)\right\}. \tag{6}$$

$[p^* \cdot n]$ is the jump of the normal derivative p^* through M.

Theorem 2

For any $v \in \mathrm{K}$, any $\lambda \in \Lambda$, and any $q^* \in H(\Omega_\pm, \mathrm{div})$ distance to minimizer rated as

$$\|\nabla(v-u)\|_A \leq \mathfrak{M}(v, q^*, \lambda, \psi), \tag{7}$$

where

$$\begin{aligned}\mathfrak{M}(v, q^*, \lambda, \psi) := & \|A\nabla v - q^*\|_{A^{-1}} + \sqrt{2}\left(\int\limits_{\mathcal{M}} \lambda(v - \Psi) d\mu\right)^{1/2} \\ & + C_1\|f + \mathrm{div}\, q^*\|_{\Omega_+} + C_2\|f + \mathrm{div}\, q^*\|_{\Omega_-} \\ & + C_3\|\lambda - [q^* \cdot \mathbf{n}]\|_{\mathcal{M}},\end{aligned}$$

where C_1, C_2, C_3 are the same constants as in the Lemma. Majorant $\mathcal{M}$ is a non-negative functional, that vanishes if and only if $v = u$ and $q^* = \nabla u$ a.e. in Ω, and $\lambda := \left[\frac{\partial u}{\partial n}\right]$ a.e. in $\mathcal{M}$.

Scheme of proof

- For any $v \in \mathrm{K}$ direct calculations show that :

$$\|\nabla(u-v)\|_A^2 \leq J(v) - J(u),$$

- then we consider the perturbed functional

$$J_\lambda(v) := J(v) - \int\limits_{\mathcal{M}} \lambda\,(v - \Psi) d\mu.$$

$$v \in \varphi + H^1_0(\Omega) := \{w = \varphi + v: \ v \in H^1_0(\Omega)\},$$
$$\lambda \in \Lambda := \{\lambda \in L^2(\mathcal{M}): \ \lambda(x) \geq 0 \text{ a.e. in } \mathcal{M}\ \}.$$

- we can swap sup and inf

$$J(u) = \inf_{v \in \varphi + H^1_0(\Omega)} \sup_{\lambda \in \Lambda} J_\lambda(v) \ \geq \sup_{\lambda \in \Lambda} \inf_{v \in \varphi + H^1_0(\Omega)} J_\lambda(v) \ \geq J_\lambda(u_\lambda),$$

$\forall\, \lambda \in \Lambda$.

Here u is the exact solution of the problem (1) .

Scheme of proof

- now we define the functional dual to $J(\lambda)$

$$\begin{aligned}J^*_\lambda(y^*) = & \int\limits_{\Omega} (y^* \cdot \nabla\varphi - \frac{1}{2}A^{-1}y^* \cdot y^* - f\varphi) dx \\ & - \int\limits_{\mathcal{M}} \lambda\,(\varphi - \Psi)\, d\mu, \ \text{if } y^* \in Q^*_{\lambda,\mathcal{M}}\end{aligned}$$

$$Q^*_{\lambda,\mathcal{M}} := \left\{ y^* \in L^2(\Omega, \mathbb{R}^n): \int\limits_{\Omega} y^* \cdot \nabla w dx = \int\limits_{\Omega} fw dx + \int\limits_{\mathcal{M}} \lambda w d\mu \quad \forall\, w \in H^1_0(\Omega) \right\}.$$

Comment: If $y^* \notin Q^*_{\lambda,\mathcal{M}}$ then $J^*_\lambda(y^*) = -\infty$ and we have $\forall\, \lambda \in \Lambda$

$$J(u) \geq J_\lambda(u_\lambda) = J^*_\lambda(y^*_\lambda)$$

where y^*_λ is the exact solution of the problem, dual to (1).

$$\begin{aligned}J(v) - J(u) \leq J(v) - J^*_\lambda(y^*_\lambda) &= J(v) - \sup_{y^* \in Q^*_{\lambda,\mathcal{M}}} J^*_\lambda(y^*_\lambda) \\ &= J(v) + \inf_{y^* \in Q^*_{\lambda,\mathcal{M}}} (-J^*_\lambda(y^*_\lambda)) = \inf_{y^* \in Q^*_{\lambda,\mathcal{M}}} (J(v) - J^*_\lambda(y^*_\lambda)).\end{aligned}$$

Therefore,

$$J(v) - J(u) \leq J(v) - J^*_\lambda(y^*_\lambda) \ \ \forall\, v \in K, \ \lambda \in \Lambda, \ y^* \in Q^*_{\lambda,\mathcal{M}}.$$

Conclusions

- An estimate for the deviation of an arbitrary approximate solution (belonging to an admissible set) is obtained $y^* \in Q^*_{\lambda,\mathcal{M}}$ from the exact solution.
- An extension of the admissible set for the free parameter included in the deviation estimate was made.

$$H(\Omega_\pm, \mathrm{div}) := \left\{ q^* \in L^2(\Omega, R^n): \mathrm{div}\,(q^*|_{\Omega_\pm}) \in \ L^2(\Omega_\pm), \ [q^* \cdot n] \in L^2(M) \right\}.$$

- A modification of the deviation estimate has been established, which is more convenient for practical use.

References

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