Sectorial Tuples of Operators and Fractional Multi-Term Linear Equations

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«O.A. Ladyzhenskaya centennial conference on PDE's»

16 – 22.07.2022

Let $\mathcal{Z}$ be a Banach space, $\mathcal{L}(\mathcal{Z})$ denote the Banach space of linear bounded operators in $\mathcal{Z}$, and by $\mathcal{C}l(\mathcal{Z})$ the set of linear closed operators with dense domains in $\mathcal{Z}$ be denoted,
$\mathbb{R}_+ := \{a \in \mathbb{R} : a > 0\}$, $h : \mathbb{R}_+ \to \mathcal{Z}$.
Consider the linear homogeneous multi-term fractional equation

$$D_t^{\alpha} z(t) = \sum_{j=1}^{m-1} A_j D_t^{\alpha-m+j} z(t) + \sum_{l=1}^{n} B_l D_t^{\alpha_l} z(t) + \sum_{s=1}^{r} C_s J_t^{\beta_s} z(t) \qquad (1)$$

with initial conditions

$$D_t^{\alpha-m+k} z(0) = 0, \;\; k = m^*, m^*+1, \dots, m-1. \qquad (2)$$

We accept the following notation $g_\beta(t) := \Gamma(\beta)^{-1} t^{\beta-1}$ for $\delta > 0$, $t > 0$, $J_t^\beta h(t) := \int\limits_0^t g_\beta(t-s)h(s)ds$ for $t > t_0$. Let $m-1 < \alpha \le m \in \mathbb{N}$, $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $\beta_1 > \beta_2 > \dots > \beta_r \ge 0$, D_t^m is an ordinary derivative, witch order is J_t^0 is an identity operator, m^* is the defect of the Cauchy type problem for multi-term equation. The Riemann — Liouville derivative of function h defined as follows

$$D_t^\beta h(t) = D_t^m J_t^{m-\alpha} h(t).$$

The defect of the Cauchy type problem for multi-term equation

To solve Equation

$$D_t^{\alpha} z(t) = \sum_{j=1}^{m-1} A_j D_t^{\alpha-m+j} z(t) + \sum_{l=1}^{n} B_l D_t^{\alpha_l} z(t) + \sum_{s=1}^{r} C_s J_t^{\beta_s} z(t).$$

conditions are necessarily satisfied [1]

$$D_t^{\alpha-m+k} z(0) = 0,\ k = 0, 1, \dots, m^* - 1, \quad D_t^{\gamma} z(0) = 0,\ \gamma \in \Lambda,$$

where Λ is a set $\{\alpha_1 - 1, \alpha_1 - 2, \dots, \alpha_1 - m_1, \alpha_2 - 1, \alpha_2 - 2, \dots, \alpha_n - m_n\}$, from which repeated numbers are thrown out (they are present if there are numbers with the same fractional part among α_l, $l = 1, 2, \dots, n$).

Thus, for (1), it makes sense to consider only the incomplete Cauchy type problem

$$D_t^{\alpha-m+k} z(0) = z_k,\ k = m^*, m^* + 1, \dots, m - 1.$$

Denote $\underline{\alpha} := \max\{\alpha_l : l \in \{1, 2, \dots, n\},\ \alpha_l - m_l < \alpha - m\}$,
$\overline{\alpha} := \max\{\alpha_l : l \in \{1, 2, \dots, n\},\ \alpha_l - m_l > \alpha - m\}$, $\underline{m} = \lceil \underline{\alpha} \rceil$, $\overline{m} = \lceil \overline{\alpha} \rceil$, $m^* := \max\{\underline{m} - 1, \overline{m}\}$.

[1] Fedorov V. E., Turov M. M. The defect of a Cauchy type problem for linear equations with several Riemann—Liouville derivatives // Siberian Mathematical Journal, 2021, Vol. 62, No. 5, pp. 925–942.

Let $A_1, A_2, \dots, A_{m-1}, B_1, B_2, \dots, B_n, C_1, C_2, \dots, C_r$ be closed linear operators with domains $D_{A_1}, D_{A_2}, \dots, D_{A_{m-1}}, D_{B_1}, D_{B_2}, \dots, D_{B_n}, D_{C_1}, D_{C_2}, \dots, D_{C_r}$ respectively. Denote

$$D := \bigcap_{j=1}^{m-1} D_{A_j} \cap \bigcap_{l=1}^{n} D_{B_l} \cap \bigcap_{s=1}^{r} D_{C_s},$$

and

$$R_\lambda := \left(\lambda^\alpha I - \sum_{j=1}^{m-1} \lambda^{\alpha-m+j} A_j - \sum_{l=1}^{n} \lambda^{\alpha_l} B_l - \sum_{s=1}^{r} \lambda^{-\beta_s} C_s\right)^{-1} : \mathcal{Z} \to D.$$

We supply the set D with the norm:

$$\|\cdot\|_D = \|\cdot\|_{\mathcal{Z}} + \sum_{j=1}^{m-1} \|A_j \cdot\|_{\mathcal{Z}} + \sum_{l=1}^{n} \|B_l \cdot\|_{\mathcal{Z}} + \sum_{s=1}^{r} \|C_s \cdot\|_{\mathcal{Z}},$$

with respect to which D is a Banach space, since it is the intersection of the Banach spaces $D_{A_1}, D_{A_2}, \dots, D_{A_{m-1}}, D_{B_1}, D_{B_2}, \dots, D_{B_n}, D_{C_1}, D_{C_2}, \dots, D_{C_r}$ with the corresponding graph norms.

Definition of Class $\mathcal{A}^{n,r}_{\alpha}(\theta_0, a_0)$

Definition

A tuple of operators $(A_1, A_2, \ldots, A_{m-1}, B_1, B_2, \ldots, B_n, C_1, C_2, \ldots, C_r) \in (\mathcal{C}l(\mathcal{Z}))^{m+n+r-1}$ *belongs to the class* $\mathcal{A}^{n,r}_{\alpha}(\theta_0, a_0)$ *for some* $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, *if*

(i) D *is dense in* $\mathcal{Z}$;

(ii) *for all* $\lambda \in S_{\theta_0,a_0} := \{\mu \in \mathbb{C} : |\arg(\mu - a_0)| < \theta_0\}$, $p = 0, 1, \ldots, m-1$ *operators*

$$R_\lambda \cdot \left(I - \sum_{j=p+1}^{m-1} \lambda^{j-m} A_j \right) \in \mathcal{L}(\mathcal{Z})$$

exist;

(iii) *for any* $\theta \in (\pi/2, \theta_0)$, $a > a_0$, *there exists such* $K(\theta, a)$, *that for all* $\lambda \in S_{\theta,a}$, $p = 0, 1, \ldots, m-1$ *we have*

$$\left\| R_\lambda \cdot \left(I - \sum_{j=p+1}^{m-1} \lambda^{j-m} A_j \right) \right\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta, a)}{|\lambda - a||\lambda|^{\alpha-1}}.$$

Resolving Operators Families

Definition

*Let $p \in \{0,1,\dots,m-1\}$; a family of operators $\{S_p(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is called p-**resolving** for equation* (1), *if next conditions are satisfied:*
(i) $S_p(t)$ *is strongly continuous for $t > 0$;*
(ii) *for $t > 0$ $S_p(t)[D_{A_j}] \subset D_{A_j}$, $S_p(t)A_j x = A_j S_p(t)x$ for all $x \in D_{A_j}$, $j = 1,2,\dots,m-1$;*
$S_p(t)[D_{B_l}] \subset D_{B_l}$, $S_p(t)B_l x = B_l S_p(t)x$ for all $x \in D_{B_l}$; $S_p(t)[D_{C_s}] \subset D_{C_s}$, $S_p(t)C_s x = C_s S_p(t)x$ for all $x \in D_{C_s}$;
(iii) *for every $z_p \in D$ $S_p(t)z_p$ is a solution of problem* (1), (2) *with $z_l = 0$ for every $l \in \{0,1,\dots,m-1\} \setminus \{p\}$.*

Definition

A p-resolving family of operators for $p \in \{0,1,\dots,m-1\}$ is called ***analytic***, *if it has the analytic extension to a sector Σ_{ψ_0} at some $\psi_0 \in (0,\pi/2]$. An analytic p-resolving family of operators $\{S_p(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ has a type (ψ_0, a_0, β) at some $\psi_0 \in (0,\pi/2]$, $a_0 \in \mathbb{R}$, $\beta \in [0,1)$, if for all $\psi \in (0,\psi_0)$, $a > a_0$ there exists such $C(\psi,a)$, that for all $t \in \Sigma_\psi$ the inequality $\|S_p(t)\|_{\mathcal{L}(\mathcal{Z})} \le C(\psi,a)|t|^{-\beta} e^{a\operatorname{Re} t}$ is satisfied.*

Theorem (The main result)

Let $m-1<\alpha\leq m\in\mathbb{N}$, $A_j\in\mathcal{C}l(\mathcal{Z})$, $j=1,2,\dots,m-1$, $\alpha_1<\alpha_2<\dots<\alpha_n<\alpha$, $m_l-1<\alpha_l\leq m_l\in\mathbb{N}$, $\alpha_l-m_l\neq\alpha-m$, $B_l\in\mathcal{C}l(\mathcal{Z})$, $l=1,2,\dots,n$, $\beta_1>\beta_2>\dots>\beta_r\geq 0$, $C_s\in\mathcal{C}l(\mathcal{Z})$, $s=1,2,\dots,r$, D *is dense in* $\mathcal{Z}$. *Then there exist analytic p-resolving operators families for equation* (1) *of type* $(\theta_0,a_0,0)$, $p=m^*,m^*+1,\dots,m-1$ *(of type* $(\theta_0,a_0,m-\alpha)$ *for* $p=m^*=0$*), if and only if* $(A_1,A_2,\dots,A_{m-1},B_1,B_2,\dots,B_n,C_1,C_2,\dots,C_r)\in\mathcal{A}^{n,r}_{\alpha}(\theta_0,a_0)$.
If $z_p\in D$, $p=m^*,m^*+1,\dots,m-1$, *then there exists a unique solution to problem* (1), (2), *and it has the form*

$$z(t)=\sum_{p=m^*}^{m-1}Z_p(t)z_p,\ \text{where } Z_p(t)=\frac{1}{2\pi i}\int_{\Gamma}\lambda^{m-1-p}R_\lambda\left(I-\sum_{j=p+1}^{m-1}\lambda^{j-m}A_j\right)e^{\lambda t}d\lambda,$$

$\Gamma:=\Gamma^+\cup\Gamma^-\cup\Gamma^0$, $\Gamma^0:=\{\lambda\in\mathbb{C}:|\lambda-a|=r_0>0,\arg\lambda\in(-\theta,\theta)\}$, $\Gamma^\pm:=\{\lambda\in\mathbb{C}:\arg(\lambda-a)=\pm\theta,|\lambda-a|\in[r_0,\infty)\}$, $\theta\in(\pi/2,\theta_0)$, $a>a_0$, $r_0>0$. *Moreover, the solution can be extended on the sector* $\Sigma_{\theta_0-\pi/2}$ *analytically.*

Corollary (1)

Let $m-1<\alpha\leq m\in\mathbb{N}$, $A_j\in\mathcal{C}l(\mathcal{Z})$, $j=1,2,\ldots,m-1$, $\alpha_1<\alpha_2<\cdots<\alpha_n<\alpha$, $m_l-1<\alpha_l\leq m_l\in\mathbb{N}$, $\alpha_l-m_l\neq\alpha-m$, $B_l\in\mathcal{C}l(\mathcal{Z})$, $l=1,2,\ldots,n$, $\beta_1>\beta_2>\cdots>\beta_r\geq 0$, $C_s\in\mathcal{C}l(\mathcal{Z})$, $s=1,2,\ldots,r$, D *is dense in* $\mathcal{Z}$. *If there exist analytic* p*-resolving operators families for equation* (1) *of type* $(\theta_0,a_0,0)$, $p=m^*,m^*+1,\ldots,m-1$ *(of type* $(\theta_0,a_0,m-\alpha)$ *for* $p=m^*=0$*), then for every* $p\in\{m^*,m^*+1,\ldots,m-1\}$ *such family is unique and for any* $q\in\{p+1,p+1,\ldots,m-1\}$, $z_q\in D$

$$S_q(t)z_q=J_t^{q-p}S_p(t)z_q+\frac{1}{2\pi i}\int\limits_{\Gamma}R_\lambda\sum_{j=p+1}^{q}\lambda^{j-1-q}A_jz_qe^{\lambda t}d\lambda.$$

Corollary (2)

Let $m-1<\alpha\leq m\in\mathbb{N}$, $A_j\in\mathcal{L}(\mathcal{Z})$, $j=1,2,\ldots,m-1$, $\alpha_1<\alpha_2<\cdots<\alpha_n<\alpha$, $m_l-1<\alpha_l\leq m_l\in\mathbb{N}$, $\alpha_l-m_l\neq\alpha-m$, $B_l\in\mathcal{L}(\mathcal{Z})$, $l=1,2,\ldots,n$, $\beta_1>\beta_2>\cdots>\beta_r\geq 0$, $C_s\in\mathcal{L}(\mathcal{Z})$, $s=1,2,\ldots,r$. *Then* $(A_1,A_2,\ldots,A_{m-1},B_1,B_2,\ldots,B_n,C_1,C_2,\ldots,C_r)\in\mathcal{A}_\alpha^{n,r}(\theta_0,a_0)$.