On the constancy of the extremal function in the embedding theorem of fractional order

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Introduction

Let $n \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Assume that $s \in (0,1)$, $2_s^* := 2n/(n-2s)_+$ and

$$q \in \begin{cases} [1, 2_s^*] & \text{if } n \geq 2 \text{ or } n = 1 \text{ and } s < 1/2; \\ [1, \infty) & \text{if } n = 1 \text{ and } s = 1/2; \\ [1, \infty] & \text{if } n = 1 \text{ and } s > 1/2. \end{cases}$$

We consider the fractional Sobolev embedding $\mathcal{H}^s(\Omega) \hookrightarrow L_q(\Omega)$:

$$\inf_{u \in \mathcal{H}^s} \mathcal{I}_{s,q}^{\Omega}[u] := \inf_{u \in \mathcal{H}^s} \frac{\|u\|^2_{\mathcal{H}^s(\Omega)}}{\|u\|^2_{L_q(\Omega)}} \equiv \inf_{u \in \mathcal{H}^s} \frac{\langle (-\Delta)^s_{Sp} u, u \rangle + \|u\|^2_{L_2(\Omega)}}{\|u\|^2_{L_q(\Omega)}} > 0. \qquad (1)$$

The spectral Neumann fractional Laplacian

The spectral Neumann fractional Laplacian $(-\Delta)^s_{Sp}$ is the s-th power of conventional Neumann Laplacian in the sense of spectral theory. Its quadratic form in a bounded domain Ω is defined by

$$\langle (-\Delta)^s_{Sp} u, u \rangle := \sum_{j=1}^{\infty} \lambda_j^s \langle u, \phi_j \rangle^2, \qquad (2)$$

where λ_j are eigenvalues and ϕ_j are ortonormal eigenfunctions of the Neumann Laplacian in Ω (we denote $\lambda_0 = 0$ and $\phi_0 = C$).

Existence results

Existence of the extremal function in (1) is a simple fact for $q \in [1, 2_s^*)$ due to the fact that the embedding $\mathcal{H}^s(\Omega) \hookrightarrow L_q(\Omega)$ is compact here. The critical non-compact case $q = 2_s^*$ is much more complex. The following result was proved in [5] (for the local case $s = 1$ this result is well-known):

Theorem (N.Ustinov, 2020, [5]):

Let $n \geq 3$, let $\partial\Omega \in \mathcal{C}^2$ and let $2s > 1$. Then there exists a non-negative non-zero extremal function for the embedding theorem (1) with $q = 2_s^*$.

Problem statement

By the variational argument, for $q < \infty$, any extremal function in (1) is a ground state solution (up to a multiplicative constant) to the following problem:

$$(-\Delta)^s_{Sp} u + u = |u|^{q-2} u, \quad u \in \mathcal{H}^s(\Omega). \qquad (3)$$

Obviously, from (2) it follows that $(-\Delta)^s_{Sp} \mathbf{1} = 0$, and the Neumann boundary value problem (3) always has a trivial solution $u = \mathbf{1}$.

The question is: *is the extremal function in* (1) *constant or it gives the ground state solution for* (3) *that differs from the trivial one?*

Structure of the extremal functions

First, let us consider the case $q \in [1, 2]$. The Hölder inequality implies $\|u\|^2_{L_q(\Omega)} \leq \|u\|^2_{L_2(\Omega)} \cdot \operatorname{meas}(\Omega)^{2/q-1}$, therefore

$$\mathcal{I}_{s,q}^{\Omega}[u] = \inf_{u \in \mathcal{H}^s} \frac{\langle (-\Delta)^s_{Sp} u, u \rangle + \|u\|^2_{L_2(\Omega)}}{\|u\|^2_{L_q(\Omega)}} \geq \operatorname{meas}(\Omega)^{1-2/q} = \mathcal{I}_{s,q}^{\Omega}[\mathbf{1}],$$

and the constant function is the only minimizer of the functional $\mathcal{I}_{s,q}^{\Omega}[u]$ for any $s \in (0, 1]$.

Let us consider the more interesting case $q \in (2, 2_s^*]$. It turns out that the result here depends on the domain size. Accordingly, we use such reformulation: let $\Omega_\varepsilon := \{\varepsilon x \mid x \in \Omega\}$ and let $u_\varepsilon(y) := u(\varepsilon^{-1} y)$, then

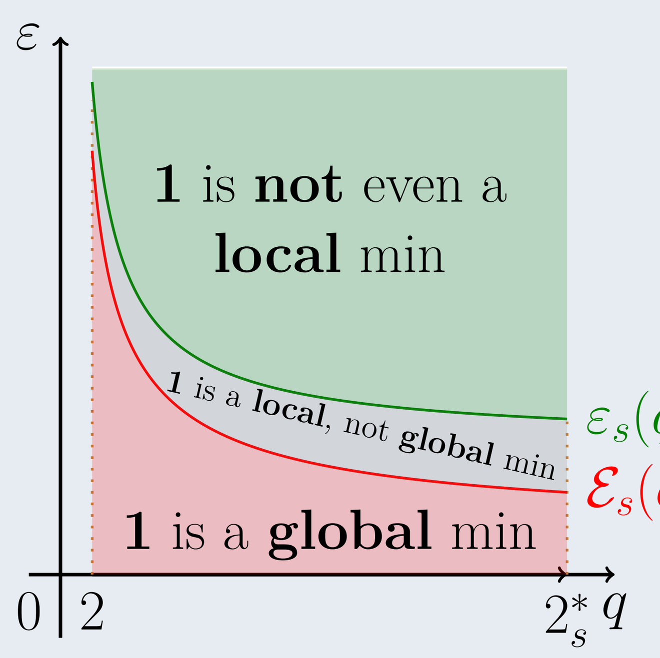
$$\mathcal{I}_{s,q}^{\varepsilon}[u] := \frac{\mathcal{I}_{s,q}^{\Omega_\varepsilon}[u_\varepsilon]}{\varepsilon^{n-2s-\frac{2n}{q}}} = \frac{\langle (-\Delta)^s_{Sp} u, u \rangle + \varepsilon^{2s} \|u\|^2_{L_2(\Omega)}}{\|u\|^2_{L_q(\Omega)}}.$$

So, the main question for the functional $\mathcal{I}_{s,q}^{\Omega}[u]$ in Ω_ε transforms into the similar question for the functional $\mathcal{I}_{s,q}^{\varepsilon}[u]$ in Ω in terms of ε.

Theorem (N.Ustinov, 2020, [6]):

Let $\operatorname{meas}(\Omega) = 1$ and let $q \in (2, 2_s^*]$. Then:

1. for $\varepsilon > \varepsilon_s(q) := (\lambda_1^s/(q-2))^{1/(2s)}$ the function $u = \mathbf{1}$ is not a ***local*** minimizer of the functional $\mathcal{I}_{s,q}^{\varepsilon}[u]$ (obviously, is not a ***global*** one);
2. for $\varepsilon < \varepsilon_s(q)$ the function $u = \mathbf{1}$ gives a ***local*** minimum for $\mathcal{I}_{s,q}^{\varepsilon}[u]$;
3. there exists an $\mathcal{E}_s(q) > 0$ such that, for all $\varepsilon \leq \mathcal{E}_s(q)$ the function $u = \mathbf{1}$ gives a ***global*** minimum for the functional $\mathcal{I}_{s,q}^{\varepsilon}[u]$, whereas for all $\varepsilon > \mathcal{E}_s(q)$ the function $u = \mathbf{1}$ is not a ***global*** minimizer for the functional $\mathcal{I}_{s,q}^{\varepsilon}[u]$. Moreover, the function $\mathcal{E}_s(q)$ is continuous and monotonically decreasing.



Whether $\mathcal{E}_s(q) = \varepsilon_s(q)$?

- Functions $\mathcal{E}_1(q)$ and $\varepsilon_1(q)$ were initially introduced in [3]; results on the constancy of the extremal function in the local case $s = 1$ are the same.
- For $n = 1$ one has $\mathcal{E}_1(q) = \varepsilon_1(q)$ (see [1, 2]); for $n \geq 2$ there exist both examples of domains with $\mathcal{E}_1(q) < \varepsilon_1(q)$ (see [3]) or with $\mathcal{E}_1(q) = \varepsilon_1(q)$ (see [4]).
- In [6] for arbitrary $s \in (0, 1)$ examples of domains with $\mathcal{E}_s(q) < \varepsilon_s(q)$ were provided for $n \geq 2$, in the analogous way to [3].

References

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