Cauchy-Dirichlet problem for quasilinear parabolic systems
with a nosmooth in time principal matrix

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(joint work with J. Stara, O. John)

ADVANCES IN NONLINEAR PDE
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We consider quasilinear system

$$u_t - div(\mathbf{a}(z,u)\nabla u) = g - div\, G, \;\; z = (x,t) \in Q = \Omega \times (-T, 0), \tag{1}$$

$$u|_{\partial_p Q} = u^0. \tag{2}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $T > 0$, and u: $Q \to \mathbb{R}^N$, $u = (u^1, \ldots, u^N)$, $N > 1$, $\nabla u = \{u^k_{x_\alpha}\}^{k \leq N}_{\alpha \leq n}$.

We assume that for almost all $z \in Q$ and all $u \in \mathbb{R}^N$

$$\|\mathbf{a}(z,u)\| \leq \mu; \;\; (\mathbf{a}(z,u)\xi, \xi) \geq \nu|\xi|^2, \;\; \forall \xi \in \mathbb{R}^{nN},$$

with some constants $0 < \nu < \mu$. We write $\mathbf{a} \in \{\nu, \mu\}$. We are interesting in **additional** conditions on the matrix $\mathbf{a}$ (for smooth enough g, G, u^0) to prove partial regularity of weak solutions $u \in V(Q) := W_2^{1,0}(Q) = L_2((-T,0); W_2^1(\Omega))$ of (1).

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parabolic case:$(\mathbf{n = 1})$-regularity; $(\mathbf{n} = 2)$ $-$? $(\mathbf{n} \geq 3)$ - counterexamples by O.John, J.Stara, J.Maly (1986), and O.John, J.Stara (1995): weak bounded solution in $Q_1 = B_1(0) \times (0,1)$ under condition $u|_{\partial_p Q} = \phi$, $\phi \in Lip$, but u is smooth on the set $\overline{Q_1} \setminus \{(0,t) : \, t \geq 1\}$.

Partial regularity results a) Local partial regularity for solutions $u \in V(Q)$ of system (1) was proved by M.Giaquinta, E.Giusti (1973)."If $\mathbf{a} \in \{\nu, \mu\}$, all elements of $\mathbf{a}$ are *uniformly continuous* functions (only continuous) then "*for any fixed* $\alpha \in (0,1)$ *there exist* θ, R_0 *such that the assumption*

$$\fint_{Q_R(z^0)} |u-(u)_{R,z^0}|^2\,dz < \theta, \quad \text{in } Q_R(z^0) \subset\subset Q, \quad R < R_0, \qquad (*)$$

supplies Hölder continuity of u *in some* $Q_r(z^0)$, $r < R$, *with* $\alpha \in (0,1)$," here $Q_r(z^0) = B_r(x^0) \times \Lambda_r(t^0)$, $\Lambda_r(t^0) = (t^0 - r^2, t^0 + r^2)$."
By Caccioppoli and Poincare inequalities, (*) is equivalent to the condition

$$\frac{1}{R^n}\int_{Q_R(z^0)} |\nabla u|^2\,dz < \theta_1, \quad R < R_0. \qquad (**)$$

We say $\mathbf{z^0 \in Reg(u)}$ if $\quad \liminf\limits_{\mathbf{R\to 0}} \mathbf{\frac{1}{R^n}\int_{Q_R(z^0)} |\nabla u|^2\,dz = 0}$.
We put $\Sigma = \{z \in Q : \liminf\limits_{r\to 0} r^{-n}\int_{Q_r} |\nabla u|^2\,dz > 0\}$, $\mathcal{H}_n(\Sigma;\delta) = 0$; $u \in C^\alpha(Q_0;\delta)$, $\forall \alpha \in (0,1)$, $Q_0 = Q \setminus \Sigma$. ($H_{n-2}(\Sigma_t) = 0$ for a.a. $t \in (0,T)$).
b) **boundary partial regul.:** Cauchy-Dir.problem -S.Campanato,1981; Cauchy-Neum. pr.- A.Arkhipova, 1992-1994.

The work I discuss today was inspired by results dedicated to L_p- theory for *scalar* nonlinear parabolic equations (in divergence and nondivergence form) and to *linear systems* with *nonsmooth in time* principal matrix. Works by N.V. Krylov (and with coauthors Dong H, Kim D., ...)

There are three approaches to study regularity of *quasilinear* systems:
1) indirect method (by contradiction)
2) direct method (to freeze coefficients + Gehring Lemma)
3) A-caloric approximation method
• We say that h is A-caloric function in $Q_R(z^0) = B_R(x^0) \times \Lambda_R(t^0)$ if it satisfies

$$h_t - A\,\nabla^2 h = 0, \;\; z \in Q_R(z^0).$$

The problem: to estimate $\|u - h\|^2_{2,Q_R(z^0)}, \qquad h|_{\partial_p Q} \neq u$, if the integral identities for u and h are similar in some sense.

The idea of the method belongs to E.De Giorgi (1961) who compared weak solutions of elliptic equations with harmonic functions. For different classes of elliptic systems, the method was successfully developed by F. Duzaar, J.F.Grotowski (2000) (A-harmonic method), and by F. Duzaar, G. Mingione (2005) for parabolic systems (A-caloric method).

We modified A-caloric method to A(t)-caloric method to prove partial regularity of $u \in V(Q)$ for systems (1) when no smoothness of the matrix $\mathbf{a}(x,t,u)$ in t is assumed.

Main assumptions

$[H1]$ The matrix $\mathbf{a}(z,u) \in \{\nu,\mu\}$ for almost all $z \in Q$ and all $u \in \mathbb{R}^N$.

$[H2]$ $\|\mathbf{a}(z,u) - \mathbf{a}(z,v)\| \le \omega(|u-v|^2)$, a.a. $z \in Q$, $\forall u,v \in \mathbb{R}^N$, where $\omega(s)$ is bounded convex function, $\omega(s) \to 0$, $s \to 0$.

$[H3]$ $\mathbf{a}(\cdot,t,u) \in VMO(\Omega)$ for a.a.$t \in (-T,0)$, $\forall u \in \mathbb{R}^N$, and

$$\sup_{z^0 \in Q, \eta \in \mathbb{R}^N} \sup_{\rho \le r} \fint_{\Lambda_\rho(t^0)} \fint_{B_\rho(x^0)\cap\Omega} |\mathbf{a}(y,t,\eta) - (\mathbf{a})_{\rho,x^0}(t,\eta)|^2 dy\, dt := q^2(r) \underset{r\to 0}{\to} 0,$$

$[H4]$

$$\underset{t\in(-T,0)}{ess\, sup} \|\mathbf{a}(x,t,u) - \mathbf{a}(y,t,v)\| \le L\,(|x-y|^\beta + |u-v|^\beta),\ \forall x,y \in \Omega,\ u,v \in \mathbb{R}^N.$$

If $g, G = 0$, local smoothness in Q: let $z^0 \in Reg(u)$ then

1) $[H1]-[H3] \Rightarrow u \in C^\alpha(Q_r(z^0))$, $\forall \alpha \in (0,1)$;

2) $[H1]$, $[H4] \Rightarrow \nabla u \in C^\beta(Q_r(z^0))$.

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Remark As a consequence, we arrive at the known result: $u \in C^{\alpha}(Q_0;\delta)$, $\forall \alpha \in (0,1)$, where open set $Q_0 = Q \setminus \Sigma$, Σ is the singular set of u, $\mathcal{H}_{n-\epsilon}(\Sigma;\delta) = 0$. Under the conditions of Theorem 2, we can additionally assert that $\nabla u \in C^{\beta}(Q_0;\delta)$.

Second step– **boundary regularity**.

Theorem *Let $[H1]-[H3]$ hold, $\partial\Omega \in C^1$, any $\alpha \in (0,1)$ be fixed. Let $g \in L^{2,n-2+2\alpha}(Q), G \in L^{2,n+2\alpha}(Q)$, $u^0 \in \mathcal{L}^{2,n+2+2\alpha}(Q)$, $\nabla u^0 \in L^{2,n+2\alpha}(Q)$, $u^0_t \in L^{2,n-2+2\alpha}(Q)$; $u \in V(Q)$ be a weak solution to problem (1),(2). If $z^0 = (x^0, t^0) \in Q \cup \partial_p Q$, $z^0 \in Reg(u)$, then there exist $r > 0$ such that $u \in C^{0,\alpha}(\overline{Q}_r(z^0))$, $\nabla u \in L^{2,n+2\alpha}(Q_r)$ and these norms are estimated by $\|u\|_{V(Q)}$ and known characteristics of data.*

- Under assumptions $[H1],[H4]$ for smooth enough g, G, u^o and $\partial\Omega$, we prove that additionally $\nabla u \in C^{\beta}(\overline{Q_r(z^0)})$, some $\beta \in (0,1)$.

The idea of the proof.

1)$v(z) = u(z) - u^0(z)$, $v|_{\partial_p Q} = 0$;

2) local coordinates $y = y(x)$, $Q_1^+ = B_1^+ \times (-1,0)$; $\Gamma_1 = \gamma_1 \times (-1,0)$, $\gamma_1 = B_1(0) \cap \{x_n = 0\}$.

We preserve all notation:

$$u_t - div(a^0(z,u)\nabla u) = f(z) - div\, F(z), \quad z \in Q_1^+, \tag{3}$$

$$u|_{\partial' Q_1^+(0)} = 0, \quad \partial' Q_1^+(0) = \Gamma_1 \cup (B_1^+ \times \{-1\}). \tag{4}$$

• *A function* $u \in V(Q_1^+)$ *is a weak solution to problem (3), (4), if* u *satisfies boundary condition (4) in the sense of traces on* Γ_1 *and the identity*

$$\int_{Q_1^+(0)} [-u\,\phi_t + (a^0(z,u)\nabla u, \nabla \phi)]\, dz = \int_{Q_1^+(0)} (f\,\phi + F, \nabla \phi))\, dz, \tag{5}$$

$\forall \phi \in W_2^1(Q_1^+(0)),\ \phi|_{\partial^* Q_1^+} = 0,\ \partial^* Q_1^+ = \partial\, Q_1^+ \setminus (B_1^+ \times \{-1\}).$

• "Function $h \in V(Q_R^+),\ h|_{\Gamma_R} = 0$, is A(t)-caloric in Q_R^+ if $A(t) \in \{\nu, \mu\}$ for a.a. $t \in \Lambda_R$ and it is a weak solution to the problem

$$h_t - A(t)\,\nabla^2 h = 0,\ \ z \in Q_R^+; \quad h|_{\Gamma_R} = 0."$$

$h \in W_2^{2,1}(Q_r^+),\ D^\alpha h \in C(\overline{Q_r^+}),\ (D^\alpha h)_t \in L^\infty(Q_r^+),\ r < R, \forall |\alpha|;$

$$\fint_{Q_\rho^+(\xi^0)} |h|^2\, dz \le c\left(\frac{\rho}{r}\right)^2 \fint_{Q_r^+(\xi^0)} |h|^2\, dz,\ \ \xi^0 \in \Gamma_1(0),\ \ \rho < r \le R;$$

$$\fint_{Q^+_\rho(\xi^0)} |h - (h_{x_n})_\rho\, x_n|^2\, dz \le c\left(\frac{\rho}{r}\right)^4 \fint_{Q^+_r(\xi^0)} |h - (h_{x_n})_r\, x_n|^2\, dz.$$

Lemma (!) *Let* $0 < \nu < \mu$ *and* $Q^+_R = Q^+_R(z^0)$, $z^0 \in \Gamma_1(0)$, *be fixed. For any* $\epsilon > 0$ *there exist* $C_\epsilon = C(\epsilon, \nu, \mu, n, N) > 0$ *such that the following holds: for any matrix* $A(t) \in \{\nu, \mu\}$ *for a.a.* $t \in \Lambda_R$, *and any function* $u \in V(Q^+_R)$, $u|_{\Gamma_R} = 0$, *there exist an* $A(t)$*-caloric function* $h \in V(Q^+_{R/2})$, $h|_{\Gamma_{R/2}} = 0$, *and* $\phi \in C^1_0(Q^+_R)$, $\sup_{Q^+_R} |\nabla\phi| \le 1$, *such that*

$$\fint_{Q^+_{R/2}} (|h|^2 + R^2|\nabla h|^2)\, dz \le 2^{n+3} \fint_{Q^+_R} (|u|^2 + R^2|\nabla u|^2)\, dz, \tag{6}$$

$$\fint_{Q^+_{R/2}} |u - h|^2\, dz \le \epsilon \fint_{Q^+_R} (|u|^2 + R^2|\nabla u|^2)\, dz + C_\epsilon\, R^2\, \mathcal{K}_R, \tag{7}$$

$$\mathcal{K}_R := \left| \fint_{Q^+_R} (u\,\phi_t - (A(t)\nabla u, \nabla\phi))\, dz \right|^2 .$$

We fix $z^0 \in \Gamma_1$, $z^0 \in Reg(u)$, and $Q_R^+(z^0)$, $Q_R(z^0) \cap \{t = -1\} = \emptyset$, and put $A(t) = a_{R,x^0}(t) = \fint_{B_R^+(x^0)} \mathbf{a}(x,t,0)\,dx$,

$$\Phi(\rho, z^0) := \frac{1}{\rho^n} \int_{Q\rho(z^0)} |\nabla u|^2\,dz, \ \ \rho \leq R.$$

Using assumptions $[H1] - [H3]$, we estimate $\mathcal{K}_R$ in the way:

$$\mathcal{K}_R \leq M(\theta, R)\,R^{-2}\,\Phi(R, z^0) + c\,R^{-2} B_f, \ B_f = \|f\|^2 + \|F\|^2.$$

where some function $M(\theta, R) \to 0$, θ, $R \to 0$.
By relations (6), (7), Caccioppoli and Poincare inequalities for u and h, and the assumptions on f, F, we derive the inequality

$$\Phi(\rho) \leq c_0 \left[\left(\frac{\rho}{R}\right)^2 + (\epsilon + C_\epsilon\,M(\theta, R)) \left(\frac{R}{\rho}\right)^{n+2} \right] \Phi(R) + C\epsilon \left(\frac{R}{\rho}\right)^{n+2} R^{2\alpha}\,B_f.$$

Then we put in the last inequality $\rho = \tau R$ with $\tau \leq 1/4$ to define below:

$$\Phi(\tau R) \leq c_0[\tau^2 + \tau^{-(n+2)}(\epsilon + C_\epsilon\,M(\theta, R))]\Phi(R) + C_\epsilon \tau^{-(n+2)} R^{2\alpha} B_f.$$

For a fixed exponent $\alpha \in (0, 1)$ we put $\beta = (1+\alpha)/2$ and choose τ, ϵ, θ, R_0 to obtain

$$\Phi(\tau R, z^0) \leq \tau^{2\beta} \Phi(R, z^0) + K\,B_f\,R^{2\alpha}.$$

Using iteration process with $R_j = \tau^j R$, $j \in \mathbb{N}$, we arrive at the inequality

$$\Phi(R_j, z^0) \leq \tau^{2\beta\,j} \Phi(R, z^0) + K\,B_f \tau^{2\alpha\,j} R^{2\alpha} \Sigma_{s=0}^{j} \tau^{2s(\beta - \alpha)}.$$

It follows that

$$\Phi(\rho, z^0) \le c\left[\left(\frac{\rho}{R}\right)^{2\alpha}\Phi(R, z^0) + \rho^{2\alpha}B_f\right],\ z^0 \in \Gamma_1. \qquad (8)$$

Note that $\Phi(R, z)$ is a continuous function in z for the fixed R. It means that there exists a cylinder $Q_r^+(z^0)$ such that condition (*) holds for $\Phi(R, \xi^0)$ with any $\xi^0 \in \overline{Q_r^+(z^0)}$. It allows us to "sew" boundary estimate (8) and corresponding estimate for inner cylinders and to assert that estimate (8) is valid for all $\xi^0 \in \overline{Q_r^+(z^0)}$. It supplies us the estimate of ∇u in $L^{2,n+2\alpha}(Q_r^+(z^0);\delta)$, and $u \in \mathcal{L}^{2,n+2+2\alpha}(\overline{Q_r^+(z^0)};\delta)$. Due to isomorphism of this space to $C^\alpha(\overline{Q_r^+(z^0)};\delta)$, we obtain the estimate of the $H\ddot{o}lder$ norm of u.

• If $Q_R(z^0) \cap \{t = -1\} \ne \emptyset$, we put $\mathbf{a}^0(x, t, u) = \mathbf{a}^0(x, -1, 0)$, and f, $F = 0$ for $t < -1$.

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To estimate $\nabla u \in \mathcal{L}^{2,n+2+2\beta}$ we assume $[H1]$ and $[H4]$ and estimate the function

$$\Psi(\rho, z^0) := \fint_{Q_\rho(z^0)} (|\nabla' u|^2 + |u_{x_n} - (u_{x_n})_{\rho,z^0}|^2)\, dz,\ z^0 \in \Gamma_1.$$

In this case A(t)-caloric lemma is applied to $v(z) = u(z) - (u_{x_n})_{R,z^0}(x_n)$.