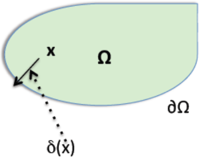
Quasilinear elliptic problems with Hardy potentials

joint work with
V. Moroz, W. Reichel & M.A. Pozio

Catherine Bandle

University of Basel, Switzerland

Introduction



$\Omega \subset \mathbb{R}^N$ bounded smooth domain, $x \in \Omega$ generic point,
$\delta(x) := \text{distance}\{x, \partial\Omega\}$, $\mu \in \mathbb{R}$, $p > 0, p \neq 1$.

Consider

$$(P) \qquad \boxed{\Delta u + \frac{\mu}{\delta^2} u = u^p \text{ in } \Omega,\ u > 0.}$$

$\frac{\mu}{\delta^2(x)}$ is called *Hardy potential*.

Driving mechanisms

nonlinear regime

$$(N) \quad \Delta U = U^p \text{ in } \Omega,$$

linear regime

$$(L) \quad \Delta h + \frac{\mu}{\delta^2} h \geq 0 \text{ in } \Omega.$$

Known results for the nonlinear problem

$\Delta u = u^p$ in Ω

- The boundary value problem with $u = \phi$ on $\partial\Omega$ has a unique solution for every positive continuous ϕ.
- Let $p > 1$.
  Then there exists a unique solution U such that $U(x) \to \infty$ as $x \to \partial\Omega$.
  1. $U(x)\delta^{\frac{2}{p-1}}(x) \to c_p$ as $x \to \partial\Omega$.
  2. $U(x) \geq u(x)$ for any other solution.
     It is therefore called the *large solution*.
- Let $p < 1$.
  1. There exist solutions which vanish identically on a sub domain $\omega \subset \Omega$ (=*dead core*) and are positive elsewhere.

Known results for the nonlinear problem

$\Delta u = u^p$ in Ω

- The boundary value problem with $u = \phi$ on $\partial\Omega$ has a unique solution for every positive continuous ϕ.
- Let $p > 1$.
  Then there exists a unique solution U such that $U(x) \to \infty$ as $x \to \partial\Omega$.
  1. $U(x)\delta^{\frac{2}{p-1}}(x) \to c_p$ as $x \to \partial\Omega$.
  2. $U(x) \geq u(x)$ for any other solution.
     It is therefore called the *large solution*.
- Let $p < 1$.
  1. There exist solutions which vanish identically on a sub domain $\omega \subset \Omega$ (=*dead core*) and are positive elsewhere.

Known results for the nonlinear problem

$\Delta u = u^p$ in Ω

- The boundary value problem with $u = \phi$ on $\partial\Omega$ has a unique solution for every positive continuous ϕ.
- Let $p > 1$.
  Then there exists a unique solution U such that $U(x) \to \infty$ as $x \to \partial\Omega$.
  1. $U(x)\delta^{\frac{2}{p-1}}(x) \to c_p$ as $x \to \partial\Omega$.
  2. $U(x) \geq u(x)$ for any other solution.
     It is therefore called the *large solution*.
- Let $p < 1$.
  1. There exist solutions which vanish identically on a sub domain $\omega \subset \Omega$ (=*dead core*) and are positive elsewhere.

Phragmen-Lindelöf alternative

$\Delta h + \frac{\mu}{\delta^2} h \geq 0,\ \mu \leq 1/4$ in a small strip near the boundary.

Theorem

(i) $\mu < 1/4$.

Let $\beta = \beta_{\pm}$ *where* $\beta(\beta - 1) + \mu = 0$

$\beta_- := 1/2 - \sqrt{1/4 - \mu} < \beta_+ := 1/2 + \sqrt{1/4 - \mu}$.

Then either

- $\limsup_{x \to \partial\Omega} h(x)/\delta^{\beta_-}(x) > 0$

or

- $h \leq c\delta^{\beta_+}$ *in* $\{\delta(x) \leq \rho\}$ *for some positive c (small subharmonics)*

(ii) $\mu = 1/4$.

The same statement holds with δ^{β_-} *replaced by* $\delta^{1/2}(x)\log(1/\delta(x))$ *and* β_+ *by* $1/2$.

Phragmen-Lindelöf alternative

$\Delta h + \frac{\mu}{\delta^2} h \geq 0$, $\mu \leq 1/4$ in a small strip near the boundary.

Theorem

(i) $\mu < 1/4$.

Let $\beta = \beta_\pm$ *where* $\beta(\beta - 1) + \mu = 0$

$\beta_- := 1/2 - \sqrt{1/4 - \mu} < \beta_+ := 1/2 + \sqrt{1/4 - \mu}$.

Then either

- $\limsup_{x \to \partial\Omega} h(x)/\delta^{\beta_-}(x) > 0$

or

- $h \leq c\delta^{\beta_+}$ *in* $\{\delta(x) \leq \rho\}$ *for some positive c (small subharmonics)*

(ii) $\mu = 1/4$.

The same statement holds with δ^{β_-} *replaced by* $\delta^{1/2}(x)\log(1/\delta(x))$ *and* β_+ *by* $1/2$.

Hardy constant

$$\boxed{C_H(\Omega) = \inf_{\mathcal{K}} \int_\Omega |\nabla\phi|^2\,dx, \text{ where } \mathcal{K} = \left\{\phi \in C_0^\infty(\Omega), \int_\Omega \frac{\phi^2}{\delta^2}\,dx = 1\right\}}.$$

Properties:

$$0 < C_H(\Omega) \le 1/4.$$

$$C_H(\Omega) = 1/4 \text{ if } \Omega \text{ is convex or for annuli in } \mathbb{R}^N,\ N > 2\ .$$

$C_H(\Omega)$ is attained if and only if $C_H(\Omega) < 1/4$.

Marcus, Mizel, Pinchover, Shafrir and Brezis 1997, 1998

1-dimensional problems

$$\boxed{u''(x) + \frac{\mu}{x^2} u(x) = u^p(x)}$$

Elementary solution: $u(x) = \left(\frac{2(p+1)}{(p-1)^2} + \mu\right)^{\frac{1}{1-p}} x^{-\frac{2}{p-1}}$

$$\boxed{h''(x) + \frac{\mu}{x^2} h = 0}$$

Solutions: $h(x) = c_1 x^{\beta_+} + c_2 x^{\beta_-}$ where $\beta_\pm$ are the roots of $\beta(\beta - 1) + \mu = 0$

$$\beta_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}.$$

$h \sim x^{\beta_+}$ small harmonic
$h \sim x^{\beta_-}$ large harmonic.

Radial solutions

$$\Omega = B_R \text{ or } \Omega = A(r_0, R) := \{r_0 < |x| < R\}$$

The solutions depending only on $r = |x|$ satisfy the ODE

$$\boxed{u''(r) + \frac{N-1}{r}u'(r) + \frac{\mu}{\delta^2(r)}u(r) = u^p(r)},$$

$$\delta = \begin{cases} R - r & \text{if } r > \frac{R+r_0}{2} \\ r - r_0 & \text{otherwise .} \end{cases}$$

Asymptotic boundary behavior

$0 < p$

Let u be a positive local solution near the boundary. Then the only possible behaviors at the boundary are

$$\lim_{\delta\to 0} \frac{u(\delta)}{\delta^{\beta_-}} = c, \text{ large harmonics },$$

$$\lim_{\delta\to 0} \frac{u(\delta)}{\delta^{\beta_+}} = c, \text{ small harmonics },$$

In addition if $\mu > -\frac{2(p+1)}{(1-p)^2}$,
($\Leftrightarrow \beta_+ < \frac{2}{1-p}$ if $p < 1$ and $\beta_- > -\frac{2}{p-1}$ if $p > 1$)

$$\lim_{\delta\to 0} \frac{u(\delta)}{\delta^{2/(1-p)}} = c_{p,\mu} \begin{cases} 0 < p < 1 \text{ dead core solution} \\ p > 1 \text{ blowup solution} \end{cases}$$

Existence of local solutions

- If $\mu > -\frac{2(p+1)}{(1-p)^2}$ there exist local solutions of all types.
- If $\mu < -\frac{2(p+1)}{(1-p)^2}$ there exist only local

small harmonics if $p > 1$,
large harmonics if $p < 1$.

Global solutions of $\Delta u + \frac{\mu}{\delta^2} u = u^p$ in arbitrary domains

Definition

A solution is called *small solution* if it belongs to $W_0^{1,2}(\Omega)$.

Theorem

• *If* $\mu < C_H(\Omega)$ *then there are no small, non trivial solutions* $u \in W_0^{1,2}(\Omega)$*.*

• *If* $p > 1$, $C_H(\Omega) < 1/4$ *and* $\mu \in (C_H(\Omega), 1/4)$ *then there exists a small solution such that* $u(\delta) \leq c\delta^{\beta_+}$.

Proof -The existence of a positive solution is incompatible with the Hardy constant.

- Construction of an upper and lower solution, Keller-Osserman type a priori bound.

Global solutions of $\Delta u + \frac{\mu}{\delta^2} u = u^p$ in arbitrary domains

Definition

A solution is called *small solution* if it belongs to $W_0^{1,2}(\Omega)$.

Theorem

• *If* $\mu < C_H(\Omega)$ *then there are no small, non trivial solutions* $u \in W_0^{1,2}(\Omega)$.

• *If* $p > 1$, $C_H(\Omega) < 1/4$ *and* $\mu \in (C_H(\Omega), 1/4)$ *then there exists a small solution such that* $u(\delta) \le c\delta^{\beta_+}$.

Proof -The existence of a positive solution is incompatible with the Hardy constant.

- Construction of an upper and lower solution, Keller-Osserman type a priori bound.

Radial solutions if $0 < p < 1$ in the ball B_R

Theorem

(i) There exist infinitely many strictly positive radial solutions.

(ii) For every $r^* < R$ there exists a solution such that $u = 0$ in $(0, r^*)$ and $u > 0$ in (r^*, R).

(iii) All solutions satisfy at the boundary

$$\lim_{r \to R} u(r)(R - r)^{-\beta_-} = v(0) > 0\,.$$

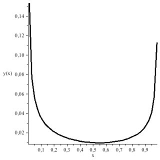
Annuli

$0 < p < 1$

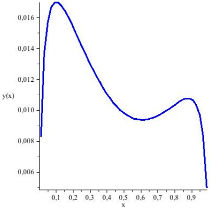
Theorem

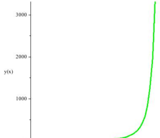
There exist local solutions of the type described above.

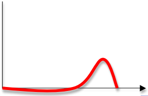
A: $\mu < 0$



B: $0 < \mu < 1/4$







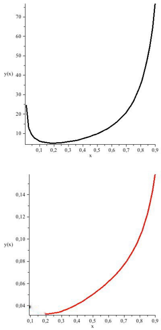
$p > 1$

The situation is very similar except that instead of vanishing with the dead core rate one has blow up

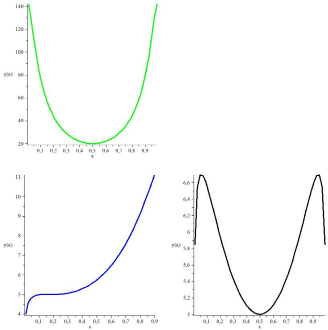
$$u(r) \sim (r - R_0)^{-\frac{2}{p-1}}.$$

$p > 1$

A: $\mu < 0$



B: $0 < \mu < 1/4$



General domains

Theorem

1. For $p > 0, \neq 1$ there exist solutions such that

$$0 < \gamma < \liminf_{\delta \to 0} \frac{u(\delta)}{\delta^{\beta_-}} \leq \limsup_{\delta \to 0} \frac{u(\delta)}{\delta^{\beta_-}} \leq \gamma^{-1}.$$

(harmonic growth at the boundary)

2. If $p > 1$ and $\mu > -\frac{2(p+1)}{(p-1)^2}$ there exist solutions such that

$$0 < \gamma < \liminf_{\delta \to 0} \frac{u(\delta)}{\delta^{-\frac{2}{p-1}}} \leq \limsup_{\delta \to 0} \frac{u(\delta)}{\delta^{-\frac{2}{p-1}}} \leq \gamma^{-1}.$$

Happy birthday,

dear Nina