Stochastic counterparts of the Cauchy problem for quasilinear systems of parabolic equations

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OUR GOAL is to apply probabilistic approach to construct a **(classical, generalized or viscosity)** solutions of the Cauchy problem for systems of nonlinear parabolic equations.

To this end we need 3 steps:

1. To reduce the Cauchy problem for a system of PDEs to a stochastic problem;
2. To solve the corresponding stochastic problem;
3. To prove that the solution of the stochastic problem gives rise to a certain (**classical, weak, viscosity**) solution of the original Cauchy problem.

Connections between linear PDEs and SDEs.

Let $s \in [0, T], x \in R^d,\ f(s,x) \in R^1$

$$f_s + \mathcal{L}(x)f = 0, f(T,x) = f_0(x) \qquad (1)$$

$$\mathcal{L}(x)f(x) = \frac{1}{2}TrA^*(x)\nabla^2 f(x)A(x) + \langle a(x), \nabla f(x)\rangle$$

Let us call a probabilistic counterpart of (1) an SDE

$$d\xi(\theta) = a(\xi(\theta))d\theta + A(\xi(\theta))dw(\theta),\ \xi(s) = x, \quad (2)$$

$w(t) \in R^d$ – Wiener process defined on $(\Omega, \mathcal{F}, P)$, $a(s,x) \in R^d, A(s,x) \in R^d \otimes R^d$ and

$$f(s,x) = E[f_0(\xi_{s,x}(T))] \qquad (3)$$

satisfies (1) in a certain sense.

At this level the above goals are achieved as follows:
1. If $f(s,x)$ is a classical solution to (1) then

$$E[f(T,\xi_{s,x}(T))] - f(s,x) = \int_s^T E[[f_\theta + \mathcal{L}f](\theta, \xi_{s,x}(\theta))]d\theta.$$

2. If we prove that the solution $\xi_{s,x}(\theta)$ of SDE (2) is smooth w.r.t. x, then we can prove that $f(s,x) = E[f_0(\xi_{s,x}(T))]$ belongs to C^2 when $f_0 \in C^2$ and defines a unique classical solution to (1).
Similar results for generalized and viscosity solutions hold but with **different diffusion processes**.

In this talk we discuss results concerning probabilistic representations of classical and generalized solutions of the Cauchy problem for systems of nonlinear parabolic equations.
In general systems of PDEs considered below have the form

$$u^m_s + F^m(x, u, \nabla u, \nabla^2 u^m) = 0, \quad u^m(T, x) = u^m_0(x), \qquad (*)$$

where $x \in R^d, u \in R^{d_1}, 0 \le s \le T,\ m = 1, \dots, d_1$
and we look for a (**classical or distributional**) solution of (*).

We specify the systems to be considered as follows:
1) **Diagonal principal part** and non diagonal lower order terms, (e.g. conservation and balance laws):
In a **linear case** it has a structure

$$F^m(x,u,\nabla u,\nabla^2 u) = \mathcal{L}(\mathbf{x})\mathbf{u}^{\mathbf{m}}$$

$$+\sum_{j=1}^{d_1}\sum_{i=1}^{d}\mathbf{B}_i^{ml}(\mathbf{x})\nabla_i\mathbf{u}^l+\sum_{l=1}^{d_1}\mathbf{c}^{ml}(\mathbf{x})\mathbf{u}^l,\quad m=1,\ldots,d_1.$$

In a **nonlinear case** all coefficients may depend on $u,\nabla u$ as well and have the form $a(x,u,\nabla u)$.

2) Systems with **non-diagonal principal part** arising in cross-diffusion problems, population dynamics, chemotaxis and some other fields, for example in financial mathematics. These are systems of the form

$$u_t = \mathrm{div}(a(u,v)\nabla u + b(u,v)\nabla v) + g(u,v),$$

$$v_t = \mathrm{div}(\alpha(u,v)\nabla v + \beta(u,v)\nabla u) + \gamma(u,v),$$

$$u(0,x) = u_0(x).$$

PDE- theory - **Ladyzenskaya, Solonnikov, Uraltzeva (1967)**, SDE -theory – (linear case D.Stroock, Yu.Dalecky 1968) nonlinear case (Yu. Dalecky , Bel. 1978,1980) – classical solutions.

i) **Semilinear case** **(summation rule)**

$$u^m_s + \mathcal{L}(x,u)u^m + B^{ml}_i(x,u)\nabla_i u^l + c^{ml}(x,u)u^l = 0, \quad (4)$$

$u(T,x) = u_0(x)$, where $B^{ml}_i = C^{ml}_k A^k_i$,

$m, l = 1, \dots, d_1, \quad i = 1, \dots, d,$

$$\mathcal{L}(\mathbf{x},\mathbf{u})\mathbf{u}^{\mathbf{m}} = \langle \mathbf{a}(\mathbf{x},\mathbf{u}), \nabla \mathbf{u}^{\mathbf{m}} \rangle + \frac{1}{2}\mathbf{Tr}\mathbf{A}^*(\mathbf{x},\mathbf{u})\nabla^2\mathbf{u}^{\mathbf{m}}\mathbf{A}(\mathbf{x},\mathbf{u})$$

Given a classical solution $u(s,x)$ to (4) one can check that its stochastic counterpart is presented as follows

$$d\xi(\theta) = a(\xi(\theta), u(\theta,\xi(\theta)))d\theta + A(\xi(\theta), u(\theta,\xi(\theta)))dw(\theta), \tag{5}$$

$$d\eta(\theta) = c(\xi(\theta), u(\theta,\xi(\theta)))\eta(\theta)d\theta$$
$$+C(\xi(\theta), u(\theta,\xi(\theta)))(\eta(\theta), dw(\theta)), \quad \eta(s) = h, \tag{6}$$

$$\langle h, u(s,x)\rangle = E[\langle \eta(T), u_0(\xi_{s,x}(T))\rangle]. \tag{7}$$

Here $B_i^{lm} = C_j^{lm} A_{ji}$ and $\langle h, u\rangle = \sum_{m=1}^{d_1} h_m u_m$.

ii) Quasilinear case

$$u^m_s + \mathcal{L}(x,u,\nabla u)u^m + B^{ml}_k(x,u,\nabla u)\nabla u^l$$
$$+c^{ml}(x,u,\nabla u)u^l = 0, \quad u(T,x) = u_0(x), \qquad (8)$$

$$\mathcal{L}(x,u,\nabla u)u = \langle a(x,u,\nabla u)\nabla u$$
$$+\frac{1}{2}TrA(x,u,\nabla u)\nabla^2 uA(x,u,\nabla u)$$

Introducing $v(s,x) = \nabla u(s,x)$ and deriving by formal differentiation an equation for $v(s,x)$, we deduce that the resulting system for $V = (u,\nabla u)$ called the 1-sf differential prolongation of (4) has the same structure as (4) itself.

This allows to include (8) into a semilinear system of PDEs. To this end we have to consider the 2-nd differential prolongation of (8) as well to get a semilinear system w.r.t. $\tilde{V} = (u, \nabla u, \nabla^2 u)$. Note that in a similar way one can reduce a fully nonlinear PDE (*) to a semilinear system w.r.t $G(v_1, v_2, v_3, v_4)$ where $v_1 = u, v_2 = \nabla u, v_3 = \nabla^2 u, v_4 = \nabla^2 u$.

Theorem

Let $u(s,x)$ be a classical solution to the system (4). Then there exists a probabilistic representation of the function $u(s,x)$ in the form (7), where processes $\xi_{s,x}(t), \eta_{s,h}(t)$ solve SDE (5), (6).

Theorem

Let there exists a solution to the stochastic system (5)–(7) and $u(s) \in C^2$. Then the function $u(s,x)$ is the unique classical solution to the Cauchy problem (4).

Theorem

Under suitable assumptions there exists an interval $[\tau, T]$ *(depending on the problem data) such that for* $s \in [\tau, T]$ *there exists a unique solution to the stochastic system (5)–(7).*

Theorem

Under suitable assumptions there exists an interval $[\tau_1, T] \subset [\tau, T]$ *such that for* $s \in [\tau_1, T]$ *the solution* $u(s, x)$ *determined by the system (5)–(7) stands for a unique classical solution of the Cauchy problem (4).*

A specific feature of (4) is the following crucial observation which can be easily revealed if one analyzes the probabilistic representation (7) of a smooth solution to (4). Namely, consider a function $\Phi(s,x,h) = \langle h, u(s,x)\rangle$ satisfying

$$\frac{\partial \Phi}{\partial s} + \frac{1}{2} TrQ^*(x,h)\nabla^2\Phi Q(x,h) + \langle q(x,h), \nabla\Phi\rangle = 0. \tag{8}$$

The above property of systems of the form (4) allows to construct stochastic counterparts for generalized and viscosity solutions of the Cauchy problem (4). The corresponding results were obtained by **Woyczynski and Bel (2007, 2012)** and **Bel (2011, 2014)**.

Say a few words about an alternative approach. Let

$$u_s + \mathcal{L}(x)u + g(u, \nabla u) = 0, u(T, x) = u_0(x) \qquad (**)$$

has a classical solution. Then by Ito's formula for $y(t) = u(t, \xi(t))$ we have

$$dy(t) = [u_t + \mathcal{L}(x)u](t, \xi_{s,x}(t))dt + [A^* \nabla u(t)](\xi_{s,x}(t))dw.$$

Set in addition $z(t) = [A^* \nabla u](t, \xi_{s,x}(t))$. Then we have $\mathbf{dy(t)} = [\mathbf{u_t} + \mathcal{L}(\mathbf{x})\mathbf{u}] + \mathbf{g} - \mathbf{g}]\mathbf{dt} + \mathbf{z(t)dw(t)}$ and $y(T) = u_0(\xi_{s,x}(T))$.

As a result we have a – **BSDE**

$$dy(t) = -g(y(t), z(t))dt + z(t)dw(t), \; y(T) = u_0(\xi(T)).$$

At the first glance we have an SDE with two unknown processes $y(t)$ and $z(t)$. Bu we know from Ito's theorem about a martingale representation that a martingale $y(t) = E[u_0(\xi(T)) + \int_0^T g(y(\theta), z(\theta))d\theta|\mathcal{F}_t]$ and $\chi = u_0(\xi(T)) + \int_0^T g(y(\theta), z(\theta))d\theta$ admits a representation $\chi = E[\chi] + \int_0^T z(\theta)dw(\theta)$ and the process $z(t)$ is defined in a unique way.

Finally, since $y(t) = u(t, \xi_{s,x}(t))$, we have that $y(s) = u(s, x)$ is nonrandom and being smooth stands for a classical solution to (**). If it is not the case then under some assumptions we can prove that $y(s) = u(s, x)$ gives a viscosity solution to (**). Note that the above possibility to consider a scalar equation for $\Phi(s, x, h) = \langle h, u(s, x)\rangle$ gives the way to obtain the required comparison results for solutions of the Cauchy problem for systems of parabolic squations from this class.

PDE - **H. Amann, Dynamic theory of quasilinear parabolic systems (1989)** Coming back to fully non-diagonal systems we consider a simple case of one dimensional cross-diffusion system

$$\partial_t u_1 = \partial_x^2[u_1(a_{11}u_1 + a_{12}u_2)] + (1 - u_1 - u_2)u_1, \quad (10)$$

$$\partial_t u_2 = \partial_x^2[u_2(a_{21}u_1 + a_{22}u_2)] + \gamma(1 - u_1 - \kappa u_2)u_2, \quad (11)$$

$u^1(0,x) = u_0^1(x), u^2(0,x) = u_0^2(x)$ and construct a probabilistic representation of a weak solution of this Cauchy problem. $W^k(R)$– Sobolev space and

$$\mathcal{W}_T^k = C([0,T]; W^k(R)) \cap L^\infty([0,T], L^\infty(R)),$$

Set $\langle\langle u,h\rangle\rangle = \int_R u(x)h(x)dx.$ A couple of (u_1,u_2) is said to be a weak solution of (10), (11) provided $u_1\ge 0$, $u_2\ge 0$, and $i=1,2$

$$\langle\langle h_i(t),u_i(t)\rangle\rangle - \langle\langle h_i(0),u_i(0)\rangle\rangle \qquad (12)$$

$$= \int_0^t \langle\langle [\partial_s h_i(s) + \frac{1}{2}\beta_i^2(u)\partial_x^2 h_i(s)], u_i(s)\rangle\rangle ds$$

$$+ \int_0^t \langle\langle (a_i - \alpha_i u_1(s) - \kappa_i u_2(s))h_i(s), u_i(s)\rangle\rangle ds.$$

$$\beta_1^2(u) = 2[a_{11}u_1 + a_{12}u_2],\ \beta_2^2(u) = 2[a_{21}u_1 + a_{22}u_2].$$

Based on this definition we can associate with (10),(11) a couple of stochastic processes satisfying to SDEs

$$\begin{cases} d\xi^1(\theta) = \beta_1(u(\theta, \xi^1(\theta)))dw_1(\theta) \\ d\xi^2(\theta) = \beta_2(u(\theta, \xi^2(\theta)))dw_2(\theta) \end{cases} \tag{13}$$

$\xi^1(0) = \xi_2(0) = y$. But these are still not exactly what we are looking for.

Consider a Markov process $\xi(\theta)$ which solves

$$d\xi(\theta) = A(\xi(\theta))dw(\theta), \quad \xi(0) = y$$

and let $p(0, y, t, x)$ be its transition probability density and $U(t)h(y) = E[h(t, \xi_{0,y}(t))]$. Let $U(t)$ and $U^+(t)$ act in $L^2(R)$

$$\int_R [U(t)h](y)u(y)dy = \int_R h(x)[U^+(t)g](x)dx$$

$$= \int_R \int_R h(x)p(0, y, t, x)dxg(y)dy. \qquad (14)$$

When h and g are continuous functions then setting $x = \xi_{s,y}(t) = \varphi_{s,t}(y)$, this relation can be rewritten in the form

$$\int_R U(t)h(y)g(y)dy = \int_R E[h(\xi_{0,y}(\theta))]g(y)dy$$

$$= E\int_R h(x)g(\hat{\xi}_{t,x}(0))\hat{J}_t(x)dx, \qquad (15)$$

where $\hat{\xi}_{t,s}(x) = \psi_{t,s}(x)$ is time reversal to $\xi(t)$, namely , $\hat{\xi}(s) = \xi(t-s)$ and $J(t)$ is the Jacobian of the map $\psi_{t,s} : R \to R$.

Let us once again assume that (u^1, u^2) is a weak solution of (10), (11) and u^i are positive bounded $\mathcal{W}^1$ functions. Then the system of SDEs

$$\begin{cases} d\xi^1(\theta) = \sqrt{2[a_{11}u^1(\theta,\xi^1(\theta)) + a_{12}u^2(\theta,\xi^1(\theta))]}dw_1(\theta), \\ d\xi^2(\theta) = \sqrt{2[a_{21}u^1(\theta,\xi^2(\theta)) + a_{22}u^2(\theta,\xi^2(\theta))]}dw_2(\theta), \end{cases} \tag{16}$$

$$\xi^1(s) = \xi^2(s) = y,$$

has a unique solution which is C^1-smooth in y. Let $\xi^i(\theta) = \varphi^i_{s,t}(y)$, $\partial_x\varphi^i_{s,t}(y) = J^i_{s,t}$ - Jacobian. Set $\hat{\xi}^i(\theta) = \xi^i(t-\theta) = \psi_{t,s}(x)$ and $\partial_x\psi^i_{t,s}(x) = \hat{J}^i_{t,s}(x)$.

Then

$$\begin{cases} \hat{\xi}^1(\theta) = x + \int_\theta^t [\partial_x [a_{11} u_1 + a_{12} u_2](\theta, \hat{\xi}^1(\theta))] d\theta \\ + \int_0^t \sqrt{2[a_{11} u_1 + a_{12} u_2](\theta, \hat{\xi}^1(\theta))} d\tilde{w}^1(\theta) \\ \hat{\xi}^2(\theta) = x + \int_\theta^t \partial_x [a_{21} u_1 + a_{22} u_2](\theta, \hat{\xi}^2(\theta)) d\theta \\ + \int_0^t \sqrt{2[a_{11} u_1 + a_{12} u_2](\theta, \hat{\xi}^2(\theta))} d\tilde{w}^2(\theta), \end{cases} \tag{17}$$

where $\tilde{w}^i(\theta) = w^i(t-\theta) - w^i(t)$.

Finally let

$$u_i(t) = E[\gamma_i(t) \circ \psi^i_{t,0}], \qquad (18)$$

where $\gamma_i(t)$ satisfies ODE

$$\frac{d}{dt}\gamma_i(t) = G_i(u(t) \circ \varphi^i_{0,t})\gamma_i(t), \quad \gamma_i(0) = u_0, \qquad (19)$$

and $\psi_{t,s} \circ \varphi_{s,t}(y) = y$. Applying Kunita theory we can check that the following relation holds

$$E[\gamma_i(t) \circ \psi^i_{t,0}(x)] = u_{0i}(x) \qquad (20)$$

$$+E\left[\int_0^t [\gamma_i(u(\theta) \circ \psi^i_{\theta,0}(x))u_i(\theta) \circ \psi^i_{\theta,0}(x)]_i d\theta\right]$$

$$+\int_0^t M_i(u)(E[u_i(\theta) \circ \psi^i_{\theta,0}(x)])d\theta,$$

Adding to (17) – (19) equations $\hat{J}_1(t) = \hat{J}_2(t) = 1$,

$$d\hat{J}_1(\theta) = \hat{J}_1(\theta)\frac{[a_{11}\partial_x u_1 + a_{12}\partial_x u_2](\theta, \hat{\xi}^1(\theta))}{\sqrt{2[a_{11}u_1 + a_{12}u_2](\theta, \hat{\xi}^1(\theta))}}dw_1 \tag{21}$$

$$+\hat{J}_1(\theta)\frac{[a_{11}\partial_x u_1 + a_{12}\partial_x u_2]^2(\theta, \hat{\xi}^1(\theta))}{2[a_{11}u_1 + a_{12}u_2](\theta, \hat{\xi}^1(\theta))}d\theta,$$

$$d\hat{J}_2(\theta) = \hat{\mathbf{J}}_{\mathbf{2}}(\theta)\frac{[a_{21}\partial_x u_1 + a_{22}\partial_x u_2](\theta, \hat{\xi}^2(\theta))}{\sqrt{2[a_{21}u_1 + a_{22}u_2](\theta, \hat{\xi}^2(\theta))}}dw_2 \tag{22}$$

$$+\hat{J}_2(\theta)\frac{[a_{21}\partial_x u_1 + a_{22}\partial_x u_2]^2(\theta, \hat{\xi}^2(\theta))}{2[a_{21}u_1 + a_{11}u_2](\theta, \hat{\xi}^2(\theta))}d\theta,$$

and

$$\partial_x u_i(t) = E\left[\partial_x[\gamma_i(t)\circ\psi^i_{t,0}]\hat{J}^i_{t,0}u_{0i}\circ\psi^i_{t,0} \right. \tag{23}$$

$$\left. +\gamma_i(t)\partial_x u_{0i}\circ\psi^i_{t,0}\hat{J}^i_{t,0}\right], i = 1, 2,$$

we obtain a closed system of equations.

We skip the proof of existence and uniqueness of a solution to (17) – (19) , (21)–(23) and instead assume once again that there exists a solution to this stochastic system and functions $u_i \in \mathcal{W}^1$
Then we prove that the function $u = (u_1, u_2)$ given by the above stochastic system defines a weak solution to (10), (11).
Given initial data $u_{0i} \in \mathcal{W}^1$ and
$\gamma_i(t) = exp\left(\int_0^t g_i(u(\theta))\right) u_{i0}] \in L^2([0, T_1]; H^1(R))$
we define as above $u_i(t) \in \mathcal{W}^1$ and $\varphi^i_{s,t}$, $\psi^i_{t,s}$.

Lemma

Mappings $U_i(t)$ and $U_i^+(t)$ given by

$$U_i(t)u_{0i} = E[\exp(\int_0^t g_i \circ \varphi^i_{0,\theta} d\theta) u_{0i} \circ \varphi^i_{0,t}],$$

$$U_i^+(t)u_{0i} = E[\gamma_i(t) \circ \psi^i_{t,0}]$$

act in $\mathcal{W}^1$ and possess a semigroup property

$$U_i(s)U_i(t) = U_i(s+t), \quad U_i^+(t)U_i^+(s) = U_i^+(t+s).$$

To evaluate generators of the above semigroups $U_i^+(t)$, $U_i(t)$ one needs a generalized Ito's formula proved by Kunita. Let

$$L_i(u) = \frac{1}{2}\beta_i^2(u)\Delta, \tag{25}$$

and $M_i(u)$ be an operator dual to $L_i(u)$, that is

$$\langle\langle M_i(u)u_i, h_i\rangle\rangle = \langle\langle u_i, L_i(u)h_i\rangle\rangle.$$

Lemma

Assume that $f(t) \in \mathcal{W}^k$ is a continuous function and $k \in \{-1, 0, 1\}$, the flows $\varphi^i_{s,t}$ are given by (16) and $\psi^i_{t,s}$ are their time reversals. Then

$$f(t) \circ \psi^i_{t,s} = f(t) + \int_s^t \partial_\theta f(\theta) \circ \psi^i_{t,\theta} d\theta \qquad (26)$$

$$- \int_s^t \nabla[f(\theta)] \circ \psi^i_{\theta,s}] \cdot d\psi^i_{\theta,s} + \int_s^t M_i(u)[f(t) \circ \psi^i_{\theta,s}] d\theta$$

and $M_i(u)$ acts in a generalized sense

Coming back to the original problem we get

$$E[u_i(t)\circ\psi_{t,0}] = u_{0i} + E\left[\int_0^t M_i(u)u_i(\theta)\circ\psi_{\theta,0}d\theta\right] \tag{27}$$

$$+E\left[\int_0^t G_i(u(\theta)\circ\psi^i_{\theta,0})u^i(\theta)\circ\psi^i_{\theta,0}d\theta\right]$$

and since

$$E\left[\langle\int_0^t M_i(u)u_i(\theta)\circ\psi^i_{\theta,0}d\theta, h_i\rangle\right] = \int_0^t\langle E[u^i(\theta)\circ\psi^i_{\theta,0}],$$

$$L_i(u)h_i\rangle d\theta] = \int_0^t\langle M_i(u)E[u^i(\theta)\circ\psi^i_{\theta,0}], h\rangle.$$

The final results are the following.

Theorem

There exist processes $\hat{\xi}_i(t) = \psi_{t,0}(x)$ such that the weak solution $u_1, u_2 \in \mathcal{W}^1$ admits a representation of the form $u_i(t) = U_i^+(t)u_{0,i} = E[\gamma_i(t) \circ \psi^i_{t,0}]$

Theorem

Let there exists a solution of the stochastic system (17)– (19), (21)– (23) and the functions $u_i(t,x)$ given by (19) belong to $\mathcal{W}^1$. Then they define a weak solution of the system (10), (11).

Thanks for your attention !

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We say that condition **C 1** holds if coefficients $a(x,u), A(x,u)$ are (locally) Lipschitz continuous in x and u and have sublinear growth in x and polynomial growth in u. Besides

$$\|u_0\| = sup_x \|u(0,x)\| \le K_0, \quad \|u_x(0,x)\| \le K_0^1.$$

We say that **C 2** holds if **C 1** holds and in addition,

$$2\langle c(x,v)h,h\rangle \le [\rho_0+\rho\|v\|^2]\|h\|^2,$$

$$\|(C(x,v)h)\|^2 \le \rho[1+\|v\|^2]\|h\|^2,$$

$$2\langle [c(x,v)-c(x_1,v_1)]h,h\rangle+\|[C(x,v)-C(x_1,v_1)]h)\|^2$$

$$\le L_v\|x-x_1\|^2+K(r)\|v-v_1\|^2]\|h\|^2,$$

where $C,\rho,L,L_1 L_{(v,v_1)}>0$ and ρ_0 are absolute constants and $r=max(\|v\|,\|v_1\|)$.
We say that condition **C 3** holds if **C 2** holds for derivatives $a_x^{(k)}, A_x^{(k)},\ c_x^{(k)},\ C_x^{(k)},\ k=1,2.$

Here

$$Tr Q^* \nabla^2 \Phi(s,x,h) Q = A^*_{ki} \frac{\partial^2 \Phi(s,x,h)}{\partial x_i \partial x_j} A_{jk}$$

$$+ 2 C_k^{lm} h_l \frac{\partial^2 \Phi(s,x,h)}{\partial x_j \partial h_m} A_{jk} + C_k^{qm} h_m \frac{\partial^2 \Phi(s,x,h)}{\partial h_q \partial h_p} C_k^{pn} h_n =$$

$$A^*_{ki} \frac{\partial^2 \Phi(s,x,h)}{\partial x_i \partial x_j} A_{jk} + 2 C_k^{lm} h_l \frac{\partial^2 \Phi(s,x,h)}{\partial x_j \partial h_m} A_{jk}.$$

Actually, due to linearity of $\Phi(s,x,h)$ in h, $\frac{\partial^2\Phi(s,x,h)}{\partial h_q \partial h_p} \equiv 0$. In addition

$$\langle q, \nabla\Phi(s,x,h)\rangle = a_j \frac{\partial \Phi(s,x,h)}{\partial x_j} + c_{lm} h_m \frac{\partial \Phi(s,x,h)}{\partial h_l},$$

$$G(s,x,h) = \langle h, g(s,x,u,A^*\nabla u)\rangle.$$

Consider mappings $U_i(t)$, $U_i^+(t)$, acting in $\mathcal{W}^1$ as follows $U_i^+(t)u_{0i} = E[\gamma_i(t) \circ \psi^i_{t,0}]$,
$U_i(t)h_{0i} = E[\exp\{\int_0^t g_i(u(\theta) \circ \varphi_{0,\theta})d\theta\}h_{0i} \circ \varphi^i_{0,t}]$.
We check that there exists a constant C, depending on $L_i(u)$, such that $\|U_i(t)u_0\|^2 \leq C_k\|u_{0i}\|^2$. Let $u_{0i} \in \mathcal{W}^{-1}$, then for any $h \in \mathcal{W}^1$ there exists a continuous linear functional U^i_t on $\mathcal{W}^{-1}$,

$$\langle U^i_{t,0}, h\rangle = E[\langle \gamma_i(t) \circ \psi^i_{t,0}, h\rangle] \qquad (24)$$

and thus $U^i_{t,0} \in \mathcal{W}^{-1}$ is called a generalized expectation of the r.v. $\gamma_i(t) \circ \psi^i_{t,0}$

$$U^i_{t,0} = u_i(t) = E[\gamma_i(t) \circ \psi^i_{t,0}]$$

.

In a similar way one can check that $U_i^+(t)u_{0i} = E[\gamma_i(t) \circ \varphi^i_{0,t}]$, acts in $\mathcal{W}^{-1}$. All families $U_i^+(t)$, $U_i(t)$ are semigroups i.e. $U_i(s)U_i(t) = U_i(s+t)$ for $s, t > 0$. By stochastic flows and conditional expectations properties

$$\langle U_i^+(s)U_i^+(t)u_0, h\rangle = E[\langle U_i^+(t)u_0, h \circ \psi_{s+t,t}\hat{J}_{s+t,t}\rangle$$

$$= E[\langle u_0, g \circ \psi_{t,0}\hat{J}_{t,0}|_{g=h\circ\psi_{t+s,s}\hat{J}_{t+s,s}}] = \dots =$$

$$= E[\langle u_0, h \circ \psi_{s+t,0}\hat{J}_{t+s,0}\rangle] = \langle U_i^+(s+t)u_0, h\rangle.$$