# On attractors of $m$-Hessian evolutions

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## 1. Introduction

Let $\Omega$ be a bounded domain in $R^n$, $Q = \Omega \times (0;\infty)$, $u \in C^{2,1}(\bar{Q})$, $u_{xx}$ be the Hesse matrix of $u$ in space variables. We denote a $p$-trace of $u_{xx}$ by $T_p[u] = T_p(u_{xx})$, $1 \leq p \leq n$ and introduce $p$-Hessian evolution operator

$$E_p[u] := -u_t T_{p-1}[u] + T_p[u], \quad (x,t) \in \bar{Q}_T. \tag{1.1}$$

Notice that by definition $T_0(u_{xx}) \equiv 1$, $T_1(u_{xx}) = \Delta u$ and (1.1) is the heat operator, when $p = 1$.

We investigate asymptotic behavior of solutions of the following initial boundary value problems:

$$E_m[u] = f, \quad u|_{\partial' Q_T} = \phi, \quad 1 \leq m \leq n, \tag{1.2}$$

where $\partial' Q_T = \Omega \times \{t = 0\} \cup \partial\Omega \times [0;T]$. In particular, we have proved

**Theorem 1.1.** *Let $f \geq \nu > 0$, $f \in C^{2,1}(\bar{Q}_T)$ for all $T \in [0;\infty)$, $\phi \in C^{2,1}(\partial' Q_T)$, $\phi = 0$ on $\partial\Omega \times [0;\infty)$, $\partial\Omega \in C^2$. Assume that $\lim_{t\to\infty} f(x,t) = \bar{f}(x)$ and there exists a solution $\bar{u} \in C^2(\bar{\Omega})$ to the Dirichlet problem*

$$T_m[u] = \bar{f}, \quad u|_{\partial\Omega} = 0.$$

*Then all solutions $u \in C^{2,1}(\bar{\Omega} \times [0;\infty))$ to the problem (1.2) tend uniformly in $C$ to the function $\bar{u}(x)$, when $t \to \infty$.*

It is of interest the following non existence theorem.

**Theorem 1.2.** *Assume that there are points $x_0, x_1 \in \Omega$ such that $\phi_{xx}(x_0, 0)$ is $(m-1)$-positive matrix, while $\phi_{xx}(x_1, 0)$ is not $(m-1)$-positive. Then there are no solutions in $C^{2,1}(\bar{Q}_T)$ to the problem (1.2), whatever $f > 0$, $\partial\Omega$, $T > 0$, $\phi$ had been.*

Eventually, we formulate the existence theorem assuming sufficiently smooth data in (1.2).

**Theorem 1.3.** *Let $f \geq \nu > 0$, $\partial\Omega$ is $(m-1)$-convex hypersurface, $\phi(x,0) \in \mathbf{K}_{m-1}(\bar{\Omega})$. Assume that compatibility conditions are satisfied. Then there exists a unique in $C^{2,1}(\bar{Q}_T)$ solution to the problem (1.2).*

## 2. Notations and definitions

We denote the space of $N \times N$ symmetric matrices by $Sym(N)$ and by $T_p(S)$ the $p$-traces of $S \in Sym(N)$, which are the sum of all principal $p$-minors of $S$, $1 \le p \le N$, $T_0(S) := 1$.

**Definition 2.1.** A matrix $S \in Sym(N)$ is $m$-positive if $S \in K_m$,

$$K_m = \{S : T_p(S) > 0, \quad p = 1, \dots, m\}. \tag{2.1}$$

The cones (2.1), $m = 1, \dots, N$ are the basis of the theory of $m$-Hessian partial differential equations and admit different equivalent definitions. Constructive Definition 1.1 has been introduced in the paper [7].

Our further proceeding will be restricted to the subspace of $Sym(N)$. Namely, we take into consideration the set

$$\mathbf{S}^{ev} = \{S^{ev} = (s_{kl})_0^n, \quad s_{00} = 1, \quad s_{0i} = s_{i0} = 0, \quad S = (s_{ij})_1^n \in Sym(n)\}. \tag{2.2}$$

In order to emphasize this restriction we introduce new notations for traces $T_p$ and cones (2.1) on the subspace (2.2)

$$E_m(s, S) := T_m(S^{ev}) = sT_{m-1}(S) + T_m(S), \quad 1 \le m \le n, \tag{2.3}$$

$$K_m^{ev} = \{s, S : E_p(s, S) > 0, \quad p = 1, \dots, m\}. \tag{2.4}$$

Let $\Omega \subset R^n$ be a bounded domain, $Q = \Omega \times (0;T)$, $\partial'' Q_T = \partial\Omega \times [0;T]$, $\partial' Q_T = (\Omega \times \{0\}) \cup \partial'' Q_T$, $u \in C^{2,1}(\bar{Q}_T)$. We introduce functional analogs of (2.2), (2.3), (2.4): $S^{ev}[u] = (s[u] = -u_t, S[u]) = u_{xx})$,

$$E_m[u] := T_m(S^{ev}[u]) = -u_t T_{m-1}(u_{xx})) + T_m(u_{xx})), \quad 1 \le m \le n, \tag{2.5}$$

$$\mathbf{K}_m^{ev}(\bar{Q}_T) = \{u \in C^{2,1}(\bar{Q}_T) : S^{ev}[u] \in K_m^{ev}, (x,t) \in (\bar{Q}_T)\}, \tag{2.6}$$

where $u_{xx}$ is Hesse matrix of $u$.

**Definition 2.2.** We say that operator (2.5) is the $m$-Hessian evolutionary operator and a function $u \in \mathbf{K}_m^{ev}(\bar{Q}_T)$ is $m$-admissible in $\bar{Q}_T$ evolution.

The development of the theory of Hessian equations has brought out some new notions in differential geometry and the first description of some may be found in [1] as necessary conditions for admissible solvability of the Dirichlet problems. In the papers [9], [8], [11] some versions of these requirements were considered independently of differential equations as the set of new geometric notions. Namely, let $\partial\Omega \in R^n$ be $C^2$-hypersurface with position-vector $X = X(\theta)$ and metric tensor $g[\partial\Omega] = (g_{ij})_1^{n-1}$, $g_{ij} = (X_i, X_j)$, $X_i = \partial X/\partial\theta^i$. In some vicinity of $M_0 \in \partial\Omega$ we introduce the set of matrices $\tau = (\tau_i^j)_1^{n-1}$ such that $g^{-1} = \tau^T\tau$ and denote

$$X_{(i)} = X_k\tau_i^k, \quad X_{(ij)} = X_{kl}\tau_i^k\tau_j^l, \quad i, j = 1, \dots, n-1. \tag{2.7}$$

Notice that $(X_{(i)}, X_{(j)}) = \delta_{ij}$ and (2.7) provides Euclidean moving frames for $\partial\Omega$. The freedom of choice of $\tau$ supplies rotations in the tangential plane.

The second item in (2.7) provides the set of symmetric matrices $\mathcal{K}[\partial\Omega]$,

$$\mathcal{K}[\partial\Omega] = (\mathcal{K}_{ij})_1^{n-1}, \quad \mathcal{K}_{ij} = (X_{(ij)}, \mathbf{n}), \tag{2.8}$$

where $\mathbf{n}$ is the interior to $\partial\Omega$ normal.

**Definition 2.3.** We say that a matrix (2.8) is the curvature matrix of $\partial\Omega$ and functions $\mathbf{k}_p(M) = T_p(\mathcal{K}[\partial\Omega])(M)$, $p = 1, \ldots, n-1$ are the $p$-curvatures of $\partial\Omega$.

By construction the curvature matrices are geometric invariant in the sense that theirs eigenvalues are the principal curvatures of $\partial\Omega$. On the other hand, $p$-curvatures are absolute geometric invariants admitting natural numbering by $p$ throughout $\partial\Omega$. It is also remarkable that if $\partial\Omega$ is $C^{2+k}$-smooth, then $\{\mathbf{k}_p\}_1^{n-1}$ are $C^k$-smooth.

Definitions 1.1, 1.3 carry out

**Definition 2.4.** A closed $C^2$-hypersurface $\Gamma$ is $m$-convex at a point $M$ if its curvature matrix is $m$-positive at this point.

Notice that $m$-positiveness of the curvature matrix does not depend on parametrization.

It follows from 2.1 that Definition 1.3 is equivalent to

**Definition 2.5.** A closed $C^2$-hypersurface $\Gamma$ is $m$-convex at a point $M$ if the first $p$-curvatures of $\Gamma$ are positive up to $m$ at $M$:

$$\mathbf{k}_p[\Gamma](M) > 0, \quad p = 1, \ldots, m. \tag{2.9}$$

As to the principal curvatures of $\Gamma \subset R^{n+1}$, it is known that at least $m$ of them are positive in the points of $m$-convexity but otherwise it is only true for $m = n$, i.e., for strictly convex hypersurfaces in common sense.

# 3. Existence and non-existence theorems

Consider in the cylinder $Q_T$ the first initial boundary value problem for the $m$-Hessian evolution equation,

$$E_m[u] = f, \quad u(x,0) = \psi, \quad u|_{\partial'' Q_T} = \phi, \quad 1 \leq m \leq n, \tag{3.1}$$

where $\psi, \phi$ are sufficiently smooth given functions, satisfying the compatibility conditions

$$\psi(x) = \phi(x,0), \quad \phi_t(x,0) = \frac{T_m(\psi_{xx}) - f(x,0)}{T_{m-1}(\psi_{xx})}, \quad x \in \partial\Omega. \tag{3.2}$$

The following proposition is a refined version of Theorem 1.2 from the paper [12].

**Theorem 3.1.** *Assume that $\partial\Omega$ is an $(m-1)$-convex hypersurface, $\partial\Omega \in C^{4+\alpha}$, $f \geq \nu > 0$, $f \in C^{2+\alpha, 1+\alpha/2}$, $\psi \in \mathbf{K}_{m-1}(\bar\Omega) \cap C^{4+\alpha}(\bar\Omega)$, $\phi \in C^{4+\alpha, 2+\alpha/2}(\partial'' Q_T)$ and $\psi, \phi$ satisfy (3.2).*

*Then there exists the unique in $\mathbf{K}_m^{ev}(\bar Q_T)$ solution $u$ to the problem (3.1) and $u \in C^{4+\alpha, 2+\alpha/2}(\bar Q_T)$.*

Restricting $\phi$ to zero we obtain

**Theorem 3.2.** *Assume that $f \geq \nu > 0$, $f \in C(\bar Q_T)$ and in (3.1) $\phi = 0$. Then there exists no more than one solution $u \in C^{2,1}(\bar Q_T)$ to the problem (3.1) and if $u$ does exist then it belongs to $\mathbf{K}_m^{ev}(\bar Q_T)$, i.e., $u$ is an $m$-admissible evolution.*

Theorems 1.2, 1.3 are a consequence of the following non existence theorem.

**Theorem 3.3.** *Let $m > 1$, $\phi = 0$ and all conditions of Theorem 2.1 are satisfied but $(m-1)$-admissibility of $\psi$, i.e., there is a point $x_0 \in \Omega$ such that $\psi_{xx}(x_0) \in Sym(n) \backslash \bar K_{m-1}$. Then there are no solutions to the problem (3.1) in $C^{2,1}(\bar Q^T)$, whatever small $T$ be.*

## 4. On asymptotic behavior of $m$-Hessian evolutions

In the paper [14] N.Trudinger and X.-J.Wang have considered the logarithmic Hessian evolution operator $P_{m,l}$, $0 \le l < m \le n$, which may be written in the form

$$P_{m,l}[u] = -u_t + \log T_{m,l}(u_{xx}), \quad T_{m,l}(u_{xx}) = \frac{T_m(u_{xx})}{T_l(u_{xx})}, \quad u \in \mathbf{K}(\bar{\Omega}). \tag{4.1}$$

We see that here $\mathbf{K}(\bar{\Omega})$ controlled by log(.) is the basis of $P_{m,l}$-admissible evolutions and it is natural to associate with operator (4.1) the set $\mathbf{K}(\bar{\Omega}) \times \mathbf{R}$. We separate Theorem 2.1 from [14] in two.

**Theorem 4.1.** *Assume that $\partial\Omega \in C^4$ is $(m-1)$-convex, $g \in C^{2,1}(\bar{Q}_T)$, $\psi \in \mathbf{K}(\bar{\Omega}) \cap C^4(\bar{\Omega})$, $\phi \in C^{4,2}(\partial'' Q_T)$. Assume also that*

$$\phi(x,0) = \log T_{m,l}(u_{xx}) - g(x,0), \quad x \in \partial\Omega.$$

*Then there exists a unique solution of the problem*

$$P_{m,l}[u] = g, \quad u(x,0) = \psi, \quad u|_{\partial'' Q_T} = \phi, \quad 0 \le l \le m \le n, \tag{4.2}$$

*which is $P_{m,l}$-admissible evolution.*

**Theorem 4.2.** *Let conditions of Theorem 3.1 be satisfied for all $T < \infty$. Assume in addition that $g(x,t)$, $\phi(x,t)$ converge uniformly as $t \to \infty$ to functions $\bar{g}(x)$, $\bar{\phi}(x)$. Then $P_{m,l}$-admissible solution to the problem (4.2) converges uniformly to the unique $m$-admissible solution $\bar{u}$ of the Dirichlet problem*

$$T_{m,l}[\bar{u}] = \exp g, \quad x \in \Omega, \quad \bar{u}|_{\partial\Omega} = \bar{\phi}. \tag{4.3}$$

Theorem 2.1 is an analog of Theorem 3.1 for the m-Hessian evolution equations. As to Theorem 3.2 the following proposition holds for $m$-Hessian evolutions.

**Theorem 4.3.** *Assume that $u = u(x,t)$ is a $m$-admissible evolution from Theorem 2.1 in which conditions are satisfied for all $T < \infty$ and assume also that $f(x,t)$, $\phi(x,t)$ converge uniformly as $t \to \infty$ to functions $\bar{f}(x)$, $\bar{\phi}(x)$. Then $u(x,t)$ converges uniformly to the $m$-admissible function $\bar{u}$, which is a solution of the Dirichlet problem*

$$T_m[\bar{u}] = f, \quad x \in \Omega, \quad \bar{u}|_{\partial\Omega} = \bar{\phi}. \tag{4.4}$$

Notice that in Theorem 2.1 an initial function $\psi$ is not required to be $m$-admissible. Comparing (4.3) with (4.4) we see that in contrast to operators (4.1) $m$-admissible evolutions may upgrade the rank of admissibility of initial functions.

## Список литературы


[1] Caffarelli L., Nirenberg L., Spruck J., The Dirichlet problem for nonlinear second order elliptic equations III. Functions of the Hessian. Acta Math. 155, 1985, 261-301.

[2] Ivochkina N. M., Trudinger N.S., Wang X.-J., The Dirichlet problem for degenerate Hessian equations. Comm. Partial Differ. Equations 29, 2004, 219-235.

[3] Trudinger N.S., The Dirichlet problem for the prescribed curvature equations. Arch. Rat. Mech. Anal. 111, 1990, 153–179.

[4] Trudinger N.S., Maximum principles for curvature quotient equations. J. Math. Sci. Univ. Tokyo 1, 1994, 551–565.

[5] Garding L., An inequality for hyperbolic polynomials. J. Math. Mech. 8 ,1959, 957-965.

[6] Ivochkina N. M., Second order equations with d-elliptic operators. Trudy Mat. Inst. Steklov, 147, 1980, 40–56; English transl. in Proc. Steclov Inst. Math. 1981, no.2.

[7] Ivochkina N. M., A description of the stability cones generated by differential operators of Monge

[8] Ivochkina N.M., Prokof'eva S.I., Yakunina G.V., The Gårding cones in the modern theory of fully nonlinear second order differential equations. J. of Math. Sci., 184, 2012, no.3, 295-315.

[9] Filimonenkova N.V., Ivochkina N.M., On the backgrounds of the theory of m-Hessian equations. Comm. Pure Appl. Anal., 12, 2013, no.4.

[10] Ivochkina N. M., Solution of the Dirichlet problem for the curvature equation order m. Algebra i Analiz 2(1990), no. 3, 192–217; English transl. in Leningrad Math. J. 2(1991)

[11] Ivochkina N. M., From Gårding cones to p-convex hypersurfaces. Proc. of the Sixth Int. Conf. on Diff. and Funct.- Diff. Equations (Moscow, August 14-21, 2011). Part 1, CMFD, 45 (2012), PFUR, M., 94-104.

[12] Ivochkina N.M., On classic solvability of the $m$-Hessian evolution equation. AMS Transl. 229 (2010), Series 2, 119-129.

[13] N.M.Ivochkina, On approximate solutions to the first initial boundary value problem for the m-Hessian evolution equations. J. Fixed Point Th. Appl. 4 (2008), no.1, 47-56.

[14] Trudinger N.S., Wang X.-J., A Poincare type inequality for Hessian integrals. Calc. Var. and PDF, 6 (1998), 315-328.