Higher regularity for the thin obstacle problem

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The thin obstacle problem

1. Ω open domain in $\mathbb{R}^n$
2. $\Gamma \subset \mathbb{R}^n$ a hyper surface
3. a^{ij} Lipschitz continuous positive definite
4. $\phi \in C^{1,1}(\Gamma)$
5. $u : \Omega \to \mathbb{R}$ a local minimizer of

$$\int \sum_{i,j} a^{ij} \partial_i u \partial_j u dx$$

with the constraint $u \geq \phi$ on Σ, i.e. for all $x \in \Omega$ there exists $r > 0$ such that

- $B_r(x) \in \Omega$
- for all $v \in H^1(B_r(x))$ with $v \geq \phi$ for $y \in B_r(x) \cap \Sigma$ and $v = u$ on $\partial B_r(x)$

$$\int_{B_r(x)} \sum_{i,j=1}^{n} a^{ij} \partial_i v \partial_j v dx \geq \int_{B_r(x)} \sum_{i,j=1}^{n} a^{ij} \partial_i u \partial_j u dx$$

General existence and regularity results

Functional analysis: Existence of minimizers for Ω bounded and suitable Dirichlet conditions at $\partial\Omega$.
Regularity theory: $u \in C^{1,s}$ for (some) $s \in (0, \frac{1}{2}]$ if $\phi \in C^2$. (Caffarelli 1979, Arkhipova & Uralceva 87& 89, $s = \frac{1}{2}$ Athanasapoulos & Caffarelli 04, Laplacian, half space, zero obstacle, Garofalo & Smets Vega Garcia 14, general).

The boundary thin obstacle problem

Study local minimizers of $\int_{B_1^+} |\nabla u|^2 dx$ subject to $u(x_1, \ldots, x_{n-1}, 0) \geq 0$.

- The contact set $\Lambda = \{y \in \mathbb{R}^{n-1} : |y| < 1, u(y,0) = 0\}$
- The positivity set $\Omega = \{y \in \mathbb{R}^{n-1} : |y| < 1, u(y,0) > 0\}$
- The free boundary $\Gamma = \partial\Lambda \cap B_1(0)$

The quantity

$$N_x(u,r) = \frac{r \int_{B_r^+(x)} |\nabla u|^2 dx}{\int_{(\partial B_r(x))} |u|^2 d\mathcal{H}^{n-1}}$$

is non decreasing and the limit

$$\kappa(x) = \lim_{r\to 0} N_x(u,r)$$

is upper semi continuous.

Blow-up

The functions

$$u_r = \frac{u(x/r)}{\|u(./r)\|_{L^2(B_1(0))}}$$

are compact in $L^2(B_1(0))$ and of norm 1. By the monotonictiy formula any blow-up is a homogeneous solution to the thin obstacle problem of homogeneity $\kappa(x)$. They are partly classified: If $x \in \Gamma$ then either $\kappa \geq 2$, or $\kappa = \frac{3}{2}$. In the second case the homogeneous solution is unique, up to symmetries, and given by

$$u_0(x) = \mathrm{Re}(x_{n-1} + ix_n)^{3/2}$$

The subset of Γ with $\kappa = \frac{3}{2}$ is called the regular part of Γ. It is open in Γ by the semi continuity of κ.

Result

Theorem (K, Petrosyan, Shi 2014)

The regular part of Γ *is analytic, i.e. it is the level set of a non degenerate analytic function.*

The Grushin Laplacian

Consider harmonic functions in the slit domain, i.e. in the unit ball, outside

$$\Lambda_0 = \{x \in \mathbb{R}^n : x_n = 0, x_{n-1} \leq 0\}$$

where we require u to be 0. We introduce new variables

$$x_{n-1} = \frac{1}{2}\left(y_n^2 - y_{n-1}^2\right), \quad x_n = y_n y_{n-1}, \quad y_i = x_i \text{ for } i \leq n-2.$$

Then

$$(x_1^2 + x_2^2)^{\frac{1}{2}} \Delta_x u = \left[\partial_{y_n}^2 + \partial_{y_{n-1}}^2 + (y_{n-1}^2 + y_n^2) \sum_{i=1}^{n-2} \partial_{y_i}^2\right] u =: \Delta_{Grushin} u$$

in $y_n > 0$ with the boundary condition $u(y) = 0$ on $\{y_n = 0\}$.
Alternatively we may neglect the boundary condition and consider odd functions on $\mathbb{R}^n$.

Facts

1. The Grushin Laplacian falls into Hörmander's theory of squares of vector fields.
2. It is hypo elliptic and analytic hypo elliptic
3. The full Calderon-Zygmund theory holds, with the proper formulation.

Consider

$$\Delta u = \nabla F \quad \text{in} \quad \mathbb{R}^n \backslash \Gamma_0, \qquad u = 0 \quad on \quad \Gamma_0.$$

Theorem

$$\| |x|^s \nabla u \|_{L^p} \leq c \| |x|^s F \|_{L^p}$$

for $s = \frac{1}{2} - \frac{1}{p}$, *or, more generally, if*

$$\frac{1}{2} - \frac{2}{p} < s < \frac{3}{2} - \frac{2}{p}.$$

The partial Hodograph transform

Suppose that (after even reflection)

$$\lim_{r\to 0} u_r \to u_0.$$

We define the map

$$x \to (x_1, \ldots x_{n-2}, \partial_{n-1}u, \partial_n u)$$

and define the Legendre function of u by

$$v(y) = u(x) - x_{n-1}y_{n-1} - x_n y_n.$$

The free boundary is the graph of the function

$$f(x_1, \ldots, x_{n-2}) = -\partial_{y_{n-1}} v(x_1, \ldots x_{n-2}, 0, 0).$$

Theorem

The function v is analytic.

The function v

The function v satisfies the Monge-Ampere type equation

$$0 = v_{n-1,n-1} + v_{nn} - \sum_{i=1}^{n-2} \det \begin{pmatrix} v_{ii} & v_{i,n-1} & v_{in} \\ v_{n-1,i} & v_{n-1,n-1} & v_{n-1,n} \\ v_{ni} & v_{nn-1} & v_{nn} \end{pmatrix}. \quad (1)$$

The determinants are the Hessian determinants of v restricted to three dimensional subspaces given by the ith, the $n-1$ and the nth coordinate. The Legendre function of u_0 is (up to constants)

$$v(y) = y_{n-1}^3 - 3y_{n-1}y_n^2.$$

It satisfies the homogeneous Grushin Laplace equation.
To prove Theorem 3 we prove that solutions to (1) coming from the construction above are analytic. *Symmetries, structure depends on third order derivatives*.

Steps of the proof

1. Γ is a Lipschitz graph.
2. There exists $s > 0$ such that Γ is a graph of a $C^{1,s}$ function.
3. There is a unique blow-up without normalization. It depends continuously on $x \in \Gamma$. (KPS, Monotonicity formulas, approximation by polynomials, .. De Silva, Savin *smoothness* 14))
4. The blow up converges in high norms away from the free boundary.
5. The nonlinear equation is a perturbation of the Grushin Laplacian. Finite differences give higher regularity. (KPS)

Extensions, work in progress (K,Rüland, Shi)

Lipschitz coefficients, $C^{1,1}$ surface, nonzero obstacle

1. Carleman inequality instead of frequency function (Compactness of blow up, blow up is a homogeneous solution).
2. 'second term'
3. Higher regularity / analyticity of regular part for analytic data