On Uniqueness in the Water Wave Theory

Vladimir Kozlov

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This talk is based on joint with N.Kuznetsov and E.Lokharu work.

Geometrical assumptions on the fluid motion. The main variables for description of fluid motion are the velocity vector, pressure and free surface profile (if there is such). We suppose that the motion is steady in a certain direction, for example x-direction, i.e. all variables depend on y, z and $x - ct$, where c is a given constant. Moreover we assume that the motion is two dimensional, i.e. it does not depend on z.

Physical assumptions. The fluid motion is described by the Euler equations. The irrotational case includes only the gravity force. In order to consider an influence of other factors on the water flow one must include vorticity into the problem.

By introducing the stream function ψ: $\mathbf{v} = (\psi_{\mathbf{y}}, -\psi_{\mathbf{x}})$ and vorticity $\omega = \omega(\psi)$ the mathematical model for this problem can be written as follows

Let water occupy the rigion

$$\mathcal{D} = \{-\infty < x < +\infty,\ 0 < y < \xi(x)\},$$

where $y = \xi(x)$ is unknown free-surface. In terms of unknowns ψ and ξ the problem can be written (after some re-scaling) in the form

$$\begin{aligned} &\psi_{xx} + \psi_{yy} + \omega(\psi) = 0, \qquad (x, y) \in \mathcal{D}; \\ &\psi(x, 0) = 0, \quad \psi(x, \xi(x)) = 1, \quad x \in \mathbb{R}; \\ &|\nabla_{x,y}\psi(x, \xi(x))|^2 + 2\xi(x) = 3r, \quad x \in \mathbb{R}, \end{aligned}$$

where $r > 0$ is Bernoulli's constant.

Stokes waves: a) (*Irrotational case*) Stokes, 1847, introduced these waves and conjectures various properties of such waves; Nekrasov (1951, 1921-trudy Ivano-Voznesenskogo Pol. Institute), Levi-Civita, Struik (1926) proved existence of Stokes waves of small amplitude; Keady, Norbury, Boffoni, Dancer, Toland ... proved existence of global branches of Stokes waves and important properties of Stokes waves.
b) (*Rotational case*) Constantin, Strauss, unidirectional flow-existence of Stokes waves of small amplitude and bifurcation branches of large Stokes waves, 2004.
Flow with counter currents: Wahlén 2009 (constant vorticity), K-Kuznetsov 2012 (arbitrary vorticity) existence of Stokes waves of small amplitude

Solitary waves: a) (*Irrotational case*) Russel (1834) discovered experimentally existence of solitary waves; Lavrentiev (1943, 1975) proved the existence of solitary waves by approximated them by Stokes waves; Friedrichs, Hyers (1954) give another more explicit proof of existence of solitary waves of small amplitude; Amick, Toland, Plotnikov, Craig, Benjamin, Bona, Bose... proved existence of global branches of solitary waves and established various their properties;
b) (*Rotational case, one-directional flow*) Groves, Wahlén, 2007, Hur, 2008 existence of solitary waves of small amplitude, Wheeler, 2013 existence of branches of solitary waves

Periodic waves (different from Stokes waves): a) (*Irrotational case*) (we note that small amplitude waves are necessary Stokes or Solitary waves) Chen and Saffman (1980), Craig, Nicholls, Buffoni, Dancer and Toland ......
b) (*Rotational case*) Ehrnstrm, Mats; Escher, Joachim; Wahln, Erik 2011 (linear vorticity, small amplitude waves)

General water waves

From Bernoulli's equation it follows that

$$0 < \xi(x) \leq 3r/2.$$

Stream solutions A pair $(u(y),\ h)$, where $h = \mathrm{const}$, is called a *stream solution* when

$$\psi(x, y) = u(y) \quad \text{and} \quad \xi(x) = h$$

satisfy

$$u'' + \omega(u) = 0, \quad u(0) = 0, \ \ u(h) = 1, \qquad (1)$$

$$|u'(h)|^2 + 2h = 3r. \qquad (2)$$

We consider here only unidirectional flows. Let us describe all such solutions. We put

$$s_0^2 = 2 \max_{0 \leq x \leq 1} \Omega(x), \ \ \Omega(x) = \int_0^x \omega(\tau) d\tau.$$

By $U(y;s)$, we denote a solution to $U'' + \omega(U) = 0$ satisfying

$$U(0;s) = 0, \quad U'(0;s) = s, \quad \text{where } s \in \mathbb{R}.$$

It is given by

$$y = \int_0^U \frac{d\tau}{\sqrt{s^2 - 2\Omega(\tau)}}.$$

This function U satisfies (1) with

$$h = h(s) = \int_0^1 \frac{d\tau}{\sqrt{s^2 - 2\Omega(\tau)}}.$$

Relations (2) are satisfied if

$$\mathcal{R}(s) = r, \;\; \mathcal{R}(s) = \frac{1}{3}(s^2 + 2h(s) - 2\Omega(1)). \tag{3}$$

Corresponding stream solutions are called conjugate flows.

Benjamin, 1971, A unified theory of conjugate flows. Phil. Trans. R. Soc. A: characterized the existence of conjugate streams as a common feature for many hydrodynamic models and emphasized that it is crucial to the understanding of observed wave phenomena.

Keady and Norbury, 1975, boundaries for Stokes wave (irrotational case); 1978, boundaries for Stokes wave (rotational case);

Benjamin, 1995, generalized results of Keady and Norbury for periodic irrotational waves and formulated Benjamin-Lighthill conjecture;

K.-Kuznetsov, 2008, 2009, 2012 established bounds for general water waves (not necessary Stokes or periodic)

We put

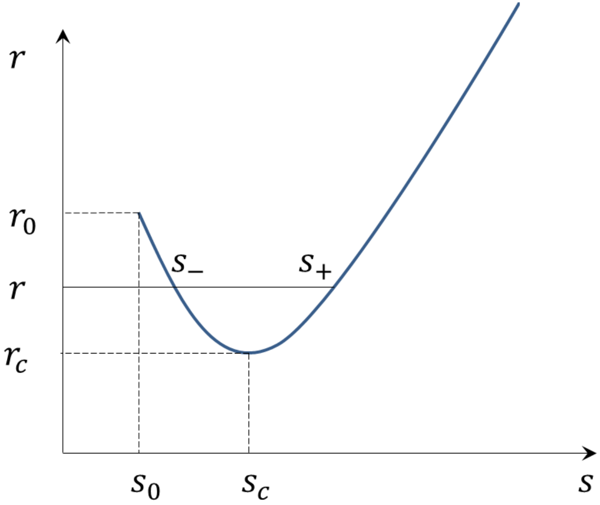
$$r_c = \inf_{s>s_0} \mathcal{R}(s) \text{ and } r_0 = \lim_{s\to s_0} \mathcal{R}(s).$$

Certainly, if $r < r_c$ then equation (3) has no solution and hence there are no unidirectional stream solutions. It appears that in this case there are no any solutions to the water wave problem.
Moreover, if $r = r_c$ then the only water-wave solution is the stream solution (Wheeler 2014).
Consider the case when $r > r_c$ and $r < r_0$. Then equation (3) has two roots $s_- < s_+$. We put $h_+ = h(s_-)$ and $h_- = h(s_+)$. Clearly, $h_+ > h_-$. Let now (ψ, ξ) be a solution to the water wave problem different from a stream solution. Then the following inequalities are valid:

$$\xi(x) > h_-, \ \inf \xi < h_+, \ \sup \xi > h_+.$$

If $r \geq r_0$ then there are no solutions different from stream ones.

Figure 1



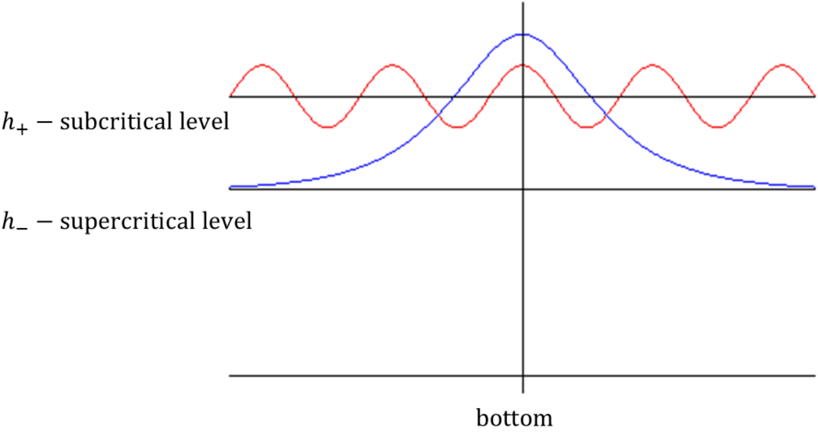
let us describe all solutions (ψ, ξ) to the water wave problem subject to

$$0 < \delta \leq \psi_y(x, y) \text{ and } |\nabla \psi| + |\xi_x| \leq M. \tag{4}$$

Then there exists $r^* = r^*(\delta, M) > r_c$ such that for $r \in (r^*, r_c)$ the following discription of all solutions to the water wave problem is valid:

1. There is a unique (up to translations) solitary wave. We denote its height by h_{sol};
2. All other nontrivial waves are Stokes waves of heights from the interval (h_+, h_{sol});
3. If $h \in (h, h_{sol})$ then there exists a unique (up to translations) Stokes wave of the height h;
4. $$\sup_{x \in \mathbb{R}} e^{-\alpha|x|} |\xi(x) - \xi_{sol}(x)|^2 dx \leq C|h - h_{sol}|^2$$

Figure 2



Change of variables $p = \psi(x, y)$ and $a = x$. Then the function $h(p, q) = y(p, q)$ satisfies the problem

$$\frac{1}{2}\Big(\frac{1 + h_q^2}{h_p^2}\Big)_p - \Big(\frac{h_q}{h_p}\Big)_q + \omega(p) = 0 \text{ on } S = (0, 1) \times \mathbb{R},$$

$$h(p, q) = 0, \quad q \in \mathbb{R}, \quad p = 0$$

and

$$1 + h_q^2 = (3r - 2h)h_p^2, \quad p = 1.$$

Maximum principle, the Harnack ineqquality, the Hopf lemma, integral identities, asymptotic analysis

THANK YOU