Criteria for the Poincaré–Hardy inequalities

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Dear Nina,

Many happy returns of the day!

This talk is based on the papers

V. Maz'ya

1. *The negative spectrum of the higher-dimensional Schrödinger operator*

Dokl. Akad. Nauk SSSR 1962 **144**, 721–722.

2. *On the theory of the higher-dimensional Schrödinger operator*.

Izv. Akad. Nauk SSSR Ser. Mat. 1964, **28**, 1145–1172.

Maz'ya, V., Verbitsky, I.

1. *The Schrödinger operator on the energy space: boundedness and compactness criteria*,
Acta Mathematica, 2002 **188**, 263–302.

Maz'ya, V., Verbitsky, I.

2. *Infinitesimal form boundedness and Trudinger's subordination for the Schrödinger operator*

Inventiones Mathematicae, 2005, **161**, 81–136

3. *Form boundedness of the general second order differential operator*

Comm. Pure Appl. Math., 2006, **59**:9, 1286–1329

Jaye, B. J., Maz'ya, V. G., Verbitsky, I. E.

Existence and regularity of positive solutions of elliptic equations of Schrödinger type

Wiener capacity

The notion of capacity appeared first in electrostatics and was introduced to mathematics by N. Wiener in the 1920s. Since then several generalizations and modifications of Wiener's capacity appeared: Riesz, Bessel, polyharmonic capacities, p-capacity and others. They are of use in potential theory, probability, function theory and partial differential equations. The capacities provide adequate terms to describe sets of discontinuities of Sobolev functions, removable singularities of solutions to partial differential equations, sets of uniqueness for analytic functions, regular boundary points in the Wiener sense, divergence sets for trigonometric series, etc.

Let Ω be an open set in $\mathbb{R}^n$ and let F be a compact subset of Ω. The Wiener (harmonic) capacity of F with respect to Ω is defined as the number

$$\mathrm{cap}_\Omega F = \inf\Big\{\int_\Omega |\nabla u|^2 dx : \; u \in C_0^\infty(\Omega), \; u \geq 1 \text{ on } F\Big\}. \qquad (1)$$

We shall use the simplified notation $\mathrm{cap} F$ if $\Omega = \mathbb{R}^n$. The capacity $\mathrm{cap}_\Omega F$ can be defined equivalently as the least upper bound of $\nu(F)$ over the set of all measures ν supported by F and satisfying the condition

$$\int_\Omega G(x, y)\, d\nu(y) \leq 1,$$

where G is the Green function of the domain Ω. If $\Omega = \mathbb{R}^3$ then it is just the electrostatic capacity of F.

It follows from the definition (1) that the capacity is a nondecreasing function of F and a nonincreasing one of Ω. We have Choquet's inequality

$$\mathrm{cap}_\Omega(F_1 \cap F_2) + \mathrm{cap}_\Omega(F_1 \cup F_2) \leq \mathrm{cap}_\Omega F_1 + \mathrm{cap}_\Omega F_2$$

for arbitrary compact sets F_1 and F_2 in Ω [**?**]. It is easy to check that the Wiener capacity is continuous from the right. This means that for each $\varepsilon > 0$ there exists a neighborhood G, $F \subset G \subset \overline{G} \subset \Omega$ such that for each compact set F_1 with $F \subset F_1 \subset G$ the inequality

$$\mathrm{cap}_\Omega F_1 \leq \mathrm{cap}_\Omega F + \varepsilon$$

holds.

Let E be an arbitrary subset of Ω. The inner and the outer capacities are defined as numbers

$$\underline{\mathrm{cap}}_\Omega E = \sup_{F \subset E} \mathrm{cap}_\Omega F, \qquad F \text{ compact in } \Omega,$$

$$\overline{\mathrm{cap}}_\Omega E = \inf_{G \supset E} \mathrm{cap}_\Omega G, \qquad G \text{ open in } \Omega.$$

It follows from the general Choquet theory that for each Borel set both capacities coincide. Their common value is called the Wiener (harmonic) capacity and will be denoted by $\mathrm{cap}_\Omega E$.

By v_n we denote the volume of the unit ball in $\mathbb{R}^n$ and let $\mathrm{mes}_n F$ stand for the n-dimensional Lebesgue measure of F. By the classical isoperimetric inequality, the following isocapacitary inequalities hold

$$\mathrm{cap}_\Omega F \geq n v_n^{2/n}(n-2)\,|(\mathrm{mes}_n\,\Omega)^{(2-n)/n} - (\mathrm{mes}_n\,F)^{(2-n)/n}|^{-1} \quad \text{if } n > 2 \tag{2}$$

and

$$\mathrm{cap}_\Omega F \geq 4\pi \left(\log \frac{\mathrm{mes}_2\,\Omega}{\mathrm{mes}_2\,F}\right)^{-1} \quad \text{if } n = 2. \tag{3}$$

In particular, if $n > 2$ then

$$\mathrm{cap} F \geq n v_n^{2/n}(n-2)(\mathrm{mes}_n\,F)^{(n-2)/n}. \tag{4}$$

If Ω and F are concentric balls, then the three preceding estimates come as identities.

Using Wiener's capacity, one can obtain two-sided estimates for the best constant in the Friedrichs inequality

$$\|u\|_{L_2(B_1)} \leq C \, \|\nabla u\|_{L_2(B_1)}, \tag{5}$$

where B_1 is a unit open ball and u is an arbitrary function in $C^\infty(\overline{B_1})$ vanishing on a compact subset F of $\overline{B_1}$.
Proposition. The best constant C in (5) satisfies

$$C \leq c(n)\big(\mathrm{cap}\, F\big)^{-1/2}, \tag{6}$$

where $c(n)$ depends only on n.
Proposition. Let

$$\mathrm{cap}\, F \leq \gamma \, \mathrm{cap}\, B_1, \tag{7}$$

where $\gamma \in (0,1)$. Then any constant C in (5) satisfies

$$C \geq c(n,\gamma)\big(\mathrm{cap}\, F\big)^{-1/2}. \tag{8}$$

Positivity of the Schrödinger operator with negative potential

Theorem 1. [Maz. 1,2] Let Ω be an open set in $\mathbb{R}^n$, $n \geq 1$, and let $\mathbb{V}$ be a nonnegative Radon measure in Ω. The inequality

$$\int_\Omega |u|^2 \, \mathbb{V}(dx) \leq \int_\Omega |\nabla u|^2 \, dx \tag{9}$$

holds for every $u \in C_0^\infty(\Omega)$ provided

$$\frac{\mathbb{V}(F)}{\mathrm{cap}_\Omega F} \leq \frac{1}{4} \tag{10}$$

for all compact sets $F \subset \Omega$.
A necessary condition for (9) is

$$\frac{\mathbb{V}(F)}{\mathrm{cap}_\Omega F} \leq 1, \tag{11}$$

Corollary The trace inequality

$$\int_\Omega |u|^2 \, \mathbb{V}(dx) \le C \int_\Omega |\nabla u|^2 \, dx \tag{12}$$

holds for every $u \in C_0^\infty(\Omega)$ if and only if

$$\sup_{F \subset \Omega} \frac{\mathbb{V}(F)}{\mathrm{cap}_\Omega F} < \infty.$$

The bounds $1/4$ and 1 in (10) and (11) are sharp. The gap between these sufficient and necessary conditions is the same as in Hille's non-oscillation criteria for the operator

$$-u'' - \mathbb{V}\, u, \qquad \mathbb{V} \ge 0,$$

on the positive semiaxis $\mathbb{R}^1_+$:

$$x\, \mathbb{V}((x,\infty)) \le 1/4 \qquad \text{and} \quad x\, \mathbb{V}((x,\infty)) \le 1 \tag{13}$$

for all $x \ge 0$. By the way, conditions (13) are particular cases of (10) and (11) with $n = 1$ and $\Omega = \mathbb{R}^1_+$.

Combining Theorem 1 with isocapacitary inequalities (2)–(4), we arrive at sufficient conditions for (9) whose formulations involve no capacity. For example, in the two-dimensional case, (9) is guaranteed by the inequality

$$\mathbb{V}(F) \le \frac{4\pi}{\log \dfrac{\mathrm{mes}_2\,\Omega}{\mathrm{mes}_2\,F}}.$$

The sharpness of this condition can be easily checked by analyzing the well known Hardy-type inequality

$$\int_\Omega \frac{|u(x)|^2}{|x|^2(\log|x|)^2}\,dx \le 4\int_\Omega |\nabla\, u(x)|^2\,dx,$$

where $u \in C_0^\infty(\Omega)$ and Ω is the unit disc.

Here is a dual assertion to Theorem 1 which is stated in terms of the Green function G of Ω and does not depend on the notion of capacity.

Theorem 2.

Let $\mathbb{V}_F$ be the restriction of the measure $\mathbb{V}$ to a compact set $F \subset \Omega$. Inequality (9) holds for every $u \in C_0^\infty(\Omega)$ provided

$$\int_\Omega \int_\Omega G(x,y)\, \mathbb{V}_F(dx)\, \mathbb{V}_F(dy) \le \frac{1}{4} \mathbb{V}(F) \tag{14}$$

for all F. Conversely, inequality (9) implies

$$\int_\Omega \int_\Omega G(x,y)\, \mathbb{V}_F(dx)\, \mathbb{V}_F(dy) \le \mathbb{V}(F). \tag{15}$$

Sketch of the proof. Let u be a nonnegative function in $C_0^\infty(\Omega)$ such that $u \geq 1$ on F. Then

$$\mathbb{V}(F) \leq \int_\Omega u(x)\,\mathbb{V}_F(dx) \leq \Big(\int_\Omega\int_\Omega G(x,y)\,\mathbb{V}_F(dx)\,\mathbb{V}_F(dy)\Big)^{1/2} \|\nabla\, u\|_{L_2(\Omega)}$$

which in combination with (14) gives (10). The reference to Theorem 1 gives the sufficiency of (14).
Let (9) hold. Then

$$\Big|\int_\Omega u\,\mathbb{V}_F(dx)\Big|^2 \leq \mathbb{V}(F)\,\|\nabla u\|^2_{L_2(\Omega)}.$$

Omitting a standard approximation argument, we put

$$u(x) = \int_\Omega G(x,y)\,\mathbb{V}_F(dy)$$

and the necessity of (15) results.

The next assertion follows directly from Theorem 2.

Corollary The trace inequality (12) holds if and only if there exists a constant $C > 0$ such that

$$\int_F \int_F G(x, y)\, \mathbb{V}_F(dx)\, \mathbb{V}_F(dy) \le C\, \mathbb{V}(F) \tag{16}$$

for all compact sets F in Ω.

Trace inequality for $\Omega = \mathbb{R}^n$

Inequality

$$\int_{\mathbb{R}^n} |u|^2 \, \mathbb{V}(dx) \le C \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \tag{17}$$

deserves to be discussed in more detail. First, (17) for $n = 2$ implies $\mathbb{V} = 0$. Let $n > 2$. Needless to say, by Theorem **??** the condition

$$\sup_F \frac{\mathbb{V}(F)}{\mathrm{cap} F} < \infty, \tag{18}$$

where the supremum is taken over all compact sets F in $\mathbb{R}^n$, is necessary and sufficient for (17). Restricting ourselves to arbitrary balls B in $\mathbb{R}^n$, we have by (18) the obvious necessary condition

$$\sup_B \frac{\mathbb{V}(B)}{(\mathrm{mes}_n B)^{1-2/n}} < \infty. \tag{19}$$

On the other hand, using the isocapacitary inequality (4), we obtain the sufficient condition

$$\sup_F \frac{\mathbb{V}(F)}{(\mathrm{mes}_n F)^{1-2/n}} < \infty, \tag{20}$$

where the supremum is taken over all compact sets F in $\mathbb{R}^n$. Moreover, the best value of C in (17) satisfies

$$C \le \frac{4v_n^{-2/n}}{n(n-2)} \sup_F \frac{\mathbb{V}(F)}{(\mathrm{mes}_n F)^{1-2/n}}$$

and the constant factor in front of the supremum is sharp.

Although (19) and (20) look similar, they are not equivalent in general. In other words, one cannot replace arbitrary sets F in (18) by balls. Paradoxically, the situation with the criterion (16) in the case $\Omega = \mathbb{R}^n$ is different. In fact, Kerman and Sawyer, 1986, showed that the trace inequality (17) holds if and only if for all balls B in $\mathbb{R}^n$

$$\int_B \int_B \frac{\mathbb{V}(dx)\,\mathbb{V}(dy)}{|x-y|^{n-2}} \le C\,\mathbb{V}(B). \tag{21}$$

Maz'ya and Verbitsky, 1995, gave another necessary and sufficient condition for (17):

$$\sup_x \frac{I_1 (I_1 \mathbb{V})^2(x)}{I_1\,\mathbb{V}(x)} < \infty, \tag{22}$$

where I_s is the Riesz potential of order s, i.e.

$$I_s \mathbb{V}(x) := \int_{\mathbb{R}^n} \frac{\mathbb{V}(dy)}{|x-y|^{n-s}}.$$

We observe that the multiplicative inequality

$$\int_{\mathbb{R}^n} u^2\,\mathbb{V}(dx) \le C\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{\tau}\left(\int_{\mathbb{R}^n} u^2\,dx\right)^{1-\tau}, \qquad 0 \le \tau < 1, \tag{23}$$

is equivalent to

$$\sup_B \frac{\mathbb{V}B}{(\mathrm{mes}_n B)^{1-2\tau/n}} < \infty \tag{24}$$

(V. Maz'ya, *Sobolev Spaces*).

Relative form boundedness and form compactness

Maz'ya and Verbitsky [MV1] gave necessary and sufficient conditions for the inequality

$$\left| \int_{\mathbb{R}^n} |u(x)|^2 \, V(x) \, dx \right| \le C \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx, \quad u \in C_0^\infty(\mathbb{R}^n) \tag{25}$$

to hold. Here the "indefinite weight" V may change sign, or even be a complex-valued distribution on $\mathbb{R}^n$, $n \ge 3$. (In the latter case, the left-hand side of (25) is understood as $| < Vu, u > |$, where $< V \cdot, \cdot >$ is the quadratic form associated with the corresponding multiplication operator V.) An analogous inequality for the Sobolev space $W_2^1(\mathbb{R}^n)$, $n \ge 1$ was also characterized in [1]:

$$\left| \int_{\mathbb{R}^n} |u(x)|^2 \, V(x) \, dx \right| \le C \int_{\mathbb{R}^n} \left(|\nabla u(x)|^2 + |u(x)|^2 \right) dx, \quad u \in C_0^\infty(\mathbb{R}^n). \tag{26}$$

Such inequalities are used extensively in spectral and scattering theory of the Schrödinger operator $H_V = -\Delta + V$ and its higher-order analogs, especially in questions of self-adjointness, resolvent convergence, estimates for the number of bound states, Schrödinger semigroups, etc.

It is worthwhile to observe that the usual “naïve” approach is to decompose V into its positive and negative parts: $V = V_+ - V_-$, and to apply the just mentioned results to both V_+ and V_-. However, this procedure drastically diminishes the class of admissible weights V by ignoring a possible cancellation between V_+ and V_-.

The following result obtained in [MV1] reflects a general principle which has much wider range of applications.

Theorem 3. Let V be a complex-valued distribution on $\mathbb{R}^n$, $n \geq 3$. Then (25) holds if and only if V is the divergence of a vector-field $\vec{\Gamma} : \mathbb{R}^n \to \mathbb{C}^n$ such that

$$\int_{\mathbb{R}^n} |u(x)|^2 \, |\vec{\Gamma}(x)|^2 \, dx \leq \mathrm{const} \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx, \qquad (27)$$

where the constant is independent of $u \in C_0^\infty(\mathbb{R}^n)$. The vector-field $\vec{\Gamma} \in \mathbf{L}_2(\mathbb{R}^n, loc)$ can be chosen as $\vec{\Gamma} = \nabla \Delta^{-1} V$. Equivalently, the Schrödinger operator H_V acting from $\mathring{L}_2^1(\mathbb{R}^n)$ to $L_2^{-1}(\mathbb{R}^n)$ is bounded if and only if (27) holds. Furthermore, the corresponding multiplication operator $V : \mathring{L}_2^1(\mathbb{R}^n) \to L_2^{-1}(\mathbb{R}^n)$ is compact if and only if the embedding

$$\mathring{L}_2^1(\mathbb{R}^n) \subset L_2(\mathbb{R}^n, |\vec{\Gamma}|^2 \, dx)$$

is compact.

As a corollary, one obtains a necessary condition for (25) in terms of Morrey spaces of negative order.

Corollary 1. If (25) holds, then, for every ball $B_r(x_0)$ of radius r,

$$\int_{B_r(x_0)} |\nabla \Delta^{-1} V(x)|^2 \, dx \le C \, r^{n-2},$$

where the constant does not depend on $x_0 \in \mathbb{R}^n$ and $r > 0$.

Corollary 2. In the statements of Theorem 3 and Corollary 1, one can put the scalar function $(-\Delta)^{-\frac{1}{2}} V$ in place of $\vec{\Gamma} = \nabla \Delta^{-1} V$. In particular, (27) is equivalent to the inequality:

$$\int_{\mathbb{R}^n} |u(x)|^2 \, |(-\Delta)^{-\frac{1}{2}} V(x)|^2 \, dx \le C \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx, \qquad (28)$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

To clarify the multi-dimensional characterizations for "indefinite weights" V presented above, we state an elementary analog of Theorem 3 for the Sturm-Liouville operator $H_V = -\frac{d^2}{dx^2} + V$ on the half-line.

Theorem 4. The inequality

$$\left| \int_{\mathbb{R}_+} |u(x)|^2 \, V(x) \, dx \right| \le C \int_{\mathbb{R}_+} |u'(x)|^2 \, dx, \tag{29}$$

holds for all $u \in C_0^\infty(\mathbb{R}_+)$ if and only if

$$\sup_{a>0} \; a \int_a^\infty \left| \int_x^\infty V(t) \, dt \right|^2 \, dx < \infty, \tag{30}$$

where $\Gamma(x) = \displaystyle\int_x^\infty V(t) \, dt$ is understood in terms of distributions.

Equivalently, $H_V : \mathring{L}_2^1(\mathbb{R}_+) \to L_2^{-1}(\mathbb{R}_+)$ is bounded if and only if (30) holds. Moreover, the corresponding multiplication operator V is compact if and only if

$$a \int_a^{\infty} |\Gamma(x)|^2 \, dx = o\,(1), \quad \text{where} \quad a \to 0^+ \quad \text{and} \quad a \to +\infty. \tag{31}$$

For nonnegative V, condition (30) is easily seen to be equivalent to the standard Hille condition:

$$\sup_{a>0} \; a \int_a^{\infty} V(x)\, dx < \infty. \tag{32}$$

A similar statement is true for the compactness criterion (31).

Semi-boundedness of the Schrödinger operator

In [JMV] we obtained a characterization of potentials $\sigma \in \mathcal{D}'(\Omega)$ satisfying the semi-boundedness property of the operator:
$\mathcal{H} = -\text{div}(\mathcal{A}\nabla\cdot) - \sigma$

$$\langle \sigma, h^2 \rangle \leq \int_\Omega (\mathcal{A}\nabla h) \cdot \nabla h \, dx, \text{ for all } h \in C_0^\infty(\Omega). \tag{33}$$

In the case of the Laplacian, it means

$$\langle \sigma, h^2 \rangle \leq \int_\Omega |\nabla h|^2 \, dx, \quad \text{for all } h \in C_0^\infty(\Omega). \tag{34}$$

Theorem 5. A real-valued distribution $\sigma \in \mathcal{D}'(\Omega)$ satisfies (34) if and only if there exists $\vec{\Gamma} \in L^2_{\text{loc}}(\Omega)^n$, so that

$$\sigma \leq \operatorname{div}(\vec{\Gamma}) - |\vec{\Gamma}|^2 \qquad \text{in } \mathcal{D}'(\Omega). \tag{35}$$

The inequality in (35) can not in general be strengthened to an equality.

It had been conjectured that a condition characterizing (34) was the following:

$$\sigma \le \operatorname{div}(\vec{\Phi}), \text{ where } \int_\Omega |h|^2 |\vec{\Phi}|^2 dx \le C \int_\Omega |\nabla h|^2 dx, \text{ for all } h \in C_0^\infty(\Omega), \tag{36}$$

for some $\vec{\Phi} \in L^2_{\mathrm{loc}}(\Omega)^n$ and $C > 0$.

However, for any $C > 0$, condition (36) is *not necessary* for (34) to hold, although it is obviously sufficient when $C = \frac{1}{4}$.

Proposition. Let $\Omega = \mathbb{R}^n$, $n \ge 1$. Let σ be the radial potential defined by

$$\sigma = \cos r + \frac{n-1}{r} \sin r - \sin^2 r,$$

where $r = |x|$. Then σ satisfies (34), but cannot be represented in the form (36).

Infinitesimal form boundedness

In [MV2], we characterized the class of potentials $V \in \mathcal{D}'(\mathbb{R}^n)$ which are $-\Delta$-form bounded with relative bound zero, i.e., for every $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$|\langle Vu,\, u\rangle| \le \epsilon\, ||\nabla u||^2_{L_2(\mathbb{R}^n)} + C(\epsilon)\, ||u||^2_{L_2(\mathbb{R}^n)}, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \quad (37)$$

In other words, we found necessary and sufficient conditions for the *infinitesimal form boundedness* of the potential energy operator V with respect to the kinetic energy operator $H_0 = -\Delta$ on $L_2(\mathbb{R}^n)$. Here V is an arbitrary real- or complex-valued potential (possibly a distribution). This notion appeared in relation to the KLMN theorem and has become an indispensable tool in mathematical quantum mechanics and PDE theory.

The preceding inequality ensures that, in case V is real-valued, a semi-bounded self-adjoint Schrödinger operator $H_V = H_0 + V$ can be defined on $L_2(\mathbb{R}^n)$ so that the domain of $Q[u, u]$ coincides with $W_2^1(\mathbb{R}^n)$. For complex-valued V, it follows that H_V is an m-sectorial operator on $L_2(\mathbb{R}^n)$ with $\mathrm{Dom}(H_V) \subset W_2^1(\mathbb{R}^n)$.

The characterization of (37) found in [MV2] uses only the functions $|\nabla(1-\Delta)^{-1}\, V|$ and $|(1-\Delta)^{-1}\, V|$, and is based on the representation:

$$V = \operatorname{div} \vec{\Gamma} + \gamma, \qquad \vec{\Gamma}(x) = -\nabla(1-\Delta)^{-1}\, V, \quad \gamma = (1-\Delta)^{-1}\, V. \tag{38}$$

In particular, it is shown that, necessarily, $\vec{\Gamma} \in L_2(\mathbb{R}^n, loc)^n, \quad \gamma \in L_1(\mathbb{R}^n, loc)$, and, when $n \geq 3$,

$$\lim_{\delta \to +0} \sup_{x_0 \in \mathbb{R}^n} \delta^{2-n} \int_{B_\delta(x_0)} \left(|\vec{\Gamma}(x)|^2 + |\gamma(x)| \right) dx = 0, \tag{39}$$

once (37) holds.

In the one-dimensional case, the infinitesimal form boundedness of the Sturm-Liouville operator $H_V = -d^2/dx^2 + V$ on $L_2(\mathbb{R}^1)$ is actually a consequence of the form boundedness.

Theorem 6. Let $V \in \mathcal{D}'(\mathbb{R}^1)$. Then the following statements are equivalent.

(i) V is infinitesimally form bounded with respect to $-d^2/dx^2$.

(ii) V is form bounded with respect to $-d^2/dx^2$, i.e.,

$$|\langle V\, u,\, u\rangle| \le C\, ||u||^2_{W^1_2(\mathbb{R}^1)}, \quad \forall u \in C_0^\infty(\mathbb{R}^1).$$

(iii) V can be represented in the form $V = d\Gamma/dx + \gamma$, where

$$\sup_{x\in\mathbb{R}^1} \int_x^{x+1} \left(|\Gamma(x)|^2 + |\gamma(x)|\right)\, dx < +\infty. \tag{40}$$

(iv) Condition (40) holds where

$$\Gamma(x) = \int_{\mathbb{R}^1} \mathrm{sign}\,(x-t)\, e^{-|x-t|}\, V(t)\, dt, \qquad \gamma(x) = \int_{\mathbb{R}^1} e^{-|x-t|}\, V(t)\, dt$$

are understood in the distributional sense.

Trudinger's subordination for the Schrödinger operator

In [MV2] inequality (37) is studied also under the assumption that $C(\epsilon)$ has power growth, i.e., there exists $\epsilon_0 > 0$ such that

$$|\langle Vu,\, u\rangle| \le \epsilon\, ||\nabla u||^2_{L_2(\mathbb{R}^n)} + c\, \epsilon^{-\beta}\, ||u||^2_{L_2(\mathbb{R}^n)}, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{41}$$

for every $\epsilon \in (0, \epsilon_0)$, where $\beta > 0$. Such inequalities appear in studies of elliptic PDE with measurable coefficients, and have been used extensively in spectral theory of the Schrödinger operator.

As it turns out, it is still possible to characterize (41) using only $|\vec{\Gamma}|$ and $|\gamma|$ defined by (38), provided $\beta > 1$. It is shown in [MV2] that in this case (41) holds if and only if both of the following conditions hold:

$$\sup_{\substack{x_0 \in \mathbb{R}^n \\ 0<\delta<\delta_0}} \delta^{2\frac{\beta-1}{\beta+1}-n} \int_{B_\delta(x_0)} |\vec{\Gamma}(x)|^2 \, dx < +\infty, \tag{42}$$

$$\sup_{\substack{x_0 \in \mathbb{R}^n \\ 0<\delta<\delta_0}} \delta^{\frac{2\beta}{\beta+1}-n} \int_{B_\delta(x_0)} |\gamma(x)| \, dx < +\infty, \tag{43}$$

for some $\delta_0 > 0$. However, in the case $\beta \leq 1$ this is no longer true. For $\beta = 1$, (42) has to be replaced with the condition that $\vec{\Gamma}$ is in the local BMO space, or respectively is Hölder-continuous of order $(1-\beta)/(1+\beta)$ if $0 < \beta < 1$.