Fractional Laplacians

Roberta Musina, University of Udine, Italy

in collaboration with **Alexander I. Nazarov**, St.Petersburg Dep. of Steklov Institute

Advances in Nonlinear PDE's, St. Petersburg, September 2014

dedicated to Prof. Nina Uraltseva

The Laplace operator

- For $u \in H^1(\mathbb{R}^n)$,

$$-\Delta u = -\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

- Fourier transform: $\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)$. Then

$$\mathcal{F}\big[-\Delta u\big] = |\xi|^2 \mathcal{F}[u]$$

- For $\Omega \subset \mathbb{R}^n$, bounded, smooth, and $u \in H_0^1(\Omega)$,

$$-\Delta u = \sum_j \lambda_j \left(\int_\Omega u\varphi_j \right) \varphi_j$$

where let λ_j, φ_j= eigenvalues/eigenvectors of $-\Delta$ on $H_0^1(\Omega)$.

The Dirichlet Laplacian of order $m > 0$

For $m > 0$, any positive real number, we introduce:

• the Dirichlet Laplacian of order m,

$$\mathcal{F}\left[\,(-\Delta)^m_D u\right] = |\xi|^{2m}\mathcal{F}[u]$$

The Dirichlet Laplacian of order $m > 0$

For $m > 0$, any positive real number, we introduce:

• the Dirichlet Laplacian of order m,

$$\mathcal{F}\left[\,(-\Delta)^m_D u\right] = |\xi|^{2m}\mathcal{F}[u]$$

• the Sobolev space $H^m(\mathbb{R}^n)$ = *closure of* $C^\infty_0(\mathbb{R}^n)$ *w.r. to the norm*

$$\|u\|^2_{H^m} = \int_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}}_D u|^2 + \int_{\mathbb{R}^n} |u|^2$$

The Dirichlet Laplacian of order $m > 0$

For $m > 0$, any positive real number, we introduce:

- the Dirichlet Laplacian of order m, $\mathcal{F}\big[\,(-\Delta)^m_D u\big] = |\xi|^{2m}\mathcal{F}[u]$

- the Sobolev space $H^m(\mathbb{R}^n)$ = *closure of* $C^\infty_0(\mathbb{R}^n)$ *w.r. to the norm*

$$\|u\|^2_{H^m} = \int_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}}_D u|^2 + \int_{\mathbb{R}^n} |u|^2$$

- for $\Omega \subset \mathbb{R}^n$ bounded, smooth,

$$\widetilde{H}^m(\Omega) = \left\{u \in H^m(\mathbb{R}^n) \mid u \equiv 0 \ \text{ on } \mathbb{R}^n \setminus \overline{\Omega}\ \right\}$$

- the norm $\|u\|^2_{\widetilde{H}^m} = \int_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}}_D u|^2$ on $\widetilde{H}^m(\Omega)$

The Navier Laplacian on $\Omega \subset \mathbb{R}^n$

$$(-\Delta)_N^m := m^{\text{th}} \text{ power of } -\Delta \text{ in the sense of Spectral Theory}$$

We introduce:

- $$(-\Delta)_N^m u := \sum_j \lambda_j^m \left(\int_\Omega u \varphi_j \right) \varphi_j$$

The Navier Laplacian on $\Omega \subset \mathbb{R}^n$

$(-\Delta)^m_N :=$ m^{th} power of $-\Delta$ in the sense of Spectral Theory

We introduce:

- $$(-\Delta)^m_N u := \sum_j \lambda_j^m \left(\int_\Omega u\varphi_j \right) \varphi_j$$
- $$H^m_N(\Omega) = \{ u \in L^2(\Omega) \mid \int_\Omega |(-\Delta)^{\frac{m}{2}}_N u|^2 < \infty \}$$
- the norm $$\|u\|^2_{H^m_N(\Omega)} = \int_\Omega |(-\Delta)^{\frac{m}{2}}_N u|^2$$ on $H^m_N(\Omega)$.

The spaces $\widetilde{H}^m(\Omega)$, $H^m_N(\Omega)$

The spaces $\widetilde{H}^m(\Omega)$, $H^m_N(\Omega)$

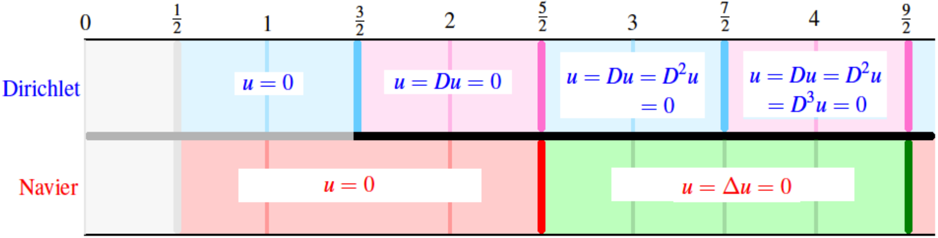
Proposition (Triebel)

$$\widetilde{H}^m(\Omega) = H^m_N(\Omega) \quad \text{if} \quad 0 < m < \tfrac{3}{2}$$

$$\widetilde{H}^m(\Omega) \subsetneq H^m_N(\Omega) \quad \text{if} \quad m \ge \tfrac{3}{2}$$

Examples: $\widetilde{H}^1(\Omega) = H^1_0(\Omega) = H^1_N(\Omega)$

$\widetilde{H}^2(\Omega) = H^2_0(\Omega)\ , \quad H^2_N(\Omega) = H^1_0(\Omega) \cap H^2(\Omega)$



Comparing the Dirichlet and the Navier norms on the "smallest" space $\widetilde{H}^m(\Omega)$

Trivial: $\displaystyle \int_\Omega |\,(-\Delta)_N^{\frac{m}{2}}u|^2 = \int_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{m}{2}}u|^2$ on $\widetilde{H}^m(\Omega)$, if $m \in \mathbb{N}$

Comparing the Dirichlet and the Navier norms on the "smallest" space $\widetilde{H}^m(\Omega)$

Trivial: $\displaystyle\int_\Omega |(-\Delta)_N^{\frac{m}{2}} u|^2 = \int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2$ on $\widetilde{H}^m(\Omega)$, if $m \in \mathbb{N}$

For NOT integer orders they are DIFFERENT! Example:
take $u \in C_0^\infty(\Omega)$, assume $0 \in \Omega$, put

$$\boxed{u_h(x) := h^{\frac{n-2m}{2}} u(hx)}, \quad u_h \in \widetilde{H}^m(\Omega), \quad (h \to \infty)$$

Comparing the Dirichlet and the Navier norms on the "smallest" space $\widetilde{H}^m(\Omega)$

Trivial: $\displaystyle\int_\Omega |(-\Delta)_N^{\frac{m}{2}}u|^2 = \int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}}u|^2$ on $\widetilde{H}^m(\Omega)$, if $m \in \mathbb{N}$

For NOT integer orders they are DIFFERENT! Example:
take $u \in C_0^\infty(\Omega)$, assume $0 \in \Omega$, put

$$\boxed{u_h(x) := h^{\frac{n-2m}{2}}u(hx)}, \quad u_h \in \widetilde{H}^m(\Omega), \quad (h \to \infty)$$

Then $$\boxed{\int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}}u_h|^2 = \int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}}u|^2}$$

Comparing the Dirichlet and the Navier norms on the "smallest" space $\widetilde{H}^m(\Omega)$

Trivial: $\int_\Omega |(-\Delta)_N^{\frac{m}{2}} u|^2 = \int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2$ on $\widetilde{H}^m(\Omega)$, if $m \in \mathbb{N}$

For NOT integer orders they are DIFFERENT! Example:
take $u \in C_0^\infty(\Omega)$, assume $0 \in \Omega$, put

$$\boxed{u_h(x) := h^{\frac{n-2m}{2}} u(hx)}, \quad u_h \in \widetilde{H}^m(\Omega), \quad (h \to \infty)$$

Then $$\boxed{\int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u_h|^2 \; = \; \int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2}$$

but $$\boxed{\int_\Omega |(-\Delta)_N^{\frac{m}{2}} u_h|^2 \; \boxed{\neq} \; \int_\Omega |(-\Delta)_N^{\frac{m}{2}} u|^2}$$

Comparing the Dirichlet and the Navier norms on the "smallest" space $\widetilde{H}^m(\Omega)$

Theorem (RM-A.I. Nazarov, CPDE 2014 and preprint arXiv 2014)

Let $m \notin \mathbb{N}_0$ $k \in \mathbb{N}_0$, $u \in \widetilde{H}^m(\Omega)$, $u \neq 0$.

- $$\int_\Omega |(-\Delta)^{\frac{m}{2}}_N u|^2 > \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}}_D u|^2 \quad \textit{if } 2k < m < 2k+1$$
- $$\int_\Omega |(-\Delta)^{\frac{m}{2}}_N u|^2 < \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}}_D u|^2 \quad \textit{if } 2k+1 < m < 2k$$

Proof.

Step 1: $0 < m < 1 \quad \Rightarrow \int_\Omega |(-\Delta)^{\frac{m}{2}}_N u|^2 > \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}}_D u|^2$

the argument is based on the "extension argument"

$$u = u(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R} \quad \rightsquigarrow \quad w = w(x,y) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$$

[Caffarelli-Silvestre, CPDE 2007] **[Stinga-Torrea, CPDE 2010]**

Proof.

Step 1: $0 < m < 1 \quad \Rightarrow \int_{\Omega} |\,(-\Delta)_N^{\frac{m}{2}} u|^2 > \int_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{m}{2}} u|^2$

the argument is based on the "extension argument"

$$u = u(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R} \quad \rightsquigarrow \quad w = w(x,y) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$$

[Caffarelli-Silvestre, CPDE 2007] [Stinga-Torrea, CPDE 2010]

Consider the energy $\mathcal{E}(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2m} |\nabla w|^2$ for functions w in

$$\mathcal{W}^D(u) = \left\{ w \mid \mathcal{E}(w) < \infty \,,\; w(\cdot, 0) = u \right\}$$

Proof.

Step 1: $$0 < m < 1 \quad \Rightarrow \int_\Omega |(-\Delta)^{\frac{m}{2}}_N u|^2 > \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}}_D u|^2$$

the argument is based on the "extension argument"

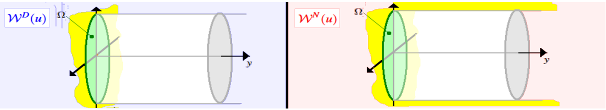
$$u = u(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R} \quad \rightsquigarrow \quad w = w(x,y) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$$

[Caffarelli-Silvestre, CPDE 2007] **[Stinga-Torrea, CPDE 2010]**

Consider the energy $\mathcal{E}(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2m} |\nabla w|^2$ for functions w in

$$\mathcal{W}^D(u) = \left\{ w \mid \mathcal{E}(w) < \infty \, , \, w(\cdot, 0) = u \right\}$$ or in the smaller

$$\mathcal{W}^N(u) = \left\{ w \in \mathcal{W}^D(u) \mid w(x, \cdot) \equiv 0 \ \text{ if } x \notin \overline{\Omega} \right\}$$



Solve the minimization problems

$$w_u^D \mapsto \inf_{\mathcal{W}^D(u)} \mathcal{E} \qquad\qquad w_u^N \mapsto \inf_{\mathcal{W}^N(u)} \mathcal{E}$$

Lemma (Caffarelli-Silvestre; Stinga-Torrea)

$$\int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2 = c_m \mathcal{E}(w_u^D) \qquad \int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 = c_m \mathcal{E}(w_u^N)$$

Solve the minimization problems

$$w_u^D \mapsto \inf_{\mathcal{W}^D(u)} \mathcal{E} \qquad\qquad w_u^N \mapsto \inf_{\mathcal{W}^N(u)} \mathcal{E}$$

Lemma (Caffarelli-Silvestre; Stinga-Torrea)

$$\int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2 = c_m \mathcal{E}(w_u^D) \qquad \int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 = c_m \mathcal{E}(w_u^N)$$

But $\mathcal{W}^N(u) \subset \mathcal{W}^D(u)$, hence, trivially,

$$\mathcal{E}(w_u^N) \geq \mathcal{E}(w_u^D) \;\Rightarrow\; \int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 \geq \int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2$$

Solve the minimization problems

$$w_u^D \mapsto \inf_{\mathcal{W}^D(u)} \mathcal{E} \qquad\qquad w_u^N \mapsto \inf_{\mathcal{W}^N(u)} \mathcal{E}$$

Lemma (Caffarelli-Silvestre; Stinga-Torrea)

$$\int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2 = c_m \mathcal{E}(w_u^D) \qquad \int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 = c_m \mathcal{E}(w_u^N)$$

But $\mathcal{W}^N(u) \subset \mathcal{W}^D(u)$, hence, trivially,

$$\mathcal{E}(w_u^N) \geq \mathcal{E}(w_u^D) \;\; \Rightarrow \;\; \int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 \geq \int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2$$

If equality holds, then $w_u^D = w_u^N =: w$ solves:

$$\begin{cases} -\mathrm{div}\left(y^{1-2m} \nabla w\right) = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ w(\cdot, 0) = u \;\text{ on } \mathbb{R}^n, \quad w \equiv 0 & \text{on } (\mathbb{R}^n \setminus \overline{\Omega}) \times \mathbb{R}_+ \end{cases}$$

Solve the minimization problems

$$w_u^D \mapsto \inf_{\mathcal{W}^D(u)} \mathcal{E} \qquad\qquad w_u^N \mapsto \inf_{\mathcal{W}^N(u)} \mathcal{E}$$

Lemma (Caffarelli-Silvestre; Stinga-Torrea)

$$\int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2 = c_m \mathcal{E}(w_u^D) \qquad \int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 = c_m \mathcal{E}(w_u^N)$$

But $\mathcal{W}^N(u) \subset \mathcal{W}^D(u)$, hence, trivially,

$$\mathcal{E}(w_u^N) \ge \mathcal{E}(w_u^D) \;\Rightarrow\; \int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 \ge \int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2$$

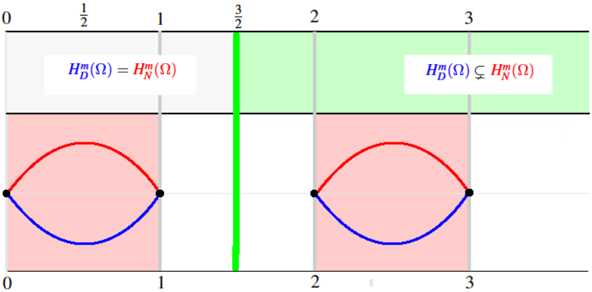
If equality holds, then $w_u^D = w_u^N =: w$ solves:

$$\begin{cases} -\mathrm{div}\big(y^{1-2m} \nabla w\big) = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ w(\cdot, 0) = u \;\text{ on } \mathbb{R}^n, \quad w \equiv 0 & \text{on } (\mathbb{R}^n \setminus \overline{\Omega}) \times \mathbb{R}_+ \end{cases}$$

Thus, $w \equiv 0$ i.e. $u \equiv 0$.

Step 2: m between an even and an odd number

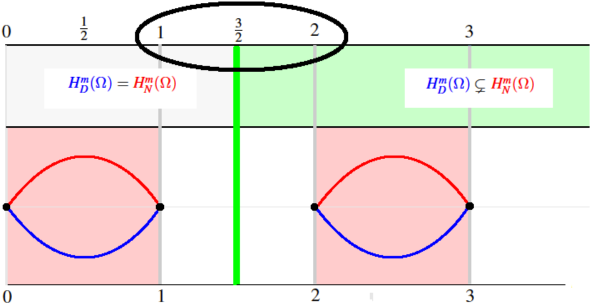
Step 2: m between an even and an odd number



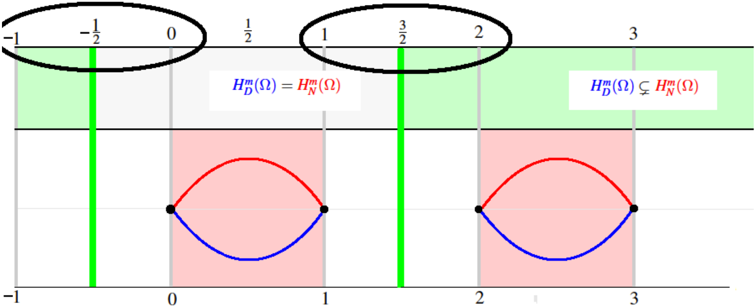
for instance, if $2 < m < 3$,

$$\int_\Omega |(-\Delta)_N^{\frac{m}{2}} u|^2 = \int_\Omega |(-\Delta)_N^{\frac{m-2}{2}}(-\Delta u)|^2 \boxed{\geq} \int_\Omega |(-\Delta)_N^{\frac{m-2}{2}}(-\Delta u)|^2 = \int_\Omega |(-\Delta)_D^{\frac{m}{2}} u|^2$$

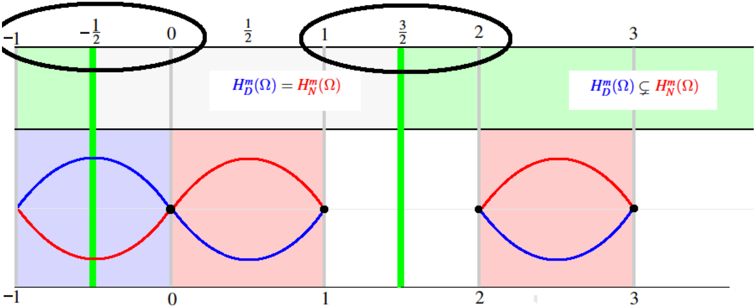
Step 3: m between an odd and an even number



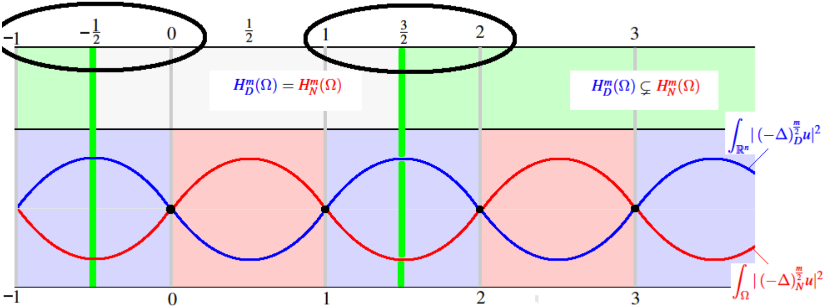
Step 3: m between an odd and an even number



Step 3: m between an odd and an even number



Step 3: m between an odd and an even number



Fracional Laplacians, of negative orders $\boxed{-1 < m < 0}$
the DUAL extension argument

Fix $u \in \widetilde{H}^m(\Omega) \hookrightarrow H^m(\mathbb{R}^m) \equiv \big(H^{-m}(\mathbb{R}^n)\big)'$

Fracional Laplacians, of negative orders $\boxed{-1 < m < 0}$
the DUAL extension argument

Fix $u \in \widetilde{H}^m(\Omega) \hookrightarrow H^m(\mathbb{R}^m) \equiv \left(H^{-m}(\mathbb{R}^n)\right)'$; consider the energy

$$\widetilde{\mathcal{E}}(w) = \underbrace{\int_0^\infty \int_{\mathbb{R}^n} y^{1-2m} |\nabla w|^2}_{\mathcal{E}(w)} - 2 \left\langle u, w\big|_{y=0} \right\rangle,$$

on the sets of functions $w = w(x,y) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$,

Fracional Laplacians, of negative orders $\boxed{-1 < m < 0}$

the DUAL extension argument

Fix $u \in \widetilde{H}^m(\Omega) \hookrightarrow H^m(\mathbb{R}^m) \equiv \big(H^{-m}(\mathbb{R}^n)\big)'$; consider the energy

$$\widetilde{\mathcal{E}}(w) = \underbrace{\int_0^\infty \int_{\mathbb{R}^n} y^{1-2m} |\nabla w|^2}_{\mathcal{E}(w)} - 2\left\langle u, w\big|_{y=0} \right\rangle,$$

on the sets of functions $w = w(x,y) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$,in

$$\widetilde{\mathcal{W}}^D(u) \subset \left\{ w \mid \mathcal{E}(w) < \infty \right\} \quad \text{suitably defined, or in}$$

$$\widetilde{\mathcal{W}}^N(u) = \left\{ w \in \widetilde{\mathcal{W}}^D \mid w(x,\cdot) \equiv 0 \ \text{ if } x \notin \overline{\Omega} \right\}$$

Solve the minimization problems

$$\widetilde{w}_u^D \mapsto \inf_{\widetilde{\mathcal{W}}^D(u)} \widetilde{\mathcal{E}} \qquad \widetilde{w}_u^N \mapsto \inf_{\widetilde{\mathcal{W}}^N(u)} \widetilde{\mathcal{E}}$$

Lemma

$$\int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2 = -\tilde{c}_m \widetilde{\mathcal{E}}(\widetilde{w}_u^D) \qquad \int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 = -\tilde{c}_m \widetilde{\mathcal{E}}(\widetilde{w}_u^N)$$

Solve the minimization problems

$$\widetilde{w}_u^D \mapsto \inf_{\widetilde{\mathcal{W}}^D(u)} \widetilde{\mathcal{E}} \qquad \widetilde{w}_u^N \mapsto \inf_{\widetilde{\mathcal{W}}^N(u)} \widetilde{\mathcal{E}}$$

Lemma

$$\int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2 = -\tilde{c}_m \widetilde{\mathcal{E}}(\widetilde{w}_u^D) \qquad \int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 = -\tilde{c}_m \widetilde{\mathcal{E}}(\widetilde{w}_u^N)$$

But $\widetilde{\mathcal{W}}^N(u) \subset \widetilde{\mathcal{W}}^D(u)$, hence, trivially,

$$0 \geq \widetilde{\mathcal{E}}(\widetilde{w}_u^N) \geq \widetilde{\mathcal{E}}(\widetilde{w}_u^D) \ \Rightarrow \ \int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2 \leq \int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2$$

...

□

Distributional inequalities

Theorem ([RM-A.I. Nazarov, CPDE 2014])

Assume $\boxed{0 < m < 1}$. *Let* $u \in \widetilde{H}^m(\Omega)$, $u \neq 0$.

$$\textit{If } u \geq 0 \textit{ then} \quad (-\Delta)^m_N u > (-\Delta)^m_D u \quad \textit{in } \mathcal{D}'(\Omega)$$

Extension to $m > -1$ in [M-Nazarov, preprint arXiv 2014]

Distributional inequalities

Theorem ([RM-A.I. Nazarov, CPDE 2014])

Assume $\boxed{0 < m < 1}$. *Let* $u \in \widetilde{H}^m(\Omega)$, $u \neq 0$.

$$\textit{If } u \geq 0 \textit{ then } \quad (-\Delta)^m_N u > (-\Delta)^m_D u \quad \textit{in } \mathcal{D}'(\Omega)$$

Extension to $m > -1$ in [M-Nazarov, preprint arXiv 2014]

Main ingredients in the proof.

1) Formula relating the Laplacians of a smooth u with the *"weighted normal derivatives"* of its extensions, [Caffarelli-Silvestre], [Stinga Torrea]

2) a fine *"boundary point lemma"* for singular elliptic equations by

Alvarado, Brigham, Maz'ya, Mitrea and Ziadé

[Probl. Mat. Anal. 2011] (Russian); [J. Math. Sci. 2011] (English tr.)

□

"How much" the Dirichlet and Navier norms differ?

Assume $0 \in \Omega$, fix $u \in \widetilde{H}^m(\Omega)$; define

$$\boxed{u_h(x) := h^{\frac{n-2m}{2}} u(hx)}\,, \quad u_h \in \widetilde{H}^m(\Omega), \quad (h \to \infty)$$

"How much" the Dirichlet and Navier norms differ?

Assume $0 \in \Omega$, fix $u \in \widetilde{H}^m(\Omega)$; define

$$\boxed{u_h(x) := h^{\frac{n-2m}{2}} u(hx)}\,, \quad u_h \in \widetilde{H}^m(\Omega), \quad (h \to \infty)$$

Then $$\boxed{\int_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{m}{2}} u_h|^2 \;=\; \int_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{m}{2}} u|^2}$$

but $$\boxed{\int_{\Omega} |\,(-\Delta)_N^{\frac{m}{2}} u_h|^2 \;\boxed{\neq}\; \int_{\Omega} |\,(-\Delta)_N^{\frac{m}{2}} u|^2} \qquad (m \notin \mathbb{N}_0)$$

"How much" the Dirichlet and Navier norms differ?

Assume $0 \in \Omega$, fix $u \in \widetilde{H}^m(\Omega)$; define

$$\boxed{u_h(x) := h^{\frac{n-2m}{2}} u(hx)}\,, \quad u_h \in \widetilde{H}^m(\Omega), \quad (h \to \infty)$$

Then $$\boxed{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}}_D u_h|^2 = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}}_D u|^2}$$

but $$\boxed{\int_{\Omega} |(-\Delta)^{\frac{m}{2}}_N u_h|^2 \boxed{\neq} \int_{\Omega} |(-\Delta)^{\frac{m}{2}}_N u|^2} \qquad (m \notin \mathbb{N}_0)$$

Theorem (RM-A.I. Nazarov, CPDE 2014 and preprint arXiv 2014)

$$\boxed{\int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}}_D u|^2 = \lim_{h\to\infty} \int_{\Omega} |(-\Delta)^{\frac{m}{2}}_N u_h|^2}$$

$n > 2m$: Sobolev and Hardy constants

The Sobolev and the Hardy constants are defined by:

$$\mathcal{S}_m := \inf_{u \in H^m(\mathbb{R}^n)} \frac{\displaystyle\int_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{m}{2}} u|^2}{\|u\|^2_{\frac{2n}{n-2m}}}\,, \quad \mathcal{H}_m := \inf_{u \in H^m_D(\mathbb{R}^n)} \frac{\displaystyle\int_{\mathbb{R}^n} |\,(-\Delta)_D^{\frac{m}{2}} u|^2}{\||x|^{-m} u\|_2^2}$$

We introduce the Navier-Sobolev and the Navier-Hardy constants by

$$\mathcal{S}_m^N(\Omega) := \inf_{u \in H^m_N(\Omega)} \frac{\displaystyle\int_{\Omega} |\,(-\Delta)_N^{\frac{m}{2}} u|^2}{\|u\|^2_{\frac{2n}{n-2m}}} \qquad \mathcal{H}_m^N(\Omega) := \inf_{u \in H^m_N(\Omega)} \frac{\displaystyle\int_{\Omega} |\,(-\Delta)_N^{\frac{m}{2}} u|^2}{\||x|^{-m} u\|_2^2}$$

$n > 2m$: Sobolev and Hardy constants

The Sobolev and the Hardy constants are defined by:

$$\mathcal{S}_m := \inf_{u \in H^m(\mathbb{R}^n)} \frac{\displaystyle\int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2}{\|u\|^2_{\frac{2n}{n-2m}}}, \quad \mathcal{H}_m := \inf_{u \in H^m_D(\mathbb{R}^n)} \frac{\displaystyle\int_{\mathbb{R}^n} |(-\Delta)_D^{\frac{m}{2}} u|^2}{\||x|^{-m} u\|_2^2}$$

We introduce the Navier-Sobolev and the Navier-Hardy constants by

$$\mathcal{S}_m^N(\Omega) := \inf_{u \in H^m_N(\Omega)} \frac{\displaystyle\int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2}{\|u\|^2_{\frac{2n}{n-2m}}} \qquad \mathcal{H}_m^N(\Omega) := \inf_{u \in H^m_N(\Omega)} \frac{\displaystyle\int_{\Omega} |(-\Delta)_N^{\frac{m}{2}} u|^2}{\||x|^{-m} u\|_2^2}$$

Theorem (RM-A.I. Nazarov, CPDE 2014 and II preprint arXiv 2014)

$$\mathcal{S}_m^N(\Omega) = \mathcal{S}_m\,, \qquad \mathcal{H}_m^N(\Omega) = \mathcal{H}_m$$