Dedicated to Nina Nikolaevna Uraltseva

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On the Cauchy problem for scalar balance laws in the class of Besicovitch almost periodic functions

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In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$ with $\mathbb{R}_+ = (0, +\infty)$, we consider the Cauchy problem

$$u_t + \mathrm{div}_x \varphi(u) = g(t, x, u), \qquad (1)$$

$$u(0, x) = u_0(x) \in L^\infty(\mathbb{R}^n). \qquad (2)$$

We suppose that the flux $\varphi(u) = (\varphi_1(u), \dots, \varphi_n(u)) \in C(\mathbb{R}, \mathbb{R}^n)$, and the source function $g(t, x, u) \in L^1_{loc}(\Pi, C(\mathbb{R}))$ is a Caratheodory function with the following properties:

$$|g(t, x, u)| \le a(t)(1 + |u|) \quad \forall u \in \mathbb{R},$$

$$|g(t, x, u) - g(t, x, v)| \le a(t)|u - v| \quad \forall u, v \in \mathbb{R},$$

where $a(t) \in L^1(\mathbb{R}_+)$.

Definition 1. (S. N. Kruzhkov)

A bounded measurable function $u = u(t,x) \in L^\infty(\Pi)$ is called an e.s. of (1), (2) if for all $k \in \mathbb{R}$

$$|u-k|_t + \operatorname{div}_x[\operatorname{sign}(u-k)(\varphi(u)-\varphi(k))] - \operatorname{sign}(u-k)g(t,x,u) \le 0 \qquad (3)$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$), and

$$\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0 \ \text{ in } L^1_{loc}(\mathbb{R}^n).$$

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Condition (3) means that $\forall f = f(t,x) \in C_0^1(\Pi), f \ge 0$

$$\int_\Pi [|u-k|f_t + \mathrm{sign}(u-k)(\varphi(u)-\varphi(k)) \cdot \nabla_x f + \mathrm{sign}(u-k)g(t,x,u)f]dtdx \ge 0$$

Theorem 1.

There exists an e.s. $u = u(t,x)$ of problem (1), (2). Moreover,

$$|u(t,x)| \le M \doteq C(1 + \|u_0\|_\infty), \qquad \text{where } C = e^{\|a\|_1}.$$

The statement of Theorem 1 readily follows from results of

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In the case of merely continuous flux vector and $n > 1$, an e.s. of (1), (2) may be nonunique, see

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Nevertheless, if the initial and source functions are space-periodic (at least in $n-1$ independent directions), an e.s. of (1), (2) is unique and space-periodic, cf [1]. We denote

$$C_R = \{\, x=(x_1,\dots,x_n)\in\mathbb{R}^n \mid |x|_\infty = \max_{i=1,\dots,n}|x_i| \le R/2 \,\},\quad R>0;$$

$$N_1(u) = \limsup_{R\to+\infty} R^{-n}\int_{C_R}|u(x)|dx,\quad u(x)\in L^1_{loc}(\mathbb{R}^n).$$

Recall that the Besicovitch space $\mathcal{B}^1(\mathbb{R}^n)$ is the closure of trigonometric polynomials (i.e., finite sums $\sum a_\lambda e^{2\pi i\lambda\cdot x}$, where $i^2=-1$, $\lambda\in\mathbb{R}^n$) in the quotient space $B^1(\mathbb{R}^n)/B^1_0(\mathbb{R}^n)$, where

$$B^1(\mathbb{R}^n) = \{u\in L^1_{loc}(\mathbb{R}^n) \mid N_1(u)<+\infty\},\ B^1_0(\mathbb{R}^n) = N_1^{-1}(0).$$

The space $\mathcal{B}^1(\mathbb{R}^n)$ equipped with the norm $\|u\|_1 = N_1(u)$ is a Banach space.

It is known that every function $u \in \mathcal{B}^1(\mathbb{R}^n)$ has the mean value

$$\fint_{\mathbb{R}^n} u(x)dx \doteq \lim_{R\to+\infty} R^{-n} \int_{C_R} u(x)dx$$

and, more generally, the Fourier coefficients

$$a_\lambda = \fint_{\mathbb{R}^n} u(x)e^{-2\pi i\lambda\cdot x}dx, \quad \lambda \in \mathbb{R}^n.$$

The set $Sp(u) = \{ \lambda \in \mathbb{R}^n \mid a_\lambda \neq 0 \}$ is called the spectrum of an almost periodic function $u(x)$. The spectrum $Sp(u)$ is known to be at most countable. We denote by $M(u)$ the additive subgroup of $\mathbb{R}^n$, generated by $Sp(u)$.

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Let $F = L^1(\mathbb{R}_+, C(\mathbb{R}))$ be a locally convex space with the topology generated by seminorms

$$\|p\|_{1,M} = \int_0^{+\infty} \max_{|u|\le M} |p(t,u)|dt, \quad p = p(t,u) \in F, \ M > 0.$$

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The following statement holds in the general case of arbitrary e.s.

Theorem 2.

Let $u(t,x), v(t,x) \in L^\infty(\Pi)$ be e.s. of (1), (2) with initial data $u_0(x), v_0(x)$, and source functions $g(t,x,u), h(t,x,u)$, respectively. Then for a.e. $t > 0$

$$N_1(u(t,\cdot) - v(t,\cdot)) \le C\left[N_1(u_0 - v_0) + N_1(\|g(\cdot,x,\cdot) - h(\cdot,x,\cdot)\|_{1,M})\right], \qquad (4)$$

where $C = e^{\|a\|_1}$, $M = \|u\|_\infty$.

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To prove Theorem 2, we utilize the relation

$$|u-v|_t + \mathrm{div}_x[\mathrm{sign}(u-v)(\varphi(u) - \varphi(v))] \le |g(t,x,u) - h(t,x,v)| \le \max_{|u|\le M} |g(t,x,u) - h(t,x,u)| + a(t)|u-v| \ \text{ in } \mathcal{D}'(\Pi), \tag{5}$$

established by Kruzhkov doubling of variables method. Integrating (5) over the parallelepiped $t \in (0,T)$, $x \in C_R$, and passing to the limit as $R \to \infty$, we obtain with the help of Gronwall lemma the required relation: for a.e. $T > 0$
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Theorem 2 readily implies **uniqueness** of e.s. $u(t,x)$ to the problem (1), (2), considered in the space $\mathcal{B}^1(\mathbb{R}^n)$ (for every fixed $t > 0$).

To establish the **existence**, we have to assume that the F-valued function $\tilde{g}(x)(t,u) = g(t,x,u)$ belongs to the space $\mathcal{B}^1(\mathbb{R}^n, F)$. Denote by M_0 the additive subgroup of $\mathbb{R}^n$ generated by $Sp(u_0) \cup Sp(\tilde{g})$.

Theorem 3.

Let $u_0(x) \in \mathcal{B}^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a bounded almost periodic function, and $u(t,x)$ be an e.s. of problem (1), (2). Then, after possible correction on a set of null measure, $u(t,\cdot) \in C([0,+\infty), \mathcal{B}^1(\mathbb{R}^n)) \cap L^\infty(\Pi)$, and for all $t > 0$ $M(u(t,\cdot)) \subset M_0$.

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Theorem 4.

Assume that

$$\forall \xi \in M_0, \xi \neq 0 \text{ functions } u \to \xi \cdot \varphi(u) \\ \text{are not affine on non-empty intervals} \tag{6}$$

Then for some constant C

$$\lim_{t \to +\infty} \fint_{\mathbb{R}^n} |u(t,x) - C| dx = 0. \tag{7}$$

If the input data $u_0(x)$, $g(t,x,u)$ are space-periodic with the common lattice of periods $L \subset \mathbb{R}^n$, then the group M_0 coincides with the dual lattice $L' = \{ \xi \in \mathbb{R}^n \mid \xi \cdot x \in \mathbb{Z} \ \forall x \in L \}$. In this case the statement of Theorem 4 reduces to the decay result for periodic e.s. recently established (in the case $g \equiv 0$) in

3. Panov E. Yu. Annales de l'Institut Henri Poincare (C) Analyse Non Lineaire, **30** 997–1007 (2013).

The non-degeneracy condition

$$\forall \xi \in L', \xi \neq 0 \ \text{ the function } u \to \xi \cdot \varphi(u) \\ \text{is not affine on non-empty intervals} \tag{8}$$

is necessary and sufficient for the decay of every x-periodic (with the lattice of periods L) e.s. $u(t,x)$:

$$\lim_{t \to +\infty} \int_{\mathbb{T}^n} |u(t,x) - C| dx = 0, \quad C = \int_{\mathbb{T}^n} u_0(x) dx. \tag{9}$$

Here $\mathbb{T}^n = \mathbb{R}^n / L$ ia an n-dimensional torus, dx is a normalized Lebesgue measure on $\mathbb{T}^n$.

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$$N_1(u(t,\cdot)-u_m(t,\cdot))\le C[N_1(u_0-u_{0m})+N_1(\|\tilde{g}-\tilde{g}_m\|_{1,M})]\underset{m\to\infty}{\to}0,\ M=\|u\|_\infty.$$

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Thus, we suppose that $u_0(x)=\sum_{\lambda\in\Lambda}a_\lambda e^{2\pi i\lambda\cdot x}$, $g(t,x,u)=\sum_{\lambda\in\Lambda}b_\lambda(t,u)e^{2\pi i\lambda\cdot x}$ are trigonometric polynomials, the set Λ is finite. The group $M_0=M(u_0)$ is a free abelian group (as a finite generated torsion free group). Therefore, we can choose a basis $\lambda_j\in M_0$, $j=1,\dots,m$. Each element $\lambda\in M_0$ is uniquely represented as

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$$u_0(x) = \sum_{\bar{k}\in J} a_{\bar{k}} e^{2\pi i \sum_{j=1}^m k_j\lambda_j\cdot x}, \quad g(t,x,u) = \sum_{\bar{k}\in J} b_{\bar{k}}(t,u) e^{2\pi i \sum_{j=1}^m k_j\lambda_j\cdot x},$$

where $a_{\bar{k}} \doteq a_{\lambda(\bar{k})}$, $b_{\bar{k}}(t,u) \doteq b_{\lambda(\bar{k})}(t,u)$, which implies that $u_0(x) = v_0(y(x))$, $g(t,x,u) = h(t,y(x),u)$, where

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are space-periodic function with the standard lattice of periods $\mathbb{Z}^m$ while $y(x)$ is a linear map from $\mathbb{R}^n$ into $\mathbb{R}^m$ defined by the equalities $y_j = \lambda_j \cdot x$, $j = 1, \dots, m$.

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$$v_t + \operatorname{div}_y \tilde{\varphi}(v) = h(t,y,v), \quad v = v(t,y),\ t > 0,\ y \in \mathbb{R}^m, \tag{10}$$

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with the flux functions $\tilde{\varphi}_j(v) = \lambda_j \cdot \varphi(v)$, $j = 1, \dots, m$. Let $v(t,y) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^m)$ be the unique (in view of periodicity of the initial data) e.s. of the Cauchy problem for equation (10) with the initial function $v_0(y)$. This solution is y-periodic: $v(t,y+e) = v(t,y)$ a.e. in $\mathbb{R}_+ \times \mathbb{R}^m$ for all $e \in \mathbb{Z}^m$. Besides, by results of

5. Panov E. Yu. J. Hyperbolic Differ. Equ. **2**, 885–908 (2005),

after possible correction on a set of null measure, $v(t,\cdot) \in C([0,+\infty), L^1(\mathbb{T}^m))$, where $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ is an m-dimensional torus.

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We introduce the function $u(t,z,x) = v(t,z+y(x)) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^{m+n})$. Since the linear operator $(z,x) \to (z+y(x),x)$ on $\mathbb{R}^{m+n}$ is invertible, the function $u(t,z,x)$ is well-defined.

Obviously, for every $k \in \mathbb{R}$

$$
\begin{aligned}
&|u-k|_t + \mathrm{div}_x[\mathrm{sign}(u-k)(\varphi(u)-\varphi(k))] - \mathrm{sign}(u-k)h(t,z+y(x),u) = \\
&\quad |v(t,z+y)-k|_t + \mathrm{div}_y[\mathrm{sign}(v(t,z+y)-k)(\tilde{\varphi}(v(t,z+y))-\tilde{\varphi}(k))] \\
&\quad - \mathrm{sign}(v(t,z+y)-k)h(t,z+y,v(t,z+y)) \le 0 \ \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^{m+n}).
\end{aligned}
$$

Obviously, for every $k \in \mathbb{R}$

$$|u-k|_t + \mathrm{div}_x[\mathrm{sign}(u-k)(\varphi(u)-\varphi(k))] - \mathrm{sign}(u-k)h(t,z+y(x),u) = \\ |v(t,z+y)-k|_t + \mathrm{div}_y[\mathrm{sign}(v(t,z+y)-k)(\tilde{\varphi}(v(t,z+y))-\tilde{\varphi}(k))] \\ -\mathrm{sign}(v(t,z+y)-k)h(t,z+y,v(t,z+y)) \le 0 \ \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^{m+n}).$$

It is also clear that

$$\lim_{t\to 0+} u(t,z,x) = u_0(z,x) \doteq v_0(z+y(x)) \ \text{ in } L^1_{loc}(\mathbb{R}^{m+n}).$$

Thus, $u(t,z,x)$ is an e.s. of the problem

$$u_t + \mathrm{div}_x\varphi(u) = h(t,z+y(x),u), \quad u(0,z,x) = v_0(z+y(x))$$

in the extended domain $\mathbb{R}_+ \times \mathbb{R}^{m+n}$.

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in the extended domain $\mathbb{R}_+\times\mathbb{R}^{m+n}$. This readily implies the following statement.

Proposition 1.

There exists a set $E_1 \subset \mathbb{R}^m$ of full measure such that for all $z \in E_1$ the function $u(t,z,x) = v(t,z+y(x))$ is an e.s. of (1), (2) with the initial data $v_0(z+y(x))$ and the source function $h(t,z+y(x),u)$.

The additive group $\mathbb{R}^n$ acts on the m-dimensional torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ by the shift transformations $S_x z = z + y(x)$. This action is measure preserving and ergodic (the latter follows from the condition that the vectors $\lambda_j, j = 1, \ldots, m$, are independent over $\mathbb{Z}$). By Birkhoff individual ergodic theorem for each $w(y) \in L^1(\mathbb{T}^m)$ for a.e. $z \in \mathbb{T}^m$

$$ ⨍_{\mathbb{R}^n} w(z + y(x))dx = \int_{\mathbb{T}^m} w(y)dy. \tag{11} $$

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Let

$$v_r(t, y) = (v(t, \cdot) * \Phi_r)(y) = \sum_{\bar{k} \in \mathbb{Z}^m, |\bar{k}|_\infty < r} a_{r\bar{k}}(t) e^{2\pi i \bar{k} \cdot y}$$

be the Fejér sums w.r.t. variables $y \in \mathbb{R}^m$. Then $v_r(t, \cdot) \to v(t, \cdot)$ as $r \to \infty$ in $L^1(\mathbb{T}^m)$ for all $t \geq 0$. By (11) with $w(y) = |v(t, y) - v_r(t, y)|$ for a.e. $(t, z) \in \mathbb{R}_+ \times \mathbb{R}^m$

$$N_1(u^z(t, \cdot) - u_r^z(t, \cdot)) = ⨍_{\mathbb{R}^n} |v(t, z + y(x)) - v_r(t, z + y(x))|dx =$$

$$\int_{\mathbb{T}^m} |v(t, y) - v_r(t, y)|dy \quad \forall r \in \mathbb{N}, \qquad (12)$$

where $u_r = u_r^z(t, x) = v_r(t, z + y(x))$, $u = u^z(t, x) = v(t, z + y(x))$.

Notice that $u_r^z(t,x)$ are trigonometric polynomials with spectra contained in M_0. In view of (12) there exists a set $E_2 \subset \mathbb{R}^m$ of full measure such that for each $z \in E_2$ the function $u^z(t,\cdot) \in \mathcal{B}^1(\mathbb{R}^n)$ and $M(u^z(t,\cdot)) \subset M_0$ for a.e. $t > 0$.

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$$⨍_{\mathbb{R}^n} |v(t',z+y(x)) - v(t,z+y(x))|dx = \int_{\mathbb{T}^m} |v(t',y) - v(t,y)|dy \underset{t'-t\to 0}{\longrightarrow} 0$$

(recall that $v(t,y) \in C([0,+\infty), L^1(\mathbb{T}^m))$). This relation shows that, after possible correction of $u^z(t,\cdot)$ on a set of null measure, the function $u^z = v(t,z+y(x)) \in C([0,+\infty), \mathcal{B}^1(\mathbb{R}^n))$.

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(recall that $v(t,y) \in C([0,+\infty), L^1(\mathbb{T}^m))$). This relation shows that, after possible correction of $u^z(t,\cdot)$ on a set of null measure, the function $u^z = v(t,z+y(x)) \in C([0,+\infty), \mathcal{B}^1(\mathbb{R}^n))$. We have proved the following

Proposition 2.

There exists a set $E_2 \in \mathbb{R}^m$ of full measure such that for $z \in E_2$ the function $u^z = v(t,z+y(x)) \in C([0,+\infty), \mathcal{B}^1(\mathbb{R}^n))$ (after possible correction on a set of null measure).

Now, we choose a sequence $z_l \in E_1 \cap E_2$, $l \in \mathbb{N}$ such that $z_l \to 0$ as $l \to \infty$. By Proposition 1 $u^{z_l}(t,x)$ is an e.s. of (1), (2) with the input functions $u_0^{z_l}(x) = v_0(z_l + y(x))$, $g^{z_l}(t,x,u) = h(t, z_l + y(x), u)$. By Theorem 2 for a.e. $t > 0$

$$N_1(u^{z_l}(t,x) - u(t,x)) \le C\,[N_1(u_0^{z_l}(x) - u_0(x)) + N_1(\|g^{z_l}(\cdot,x,\cdot) - g(\cdot,x,\cdot)\|_{1,M})] =$$

$$C\int_{\mathbb{T}^m}\left[|v_0(z_l+y) - v_0(y)| + \int_0^{+\infty}\max_{|u|\le M}|h(t,z_l+y,u) - h(t,y,u)|dt\right]dy \underset{l\to\infty}{\to} 0. \quad (13)$$

Here $M = \|u\|_\infty$, $C = e^{\|a\|_1}$. Relation (13) implies convergence $u^{z_l} \to u$ in $C([0,+\infty), \mathcal{B}^1(\mathbb{R}^n))$ and completes the proof of Theorem 3.

Now, we choose a sequence $z_l \in E_1 \cap E_2$, $l \in \mathbb{N}$ such that $z_l \to 0$ as $l \to \infty$. By Proposition 1 $u^{z_l}(t,x)$ is an e.s. of (1), (2) with the input functions $u_0^{z_l}(x) = v_0(z_l + y(x))$, $g^{z_l}(t,x,u) = h(t, z_l + y(x), u)$. By Theorem 2 for a.e. $t > 0$

$$N_1(u^{z_l}(t,x) - u(t,x)) \le C\left[N_1(u_0^{z_l}(x) - u_0(x)) + N_1(\|g^{z_l}(\cdot,x,\cdot) - g(\cdot,x,\cdot)\|_{1,M})\right] =$$

$$C\int_{\mathbb{T}^m}\left[|v_0(z_l + y) - v_0(y)| + \int_0^{+\infty} \max_{|u|\le M} |h(t, z_l + y, u) - h(t,y,u)|dt\right] dy \underset{l\to\infty}{\to} 0. \quad (13)$$

Here $M = \|u\|_\infty$, $C = e^{\|a\|_1}$. Relation (13) implies convergence $u^{z_l} \to u$ in $C([0,+\infty), \mathcal{B}^1(\mathbb{R}^n))$ and completes the proof of Theorem 3.
To prove Theorem 4, we notice that for all $\bar{k} \in \mathbb{Z}^m$, $\bar{k} \ne 0$

$$\bar{k} \cdot \tilde{\varphi}(u) = \sum_{j=1}^m \sum_{k=1}^n k_j \lambda_{jk} \varphi_k(u) = \lambda(\bar{k}) \cdot \varphi(u),$$

where $\lambda(\bar{k}) = \sum_{j=1}^m k_j \lambda_j \in M_0$, $\lambda(\bar{k}) \ne 0$. By our condition (6), we find that the functions $u \to \bar{k} \cdot \tilde{\varphi}(u)$ are not affine on non-empty intervals, that is, condition (8) is satisfied (with $L' = L = \mathbb{Z}^m$).

By the decay result [3], we claim that the periodic e.s. $v(t,y)$ satisfies the decay relation

$$\lim_{t\to+\infty}\int_{\mathbb{T}^m}|v(t,y)-C|dy=0, \tag{14}$$

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We choose a sequence $z_l\in E_1\cap E_2$, $z_l \underset{l\to\infty}{\to} 0$. In view of (11) with $w(y)=|v(t,y)-C|$ we find

$$⨍_{\mathbb{R}^n}|u^{z_l}(t,x)-C|dx=⨍_{\mathbb{R}^n}|v(t,z_l+y(x))-C|dx=\int_{\mathbb{T}^m}|v(t,y)-C|dy.$$

By the decay result [3], we claim that the periodic e.s. $v(t,y)$ satisfies the decay relation

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Passing in this equality to the limit as $l\to\infty$ and taking into account (13), we obtain that for all $t>0$

$$⨍_{\mathbb{R}^n}|u(t,x)-C|dx=\int_{\mathbb{T}^m}|v(t,y)-C|dy.$$

Decay property (7) now follows from (14):

$$\lim_{t\to+\infty}⨍_{\mathbb{R}^n}|u(t,x)-C|dx=0.$$

The proof of Theorem 4 is complete. □

Suppose that $g \equiv 0$ and non-degeneracy condition (6) fails. Then one can find a nonzero vector $\xi \in M_0$ such that the function $\xi \cdot \varphi(u) = \tau u + c$ on some segment $[a, b]$, where $\tau, c \in \mathbb{R}$. As is easy to verify, the function

$$u(t,x) = \frac{a+b}{2} + \frac{b-a}{2}\sin(2\pi(\xi \cdot x - \tau t))$$

is an e.s. of (1), (2) with the periodic initial function

$$u(0,x) = \frac{a+b}{2} + \frac{b-a}{2}\sin(2\pi(\xi \cdot x))$$

such that $Sp(u_0) = \{-\xi, \xi\} \subset M_0$. It is clear that this e.s. does not satisfy the decay property.

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The detailed proofs of presented results in the case $g \equiv 0$ can be found in recent preprints

http://arxiv.org/pdf/1406.4838.pdf

http://arxiv.org/pdf/1408.0658.pdf

Thank you for attention!