Homogenization of Monotone Operators with Oscilating Exponent of Nonlinearity

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Joint result with V. V. Zhikov

The simplest problems of Homogenization.

Problem 1
$$\begin{cases} \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = \operatorname{div} g \text{ in } \Omega, \quad g \in L^2(\Omega) \\ u^\varepsilon|_{\partial\Omega} = 0 \end{cases}$$

Ω is a Lipschitz bounded domain in $\mathbb{R}^d$

$a(y)$ is a symmetric measurable matrix,
1-periodic in $y_1, \dots, y_d$; $\square = [0,1)^d$ is a cell of periodicity;

$\exists \lambda: \quad \lambda|\xi|^2 \le a(y)\xi\cdot\xi \le \lambda^{-1}|\xi|^2 \quad \forall \xi \in \mathbb{R}^d$

$a\left(\frac{x}{\varepsilon}\right)$ is highly oscilating as $\varepsilon \downarrow 0$

$$u^\varepsilon \in W_0^{1,2}(\Omega), \quad \int_\Omega a\left(\tfrac{x}{\varepsilon}\right)\nabla u^\varepsilon\cdot\nabla\varphi\, dx = \int_\Omega g\cdot\nabla\varphi\, dx \quad \forall \varphi \in C_0^\infty(\Omega) \quad (1)$$
$$(\varphi \in W_0^{1,2}(\Omega))$$

$u^\varepsilon \rightharpoonup u$ in $W^{1,2}(\Omega)$

$u = ?$ How to pass to the limit in (1)?

The main question: $a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon \xrightarrow{L^2}$?

(the convergence of fluxes)

Homogenized (limit) problem:

$$\begin{cases} \operatorname{div} a^0 \nabla u = \operatorname{div} g & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

a^0 is a constant symmetric positive matrix, s.t.

$$a^0 e_j = \int_{\square} a(y)\left(e_j + \nabla N^j(y)\right) dy, \quad j = 1, \dots, d$$

$$N^j \in W^{1,2}_{per}(\square), \quad \operatorname{div}\left(a(y)\left(e_j + \nabla N^j(y)\right)\right) = 0 \qquad \text{(Cell Problem)}$$

$e_1, \dots, e_d$ is a canonical basis in $\mathbb{R}^d$.

$$u \in W^{1,2}_0(\Omega), \quad \int_\Omega a^0 \nabla u \cdot \nabla \varphi \, dx = \int_\Omega g \cdot \nabla \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

Problem 2

$$\begin{cases} \operatorname{div} A\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) = \operatorname{div} g \text{ in } \Omega, \quad g \in L^{\alpha'}(\Omega)^d \\ u^\varepsilon|_{\partial\Omega} = 0 \end{cases}$$

$\alpha' = \frac{\alpha}{\alpha - 1}$, $\alpha > 1$

$A(y, \xi): \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function,

1-periodic in $y_1, \dots, y_d$,

satysfying the conditions (monotonicity, coerciveness, boundedness)

$\left(A(y,\xi) - A(y,\eta)\right) \cdot (\xi - \eta) > 0$, $\xi \neq \eta$,

$A(y,\xi) \cdot \xi \geq c_1 |\xi|^\alpha - c_0$,

$|A(y,\xi)|^{\alpha'} \leq c_2 |\xi|^\alpha + c_0$,

(CB-conditions with exponent α)

$c_1, c_2 > 0$, $c_0 \geq 0$.

$u^\varepsilon \in W_0^{1,\alpha}(\Omega)$, $\int_\Omega A\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) \cdot D\varphi \, dx = \int_\Omega g \cdot D\varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega)$
$(\varphi \in W_0^{1,\alpha}(\Omega))$

$\|u^\varepsilon\|_{W^{1,\alpha}(\Omega)}, \ \|A\left(\frac{x}{\varepsilon}, Du^\varepsilon\right)\|_{L^{\alpha'}(\Omega)} \leq C$

$u^\varepsilon \rightharpoonup u$ in $W^{1,\alpha}(\Omega)$, $A\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) \rightharpoonup w$ in $L^{\alpha'}(\Omega)^d$

What is a relation between u and w?

(Problem of convergence of the fluxes).

$$w = a(\nabla u), \text{ i.e.}$$

$$A\left(\tfrac{x}{\varepsilon}, \nabla u^{\varepsilon}\right) \xrightarrow{L^{\alpha'}} a(\nabla u)$$

$$\begin{cases} \operatorname{div} a(\nabla u) = \operatorname{div} g \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$ (Hom. Problem)

$a(\xi): \mathbb{R}^d \to \mathbb{R}^d$ is continuous and satisfies:

$$(a(\xi) - a(\eta)) \cdot (\xi - \eta) > 0, \quad \xi \neq \eta$$

$$a(\xi) \cdot \xi \geq c_1 |\xi|^{\alpha} - c_0,$$

$$|a(\xi)|^{\alpha'} \leq c_2 |\xi|^{\alpha} + c_0,$$

(CB-conditions with the exponent α)

$$a(\xi) = \int_{\square} A(y, \xi + \nabla_y N(y))\, dy$$

$N(y) = N(y, \xi)$ is a solution of the CP

$N \in W^{1,\alpha}_{per}(\square)$, $\operatorname{div} A(y, \xi + \nabla N(y)) = 0$ (Cell Problem)

Problem 3 (the main problem)

$$\begin{cases} \operatorname{div} A\left(\frac{x}{\varepsilon}, \nabla u^\varepsilon\right) = \operatorname{div} g \text{ in } \Omega, \quad g \in L^\infty(\Omega)^d, \\ u^\varepsilon \big|_{\partial\Omega} = 0 \end{cases} \qquad (1)$$

$A(y, \xi)$ is a Carathèodory function, monotone in ξ, 1-periodic in $y_1, \dots, y_d$

$$\begin{cases} A(y, \xi)\cdot\xi \geq C_1 |\xi|^{p(y)} - C_0, \\ |A(y,\xi)|^{p'(y)} \leq C_2 |\xi|^{p(y)} + C_0, \quad p'(y) = \frac{p(y)}{p(y)-1} \end{cases}$$

(CB - conditions with the variable exponent $p(y)$)

where

$p(y)$ is measurable on $\cancel{\ }\ \mathbb{R}^d$, 1-periodic in $y_1, \dots, y_d$,

$$1 < \alpha \leq p(y) \leq \beta < \infty$$

$$A\left(\tfrac{x}{\varepsilon}, \xi\right)\cdot\xi \geq C_1 |\xi|^{p(\frac{x}{\varepsilon})} - C_0,$$
$$\left|A\left(\tfrac{x}{\varepsilon}, \xi\right)\right|^{p'(\frac{x}{\varepsilon})} \leq C_2 |\xi|^{p(\frac{x}{\varepsilon})} + C_0,$$

(CB - conditions with the oscilating exponent $p^\varepsilon(x) = p(\frac{x}{\varepsilon})$)

Setting of the problem ($\varepsilon = 1$)

1) Lebesque-Orlicz and Sobolev-Orlicz spaces

$$L^{p(\cdot)}(\Omega) = \{ v \in L^1(\Omega)^d : \int_\Omega |v(x)|^{p(x)} dx < \infty \}$$

$$\|v\|_{L^{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \int_\Omega |\tfrac{v}{\lambda}|^p dx \le 1 \}$$

✓ $$W = W_0^{1,p(\cdot)}(\Omega) = \{ u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^p dx < \infty \}$$

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

$$L^\beta(\Omega) \subset L^{p(\cdot)}(\Omega) \subset L^\alpha(\Omega)$$

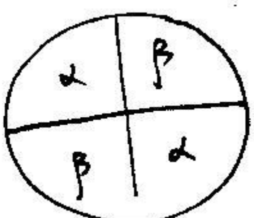
$$W_0^{1,\beta}(\Omega) \subset W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,\alpha}(\Omega)$$

$C_0^\infty(\Omega)$ is not necessarily dense in $W_0^{1,p(\cdot)}(\Omega)$ $\| \nabla$!

✓ $H = H_0^{1,p(\cdot)}(\Omega)$ is a closure of $C_0^\infty(\Omega)$ in W

$$H \neq W$$

Example (Zhikov, 1984) $d = 2$, $\Omega = \{ |x| < 1 \}$



$$p(x) = \begin{cases} \alpha, & x_1 x_2 \le 0 \\ \beta, & x_1 x_2 > 0 \end{cases}$$

$$\alpha < 2 < \beta$$

$u \in W \setminus H$

$H = W$ if $p(x)$ satisfies log-condition (Fan X., Zhikov 199

$$\exists k : \ |p(x) - p(y)| \le \frac{k}{\log \frac{1}{|x-y|}} \quad \forall x, y \in \Omega, \ |x-y| < \frac{1}{2}$$

2) Classification of solutions

<u>Definition 1</u> u is a W-solution (or solution of the 1^{st} type), if

$$u \in W, \quad \int_\Omega A(x, Du) \cdot D\varphi \, dx = \int_\Omega g \cdot D\varphi \, dx \quad \forall \varphi \in W \quad (\cancel{\varphi \in C_0^\infty(\Omega)})$$

<u>Definition 2</u> u is a H-solution (or solution of the 2^{nd} type), if

$$u \in H, \quad \int_\Omega A(x, Du) \cdot D\varphi \, dx = \int_\Omega g \cdot D\varphi \, dx \quad \forall \varphi \in H \quad (\varphi \in C_0^\infty(\Omega))$$

<u>Lavrentiev phenomenon:</u> $\exists g$ such that H-solution $\neq$ W-solution

Let V be intermediate: $H \subseteq V \subseteq W$, V-solution

All V-solutions are "variational"

<u>Theorem</u> $\exists!$ V-solution

There are other solutions, different of "variational"

<u>Definition</u> u is a weak solution, if

$$u \in W, \quad \int_\Omega A(x, Du) \cdot D\varphi \, dx = \int_\Omega g \cdot D\varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

Let $\varepsilon \to 0$

<u>Theorem</u> The Direchlet problem (1) has unique W-solution (H-solution), s.t.

$$\int_\Omega |Du^\varepsilon|^{p_\varepsilon(x)} dx \leq C, \quad \int_\Omega \left|A\left(\tfrac{x}{\varepsilon}, Du^\varepsilon\right)\right|^{p'_\varepsilon(x)} dx \leq C,$$

$$C = \mathrm{const}(c_0, c_1, c_2, \|g\|_{L^{\alpha'}(\Omega)})$$

$u^\varepsilon \to u$ in $W^{1,\alpha}(\Omega)$

$A\left(\frac{x}{\varepsilon}, Du^\varepsilon\right) \to w$ in $L^{\beta'}(\Omega)$

What is a relation between u and w? (problem of convergence of fluxes)

$$f_i(\xi) = \Gamma_i - \lim_{\varepsilon \to 0} f_\varepsilon(x,\xi), \quad f_\varepsilon(x,\xi) = f\left(\frac{x}{\varepsilon}, \xi\right), \quad f(y,\xi) = |\xi|^{p(y)}$$

Zh 1992

$$f_i(\xi) = \min_{N \in V_i(\square)} \int_\square f(y, \xi + \nabla N(y))\,dy = \int_\square f(y, \xi + \nabla N_i(y))\,dy$$

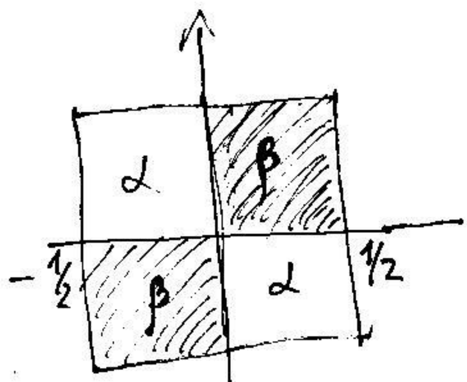
$$V_1(\square) = W^{1,p(\cdot)}_{per}(\square) = \left\{ u \in W^{1,1}_{per}(\square) : \int_\square |\nabla u|^{p(y)} dy < \infty, \int_\square u\,dy = 0 \right\}$$

$V_2(\square)$ is a closure of $C^\infty_{per}(\square)$ in $W^{1,p(\cdot)}_{per}(\square)$

$$f_1(\xi) \le f_2(\xi)$$

Example: $f_1 < f_2$

$d=2$, $p(y)$ has a chessboard structure



$\alpha < 2 < \beta$

limin*f*-property for Γ_i-limit:

$$\liminf_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^{p_\varepsilon(x)} dx \ge \int_\Omega f_i(\nabla u)\,dx$$

whenever $u_\varepsilon \xrightarrow{W^{1,\alpha}} u$, $u_\varepsilon \in V_i$

Properties of $f = \Gamma\text{-}\lim\limits_{\varepsilon\to 0} f_\varepsilon$, $f_\varepsilon(x,\xi) = |\xi|^{p(\frac{x}{\varepsilon})}$

- $f = f(\xi)$ is continuous, convex, even in $\mathbb{R}^d$
- $f \geq 0$, $f(\xi) = 0 \Longleftrightarrow \xi = 0$
- Δ_2-condition: $f(2\xi) \leq c\, f(\xi)$

$$L^f = \{ \psi \in L^1(\Omega)^d : \int_\Omega f(\psi)\,dx < \infty \}$$

$$\|\psi\|_{L^f} = \inf \{ \lambda > 0 : \int_\Omega f(\lambda^{-1}\psi)\,dx \leq 1 \}$$

$$W_0^f = \{ \varphi \in W_0^{1,1}(\Omega) : \int_\Omega f(\nabla\varphi)\,dx < \infty \}, \quad \|\varphi\|_{W^f} = \|\nabla\varphi\|_{L^f}$$

$C_0^\infty(\Omega)$ is dense in W_0^f

f is uniformly convex $\Rightarrow$ L^f, W_0^f are $\left\{ \begin{array}{l} \text{uniformly convex} \\ \text{reflexive} \end{array} \right.$

$$(L^f)^* = L^{f^*}, \quad f^*(\eta) = \sup_\xi [\eta\cdot\xi - f(\xi)]$$

Return to the monotone equation

$$u^\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega), \quad \int_\Omega A(\tfrac{x}{\varepsilon}, \nabla u^\varepsilon)\cdot \nabla\varphi\, dx = \int_\Omega g\cdot\nabla\varphi\, dx \quad \forall \varphi \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)$$

$$\int_\Omega |\nabla u_\varepsilon|^{p_\varepsilon}\, dx \le C, \qquad \int_\Omega |A(\tfrac{x}{\varepsilon}, \nabla u^\varepsilon)|^{p_\varepsilon'}\, dx \le C$$

$$u_\varepsilon \xrightarrow{W^{1,\alpha}} u \qquad\qquad A(\tfrac{x}{\varepsilon}, \nabla u^\varepsilon) \xrightarrow{L^{\beta'}} w$$

$$u \in W_0^{f} \qquad\qquad w \in L^{f^*}$$

$$a(\xi) = \int_\square A(y, \xi + \nabla N(y))\, dy$$

$$N \in W_{per}^{1,p(\cdot)}(\square), \quad \int_\square A(y, \xi + \nabla N)\cdot\nabla\varphi\, dy = 0 \quad \forall \varphi \in W_{per}^{1,p(\cdot)}(\square)$$

$a(\xi)$ is continuous

$$(a(\xi) - a(\eta))\cdot(\xi - \eta) > 0, \quad \xi \ne \eta$$

$$\left.\begin{aligned} a(\xi)\cdot\xi &\ge C_1 f(\xi) - C_0 \\ f^*(a(\xi)) &\le C_2 f(\xi) + C_0 \end{aligned}\right\} \text{CB-conditions of the new type}$$

<u>Limit Problem</u>: for $g \in L^{f^*}(\Omega)$ find

$u \in W_0^f$, $\operatorname{div} a(\nabla u) = \operatorname{div} g$ (in the sense of distributions)

<u>Theorem</u> $\forall f \in L^{f^*}$ $\exists!$ solution of the limit problem

<u>Theorem of homogenization</u> (Zh Pa 2011)

Let u^ε be a W-solution of the problem (*).

Then $u^\varepsilon \rightharpoonup u$ in $W^{1,\alpha}(\Omega)$, $A(\frac{x}{\varepsilon}, \nabla u^\varepsilon) \rightharpoonup a(\nabla u)$ in $L^{\beta'}(\Omega)^d$

$$\int_\Omega A(\tfrac{x}{\varepsilon}, \nabla u^\varepsilon)\cdot \nabla u^\varepsilon dx \to \int_\Omega a(\nabla u)\cdot \nabla u dx,$$

where u is a solution of the limit problem

To prove this result we use monotonicity method + compensated compactness arguments

<u>Compensated Compactness Lemma</u> Assume

a) $u_\varepsilon \in H_0^{1,p_\varepsilon(\cdot)}(\Omega)$, $w_\varepsilon \in L^{p_\varepsilon'(\cdot)}_{sol}(\Omega)$;

b) $\int_\Omega |\nabla u_\varepsilon|^{p_\varepsilon} dx$, $\int_\Omega |w_\varepsilon|^{p_\varepsilon'} dx \le C < \infty$;

c) $\nabla u_\varepsilon \xrightarrow{L^\alpha} \nabla u$, $w_\varepsilon \xrightarrow{L^{\beta'}} w$;

d) $\{u_\varepsilon\}_\varepsilon$ is compact in $L^\beta(\Omega)$.

Then there holds the convergence of measures

$$w^\varepsilon\cdot \nabla u^\varepsilon dx \to w\cdot \nabla u dx \qquad (*)$$

The convergence (*) holds also if a) is replaced with

a') $u_\varepsilon \in W_0^{1,p_\varepsilon(\cdot)}(\Omega)$, $w_\varepsilon \in L^{p_\varepsilon'(\cdot)}_{sol,2}(\Omega)$

If $p_\varepsilon \equiv$ const, this is Tartar–Murat Lemma

<u>Extensions</u> for $\left\{\begin{array}{l}\text{vector problems} \\ \text{Elasticity Theory problems}\end{array}\right\}$ demand

additional condition:

$$1 < \alpha < \beta < \alpha_* = \begin{cases} \dfrac{\alpha(d-1)}{d-1-\alpha} & \text{for } \alpha < d-1 \\ +\infty & \text{for } \alpha \geq d-1 \end{cases}$$

<u>Literature</u>

[1] V.V. Zhikov, S.E. Pastukhova Mat. Zametki (2011) 90:1

[2] —— " —— Trudy MMO (2014) 75:2

[3] —— " —— Doklady RAS (2010) 433:5