Viscous incompressible free-surface flow down an inclined perturbed plane

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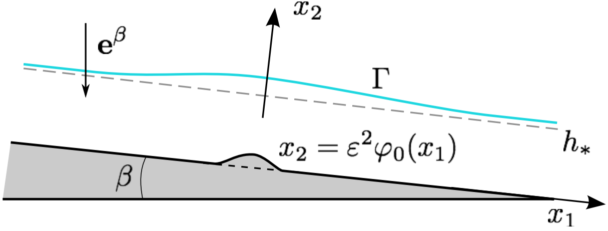


$$S = \{x \in \mathbb{R}^2 : x_2 = \varepsilon^2\varphi_0(x_1)\}, \quad \mathrm{supp}\varphi_0 \subset (-1,1),\ \varepsilon > 0,$$
$$\Gamma = \{x \in \mathbb{R}^2 : x_2 = \psi(x_1) = 1 + \varepsilon\Psi(x_1)\},$$
$$\mathbf{e}^\beta = (\cos\beta, -\sin\beta),\ \beta \in (0, \frac{\pi}{2})$$

$$
\begin{cases}
-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = -g\mathbf{e}^{\beta} \quad \text{in } \Omega, \\
div\,\mathbf{u} = 0 \quad \text{in } \Omega, \\
\mathbf{u} = 0 \quad \text{on } S, \\
\mathbf{u}\cdot\mathbf{n} = 0, \quad \boldsymbol{\tau}\cdot\mathcal{S}(\mathbf{u})\mathbf{n} = 0 \quad \text{on } \Gamma, \\
\Big(\frac{\psi'(x_1)}{\sqrt{1+\psi'(x_1)^2}}\Big)' = \sigma^{-1}\big(-p(x) + \mathbf{n}\cdot\mathcal{S}(\mathbf{u})\mathbf{n}\big)\big|_{\Gamma}, \\
\lim_{|x_1|\to\infty} \psi(x_1) = 1, \\
\int_{\sigma_t} u_1(x)dx_2 = \frac{g\sin\alpha}{3\nu},
\end{cases}
$$

$\mathbf{u}$ – velocity of the fluid, p -pressure, $\alpha = \frac{\pi}{2} - \beta$, $\boldsymbol{\tau}$ and $\mathbf{n}$ are unit vectors of the tangent and the outward normal to the free boundary Γ, $\nu > 0$ and $\sigma > 0$ are the coefficients of viscosity and surface tension, g - acceleration of the gravity, $\mathcal{S}(\mathbf{u})$ is the deformation tensor, σ_t is the cross-section of the domain Ω by the line $x_1 = t$.

V.V. Pukhnachev scheme. Systems of equations are divided in two problems: NS problem in fixed domain

$$\begin{cases} -\nu\Delta\mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\mathbf{u}_k + \nabla p_k = -g\mathbf{e}^{\beta} \quad \text{in } \Omega_k, \\ div\,\mathbf{u}_k = 0 \quad \text{in } \Omega_k, \\ \mathbf{u}_k = 0 \quad \text{on } S, \\ \mathbf{u}_k\cdot\mathbf{n} = 0, \quad \boldsymbol{\tau}\cdot\mathcal{S}(\mathbf{u}_k)\mathbf{n} = 0 \quad \text{on } \Gamma_k, \\ \int\limits_{\sigma_t} u_{k1}(x)dx_2 = \dfrac{g\sin\alpha}{3\nu}, \end{cases}$$

and the problem of finding the free boundaries Γ from the equations

$$K_{k+1}(x) = \sigma^{-1}\big(- p_k(x) + \mathbf{n}\cdot\mathcal{S}(\mathbf{u}_k)\mathbf{n}\big)|_{\Gamma_j}$$

$$\Gamma_0 \Rightarrow \Omega_0 \Rightarrow (\mathbf{u}_0, p_0) \Rightarrow \Gamma_1 \Rightarrow \Omega_1 \Rightarrow (\mathbf{u}_1, p_1) \Rightarrow \dots$$
$$\dots\Gamma_k \Rightarrow \Omega_k \Rightarrow (\mathbf{u}_k, p_k) \Rightarrow \dots$$

Finding the free boundary Γ_{k+1} we have to solve the boundary value problem for the linear ordinary differential equation

$$\begin{cases} \Psi''_{k+1}(x_1) - g\sigma^{-1}\cos\alpha\Psi_{k+1}(x_1) = \\ = \sigma^{-1}\Big(- q_k(x) + \mathbf{n}\cdot\mathcal{S}(\mathbf{u}_k)\mathbf{n}\Big)\Big|_{\Gamma_k} + \ldots \equiv \Phi_{k+1}(x_1) \\ \lim\limits_{|x_1|\to\infty} \Psi_{k+1}(x_1) = 0. \end{cases}$$

The pressure $q_k(x)$ tends to the constant q_k^+ and q_k^- as $x_1 \to +\infty$ and $x_1 \to -\infty$. Since the pressure is defined up to an additive constant we always may normalize $q_k(x)$ so that $q_k^+ = 0$. However, the pressure drop $q_{*\,k} = q_k^+ - q_k^+$ is the functional on the right-hand sides of the the Stokes problem and, in general, could be nonzero

$$q_{*\,k} = q_k^+ - q_k^+ \neq 0.$$

A unique solution $\Psi_{k+1}(x_1)$ exists if and only if the right-hand side Φ_{k+1} vanishes as $x_1 \to \pm\infty$. However, this is possible only if $q_{*\,k} = 0$.

New scheme: (Abergel, Bona, 1992 & Nazarov, K.P., 1993)
Transformation of the flow domain Ω to the unperturbed "uniform" domain $\Omega_0 = \{x \in \mathbb{R}^2 : 0 < x_2 < 1\}$ and linearization of the problem on an appropriate exact solution in Ω_0 .
On each step of iterations the determination of the velocity vector $\mathbf{u}$ and the pressure function p is not separated from the determination of the free boundary Γ (i.e. from the determination of the functions Ψ describing Γ) and all problems are solved in the same fixed domain Ω_0.

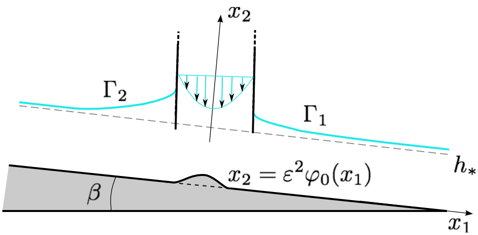
$$(\mathbf{u}_0, p_0, \Psi_0) \Rightarrow (\mathbf{u}_1, p_1, \Psi_1) \Rightarrow \cdots \Rightarrow (\mathbf{u}_k, p_k, \Psi_k) \Rightarrow \dots$$

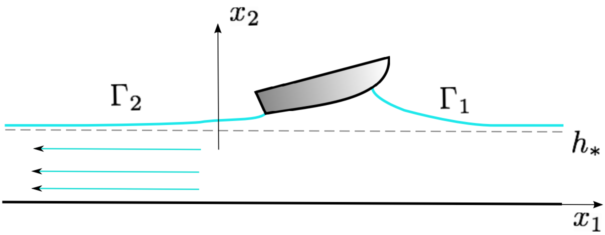
Linearised FVB problem

$$
\begin{cases}
-\nu\Delta \mathbf{v} + \nabla q = -(\mathbf{v}^0 \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v}^0 + \mathbf{F}(\mathbf{v}, q, \Psi) \quad \text{in } \Omega_0, \\
div\, \mathbf{v} = 0 \quad \text{in } \Omega_0, \\
v_1\big|_{y_2=0} = A_1, \quad v_2\big|_{y_2=0} = A_2 \\
v_2|_{y_2=1} = \dfrac{g \sin\alpha}{2\nu}\Psi' + B(\Psi), \\
\nu\Big(\dfrac{\partial v_1}{\partial y_2} + \dfrac{\partial v_2}{\partial y_1}\Big)\Big|_{y_2=1} = g \sin\alpha \Psi + D(\mathbf{v}, \Psi), \\
\Psi'' - g\sigma^{-1}\cos\alpha\Psi = \sigma^{-1}\Big(- q(y) + 2\nu\dfrac{\partial v_2}{\partial y_2}\Big)\Big|_{y_2=1} + \Phi(\mathbf{v}, \Psi), \\
\lim\limits_{|y_1|\to\infty} \Psi(y_1) = 0.
\end{cases}
$$

Difficulties: linearized problems contain more boundary conditions as it is allowed by usual ADN-elliptic theory and contains additionally the unknown functions Ψ_k defined on the "free surface" $\{x \in \mathbb{R}^2 : x_2 = 1\}$ of the "uniform" domain Ω_0. In *Nazarov* & *K.P.* the proofs are based on L^2-theory for such generalized elliptic problems[1], and in *Abergel* & *Bona* – on the detailed investigation of the pseudo-differential operator corresponding to the linearized problem.

[1] see Kozlov V.A., Maz'ya V.G., Rossmann J., Elliptic boundary value problems in domains with point singularities, *Math. Survays and Monographhs*, **52**, Amer. Math. Soc., 1997


x_2
Γ_2
Γ_1
h_*
$x_2 = \varepsilon^2\varphi_0(x_1)$
β
x_1


x_2
Γ_2
Γ_1
h_*
x_1

Modified scheme consists in the following: the unknown flow domain is mapped onto Ω_0 and consider the problem in the fixed domain. However, now we separate the finding of the solutions $(\mathbf{v}_k, q_k)$ of the Stokes problem from the finding of the functions Ψ_k describing the free boundary. In order to insure that on every step of iteration the pressure drop $q_{*\,k} = 0$, we introduce a smooth function $H_0(x_1)$ and we look for Ψ_k is the form $\Psi_k(x_1) = \chi_k H_0(x_1) + \Upsilon_k(x_1)$. The constants χ_k are chosen so that the pressure $q_k(x)$ of k-iteration satisfies the condition $q_{*\,k} = 0$. This gives the possibility to solve the problem for $(k+1)$-iteration. Finally, the iterations

$$\Big\{\mathbf{u}_k(x), p_k(x), \psi_k(x_1)\Big\}, \quad \psi_k(x_1) = 1 + \varepsilon\big(\chi_k H_0(x_1) + \Upsilon_k(x_1)\big),$$

converge to the solution $\big(\mathbf{u}(x), p(x), \psi(x_1)\big)$.

Transformation of the domain

Let

$$\omega(y_1, y_2; \Psi) = \zeta(y_2)\varepsilon \int_{-1}^{1} K(\tau)\, \varphi_0(y_1 + \tau y_2) d\tau$$

$$+(1 - \zeta(y_2)) \int_{-1}^{1} K(\tau) \Psi(y_1 + \tau y_2) d\tau,$$

where $K(\tau)$ is an infinitely smooth function such that

$$\operatorname{supp} K \subset (-1, 1), \quad \int_{-1}^{1} K(\tau) d\tau = 1, \quad \int_{-1}^{1} \tau K(\tau) d\tau = 0,$$

and ζ is an infinitely smooth cut-off function with $\zeta(y_2) = 1$ for $|y_2| \leq \frac{1}{4}$ and $\zeta(y_2) = 0$ for $|y_2| \geq \frac{1}{2}$.
Define the transformation $X(y)$:

$$x_1 = y_1, \quad x_2 = y_2 + \varepsilon\omega(y; \Psi)$$

which maps $\Omega_0 = \{x \in \mathbb{R}^2 : 0 < x_2 < 1\}$ onto the domain $\Omega = \{x \in \mathbb{R}^2 : \varepsilon^2 \varphi_0(x_1) < x_2 < 1 + \varepsilon \Psi_0(x_1)\}$.

Linearization of FVB problem

$$v_1^0(x) = \frac{g\sin\alpha}{2\nu}x_2(2-x_2),\ \ v_2^0(x) = 0,\ \ p^0(x) = g\cos\alpha(1-x_2),\ \ \psi_0(x_1) = 1$$

be the exact Poiseuille type solution for $\varepsilon = 0$, i.e., in Ω_0. Substituting

$$\mathbf{u}(x) = \mathbf{v}^0(x)+\varepsilon\mathbf{V}(x), \quad p(x) = p^0(x)+\varepsilon q(x), \quad \psi(x_1) = 1+\varepsilon\Psi(x_1),$$

introducing a new vector-field $\mathbf{v}$ with components

$$v_1(y) = V_1(X(y))\big(1+\varepsilon\partial_{y_2}\omega(y)\big),\ \ v_2(y) = V_2(X(y))-\varepsilon V_1(X(y))\partial_{y_1}\omega(y),$$

and making the change of variables $x = X(y)$, we get the following FBV problem in the strip Ω_0:

Linearization of FVB problem

$$
\begin{cases}
-\nu\Delta\mathbf{v} + \nabla q = -(\mathbf{v}^0\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v}^0 + \mathbf{F}(\mathbf{v}, q, \Psi) \quad \text{in } \ \Omega_0, \\
div\,\mathbf{v} = 0 \quad \text{in } \ \Omega_0, \\
v_1\big|_{y_2=0} = A_1, \quad v_2\big|_{y_2=0} = A_2 \\
v_2|_{y_2=1} = \dfrac{g\sin\alpha}{2\nu}\Psi' + B(\Psi), \\
\nu\Big(\dfrac{\partial v_1}{\partial y_2} + \dfrac{\partial v_2}{\partial y_1}\Big)\Big|_{y_2=1} = g\sin\alpha\Psi + D(\mathbf{v}, \Psi), \\
\Psi'' - g\sigma^{-1}\cos\alpha\Psi = \sigma^{-1}\Big(-q(y) + 2\nu\dfrac{\partial v_2}{\partial y_2}\Big)\Big|_{y_2=1} + \Phi(\mathbf{v}, \Psi), \\
\lim\limits_{|y_1|\to\infty} \Psi(y_1) = 0.
\end{cases}
$$

Stokes problem

Consider in Ω_0 the Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{w} + \nabla s = \mathbf{f} & \text{in } \Omega_0, \\ div\,\mathbf{w} = 0 & \text{in } \Omega_0, \\ \mathbf{w} = \mathbf{a} & \text{on } S_0, \\ \mathbf{w}\cdot\mathbf{n} = b, \quad \boldsymbol{\tau}\cdot\mathcal{S}(\mathbf{w})\cdot\mathbf{n} = d & \text{on } \Gamma_0. \end{cases}$$

Denote by $C^{l+\delta}(\Omega_0;\beta)$ a Banach space of functions having the finite norm

$$\|v\|_{C^{l+\delta}(\Omega_0;\beta)} = \|\exp\left(\beta\sqrt{1+x_1^2}\right) v\|_{C^{l+\delta}(\Omega_0)}.$$

Stokes problem

Theorem 1. *Let* $\mathbf{f} \in C^{\delta}(\Omega_0; \beta)$, $\mathbf{a} \in C^{2+\delta}(\mathbb{R}; \beta)$, $b \in C^{2+\delta}(\mathbb{R}; \beta)$, $d \in C^{1+\delta}(\mathbb{R}; \beta)$, *where* $\beta \in (0, \beta_*)$ *with sufficiently small* β_*, *and the following compatibility condition*

$$\int_{\mathbb{R}} b(y_1)dy_1 - \int_{\mathbb{R}} a_2(y_1)dy_1 = 0$$

holds.
(i) *There exists a unique solution* $\mathbf{w} \in C^{2+\delta}(\Omega_0; \beta)$, $\nabla s \in C^{\delta}(\Omega_0; \beta)$ *satisfying the estimate*

$$\|\mathbf{w}\|_{C^{2+\delta}(\Omega_0;\beta)} + \|\nabla s\|_{C^{\delta}(\Omega_0;\beta)} \leq c\Big(\|\mathbf{f}\|_{C^{\delta}(\Omega_0;\beta)} + \|\mathbf{a}\|_{C^{2+\delta}(\mathbb{R};\beta)}$$

$$+\|b\|_{C^{2+\delta}(\mathbb{R};\beta)} + \|d\|_{C^{1+\delta}(\mathbb{R};\beta)}\Big).$$

Moreover, the pressure function s *exponentially tends to certain constant limits* s^+ *and* s^- *as* $y_1 \to +\infty$ *and* $y_1 \to -\infty$.

Stokes problem

(ii) *If* $s^+ = s^-$*, then the pressure* s *can be normalized so that* $\lim\limits_{|y_1|\to\infty} s(y) = 0$ *and*

$$\|s\|_{C^{1+\delta}(\Omega_0;\beta)} \leq c\Big(\|\mathbf{f}\|_{C^{\delta}(\Omega_0;\beta)}+\|\mathbf{a}\|_{C^{2+\delta}(\mathbb{R};\beta)}+\|b\|_{C^{2+\delta}(\mathbb{R};\beta)}+\|d\|_{C^{1+\delta}(\mathbb{R};\beta)}\Big).$$

(iii) *The difference* $s_* = s^+ - s^-$ *is uniquely determined by the data of the Stokes problem:*

$$s_* = s^+ - s^- = \int\limits_{\Omega} \mathbf{f}\cdot\mathbf{W}^0 dy + \int\limits_{S_0} \Big(3\nu a_1 - a_2 Q^0\Big)\, dy_1 + \int\limits_{\Gamma_0} \Big(bQ^0 + \frac{3}{2}d\Big)\, dy_1,$$

where $W_1^0(y) = \dfrac{3y_2(2-y_2)}{2}, \quad W_2^0(y) \equiv 0, \quad Q^0(y) = -3\nu y_1$ *is the Poiseuille solution in* Ω_0 *satisfying the boundary conditions* $\mathbf{W}^0(y)|_{y_2=0} = 0, W_2^0(y)|_{y_2=1} = 0$ *and having the unit flux.*

Consider the following problem

$$\begin{cases} \Upsilon''(y_1) - \gamma_0 \Upsilon(y_1) = G(y_1), \quad y_1 \in \mathbb{R}, \\ \lim\limits_{|y_1|\to\infty} \Upsilon(y_1) = 0, \end{cases}$$

where $\gamma_0 = g\sigma^{-1}\cos\alpha > 0$.

Theorem 2. *Let $G \in C^{1+\delta}(\mathbb{R};\beta)$, $\beta > 0$. Then there exists a unique solution $\Upsilon \in C^{3+\delta}(\mathbb{R};\beta)$ and the following estimate*

$$\|\Upsilon\|_{C^{3+\delta}(\mathbb{R};\beta)} \leq c\|G\|_{C^{1+\delta}(\mathbb{R};\beta)}$$

holds.

Successive approximations

Let a function $H_0 \in C^{l+3+\delta}(\mathbb{R};\beta)$ be such that

$$\int_{-\infty}^{\infty} \Big(\frac{1}{2\nu} H_0'(y_1) Q^0(y_1) + \frac{3}{2} H_0(y_1) \Big)\, dy_1 = \kappa_0 \neq 0. \qquad (1)$$

We chose H_0 as a solution of the problem

$$\begin{cases} H_0''(y_1) - \gamma_0 H_0(y_1) = h_0(y_1), \\ \lim\limits_{|y_1| \to \infty} H_0(y_1) = 0, \end{cases} \qquad (2)$$

where $\gamma_0 = g\sigma^{-1}\cos\alpha$, $h_0 \in C^{l+3+\delta}(\mathbb{R};\beta)$,

$$\|h_0\|_{C^{l+1+\delta}(\mathbb{R};\beta)} \leq c\varepsilon \ \text{ and } \int_{\mathbb{R}} h_0(y_1)\, dy_1 = -1. \qquad (3)$$

Relations (3) are possible if there holds the following condition

$$0 < \beta \leq c_0 \varepsilon \leq \beta_*.$$

Successive approximations

Then

$$\|H_0\|_{C^{l+3+\delta}(\mathbb{R};\beta)} \le c\|h_0\|_{C^{l+1+\delta}(\mathbb{R};\beta)} \le c\varepsilon.$$

Integrating (2) we get

$$-1 = \int_{\mathbb{R}} h_0(y_1)\,dy_1 = \int_{\mathbb{R}} \big(H_0''(y_1) - \gamma_0 H_0(y_1)\big)\,dy_1$$

$$= -g\sigma^{-1}\cos\alpha \int_{\mathbb{R}} H_0(y_1)\,dy_1.$$

On the other hand, integrating by parts we obtain

$$\kappa_0 = \int_{-\infty}^{\infty} \Big(\frac{1}{2\nu}H_0'(y_1)Q^0(y_1) + \frac{3}{2}H_0(y_1)\Big)\,dy_1 = 3\int_{\mathbb{R}} H_0(y_1)\,dy_1$$

$(Q_0(y_1) = -3\nu y_1)$. From this formula we conclude that

$$\kappa_0 = \frac{3\sigma}{g\cos\alpha} \neq 0.$$

Take $\Psi = \chi_* H_0 + \Upsilon$, where the constant χ_* will be defined later:

$$\begin{cases} -\nu\Delta \mathbf{v} + \nabla q = -(\mathbf{v}^0 \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{v}^0 + \\ +\mathbf{F}(\mathbf{u}, q, \chi_* H_0 + \Upsilon) \quad \text{in } \Omega_0, \\ div\, \mathbf{v} = 0 \quad \text{in } \Omega_0, \\ v_1\big|_{y_2=0} = A_1, \quad v_2\big|_{y_2=0} = A_2, \\ v_2|_{y_2=1} = \chi_* \frac{g \sin\alpha}{2\nu} H_0' + \frac{g\sin\alpha}{2\nu}\Upsilon' + B(\chi_* H_0 + \Upsilon), \\ \nu\Big(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\Big)\Big|_{x_2=1} = g\sin\alpha\chi_* H_0 + g\sin\alpha\Upsilon + \\ +D(\mathbf{u}, \chi_* H_0 + \Upsilon), \\ \Upsilon'' - g\sigma^{-1}\cos\alpha\Upsilon = \chi_*(-H_0'' + g\sigma^{-1}\cos\alpha H_0) + \\ +\sigma^{-1}\Big(-q(y) + 2\nu\frac{\partial v_2}{\partial y_2}\Big)\Big|_{y_2=1} + \Phi(\mathbf{u}, \chi_* H_0 + \Upsilon), \\ \lim\limits_{|y_1|\to\infty} \Upsilon(y_1) = 0. \end{cases}$$

Successive approximations

Take as zero approximation $(\mathbf{u}_0, p_0, \psi_0, \chi_0) = (\mathbf{v}^0, p^0, 1, 0)$. Define

$$\chi_1 = -\frac{1}{g\sin\alpha\, k_0}\int\limits_{S_0}\left(3\nu A_1 - A_2 Q^0\right)dy_1,$$

$\mathbf{u}_1 = \mathbf{v}^0 + \varepsilon\mathbf{v}_1$, $p_1 = p^0 + \varepsilon q_1$, where

$$\begin{cases} -\nu\Delta\mathbf{v}_1 + \nabla q_1 = 0 \quad \text{in } \Omega_0, \\ div\,\mathbf{v}_1 = 0 \quad \text{in } \Omega_0, \\ \mathbf{v}_1\big|_{y_2=0} = \mathbf{A} = (A_1, A_2), \\ v_{2\,1}|_{y_2=1} = \chi_1 \dfrac{g\sin\alpha}{2\nu}H_0', \\ \nu\left(\dfrac{\partial v_{1\,1}}{\partial y_2} + \dfrac{\partial v_{2\,1}}{\partial y_1}\right)\Big|_{y_2=1} = \chi_1 g\sin\alpha H_0; \end{cases}$$

and $\psi_1 = 1 + \varepsilon(\chi_1 H_0 + \Upsilon_1)$, where

$$\begin{cases} \Upsilon_1'' - g\sigma^{-1}\cos\alpha\Upsilon_1 = \chi_1(-H_0'' + g\sigma^{-1}\cos\alpha H_0) + \\ +\sigma^{-1}\left(-q_1(y) + 2\nu\dfrac{\partial v_{2\,1}}{\partial y_2}\right)\Big|_{y_2=1} + \Phi(\mathbf{u}_1, \psi_0), \\ \lim\limits_{|y_1|\to\infty}\Upsilon_1(y_1) = 0. \end{cases}$$

Successive approximations

The following approximations are defined by

$$\mathbf{u}_{n+1} = \mathbf{v}^0 + \varepsilon \mathbf{v}_{n+1}, \quad p_{n+1} = p^0 + \varepsilon q_{n+1}, \quad \psi_{n+1} = 1 + \varepsilon(\chi_{n+1} H_0 + \Upsilon_{n+1}),$$

where

$$\chi_{n+1} = -(g \sin \alpha k_0)^{-1} \Big[\int\limits_{\Omega_0} \big(-(\mathbf{v}^0 \cdot \nabla)\mathbf{v}_n - (\mathbf{v}_n \cdot \nabla)\mathbf{v}^0 \big) \cdot \mathbf{W}^0 \, dy$$
$$+ \int\limits_{\Omega} \mathbf{F}(\mathbf{u}_n, p_n, \chi_n H_0 + \Upsilon_n) \cdot \mathbf{W}^0 dy + \int\limits_{S_0} \Big(3\nu A_1 - A_2 Q^0 \Big) \, dy_1$$
$$+ \int\limits_{\Gamma_0} \Big(B(\chi_n H_0 + \Upsilon_n) Q^0 + \tfrac{3}{2} D(\mathbf{u}_n, \chi_n H_0 + \Upsilon_n) \Big) \, dy_1$$
$$+ g \sin \alpha \int\limits_{\Gamma_0} \big(\tfrac{1}{2\nu} \Upsilon_n' Q^0 + \tfrac{3}{2} \Upsilon_n \big) dy_1 \Big],$$

Successive approximations

$(\mathbf{v}_{n+1}, q_{n+1})$ are solutions of the following problems:

$$\begin{cases} -\nu\Delta\mathbf{v}_{n+1} + \nabla q_{n+1} = -(\mathbf{v}^0\cdot\nabla)\mathbf{v}_n - (\mathbf{v}_n\cdot\nabla)\mathbf{v}^0 + \\ +\mathbf{F}(\mathbf{u}_n, p_n, \chi_n H_0 + \Upsilon_n) \quad \text{in } \Omega_0, \\ div\, \mathbf{v}_{n+1} = 0 \quad \text{in } \Omega_0, \\ \mathbf{v}_{n+1}\big|_{y_2=0} = \mathbf{A}, \\ v_{2\,n+1}|_{y_2=1} = \chi_{n+1}\dfrac{g\sin\alpha}{2\nu}H_0' + \dfrac{g\sin\alpha}{2\nu}\Upsilon_n' + B\big(\chi_n H_0 + \Upsilon_n\big), \\ \nu\Big(\dfrac{\partial v_{1\,n+1}}{\partial y_2} + \dfrac{\partial v_{2\,n+1}}{\partial y_1}\Big)\Big|_{y_2=1} = \chi_{n+1} g\sin\alpha\, H_0 + g\sin\alpha\,\Upsilon_n + \\ +D\big(\mathbf{u}_n, \chi_n H_0 + \Upsilon_n\big). \end{cases}$$

Finally, Υ_{n+1} are solutions of

$$\begin{cases} \Upsilon''_{n+1} - g\sigma^{-1}\cos\alpha\Upsilon_{n+1} = \chi_{n+1}(-H_0'' + g\sigma^{-1}\cos\alpha H_0) + \\ +\sigma^{-1}\Big(-q_{n+1}(y) + 2\nu\dfrac{\partial v_{2\,n+1}}{\partial y_2}\Big)\Big|_{y_2=1} + \Phi(\mathbf{u}_{n+1}, \chi_n H_0 + \Upsilon_n), \\ \lim\limits_{|y_1|\to\infty} \Upsilon_{n+1}(y_1) = 0. \end{cases}$$

Successive approximations

Notice that χ_{n+1} are chosen so that

$$q_{*\,n+1} = \lim_{x_1\to\infty} q_{n+1}(x) - \lim_{x_1\to-\infty} q_{n+1}(x) = 0.$$

Indeed,

$$q^+_{n+1} - q^-_{n+1} = \int\limits_{\Omega_0} \big(-(\mathbf{v}^0\cdot\nabla)\mathbf{v}_n - (\mathbf{v}_n\cdot\nabla)\mathbf{v}^0 + \mathbf{F}(\mathbf{u}_n, p_n, \chi_n H_0 + \Upsilon_n)\big)\cdot \mathbf{W}^0\, dy$$

$$+ \int\limits_{S_0} \Big(3\nu A_1 - A_2 Q^0\Big)\, dy_1 + \int\limits_{\Gamma_0} \Big(B(\chi_n H_0 + \Upsilon_n)Q^0 + \frac{3}{2} D(\mathbf{u}_n, \chi_n H_0 + \Upsilon_n)\Big)\, dy_1$$

$$+ \frac{3}{2}\int\limits_{\Gamma_0} (\chi_{n+1} g \sin\alpha H_0 + g\sin\alpha\Upsilon_n + D(\mathbf{u}_n, \chi_n H_0 + \Upsilon_n))dy_1,$$

or, equivalently,

Successive approximations

$$q^{+}_{n+1} - q^{-}_{n+1} = \chi_{n+1} g \sin\alpha\kappa_0 +$$

$$+ \int\limits_{\Omega_0} \big(-(\mathbf{v}^0 \cdot \nabla)\mathbf{v}_n - (\mathbf{v}_n \cdot \nabla)\mathbf{v}^0 + \mathbf{F}(\mathbf{u}_n, p_n, \chi_n H_0 + \Upsilon_n)\big) \cdot \mathbf{W}^0 \, dy$$

$$+ \int\limits_{S_0} \Big(3\nu A_1 - A_2 Q^0\Big)\, dy_1 + \int\limits_{\Gamma_0} \Big(B(\chi_n H_0 + \Upsilon_n) Q^0 + \frac{3}{2} D(\mathbf{u}_n, \chi_n H_0 + \Upsilon_n)\Big)\, dy_1$$

$$+ g \sin\alpha \int\limits_{\Gamma_0} \big(\frac{1}{2\nu} \Upsilon'_n Q^0 + \frac{3}{2} \Upsilon_n\big) dy_1,$$

Convergence of successive approximations

Assume that

$$\varepsilon|\sin\alpha|^{-1} \ll 1 \;\; as \;\; \varepsilon,\, \alpha \to 0.$$

Denote

$$Z_n = \|\mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)} + \|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + |\chi_n|.$$

For sufficiently small ε and $|\sin\alpha|$ it follows that

$$Z_{n+1} \le \varrho\big(Z_n + Z_{n-1}\big) + c\varepsilon\|\varphi_0\|_{C^{l+3+\delta}(-1,1)} \quad \text{with} \quad \varrho < \frac{1}{2}.$$

Hence if

$$Z_m \le c\varepsilon(1-2\varrho)^{-1}\|\varphi_0\|_{C^{l+3+\delta}(-1,1)} \equiv A_*, \quad m = n-1,\; m = n,$$

then

$$Z_{n+1} \le A_*.$$

Since $Z_0 = 0$ and Z_1 satisfies this inequality, it holds for all $m \ge 1$.

Convergence of successive approximations

Let us estimate the differences

$$R_n = \|\mathbf{v}_{n+1} - \mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_{n+1} - q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)}$$
$$+\|\Upsilon_{n+1} - \Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + |\chi_{n+1} - \chi_n|.$$

For sufficiently small ε and $|\sin\alpha|$ holds the estimate

$$R_n = \|\mathbf{v}_{n+1} - \mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_{n+1} - q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)}$$
$$+\|\Upsilon_{n+1} - \Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + |\chi_{n+1} - \chi_n|$$
$$\leq \varrho\Big(\|\mathbf{v}_n - \mathbf{v}_{n-1}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\mathbf{v}_{n-1} - \mathbf{v}_{n-2}\|_{C^{l+2+\delta}(\Omega_0;\beta)}$$
$$+\|q_n - q_{n-1}\|_{C^{l+1+\delta}(\Omega_0;\beta)} + \|q_{n-1} - q_{n-2}\|_{C^{l+1+\delta}(\Omega_0;\beta)}$$
$$+\|\Upsilon_n - \Upsilon_{n-1}\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + \|\Upsilon_{n-1} - \Upsilon_{n-2}\|_{C^{l+3+\delta}(\mathbb{R};\beta)}$$
$$+|\chi_n - \chi_{n-1}| + |\chi_{n-1} - \chi_{n-2}|\Big) = \varrho\big(R_{n-1} + R_{n-2}\big)$$

with $\varrho < \dfrac{1}{2}$.

Convergence of successive approximations

We have two inequalities

$$R_n \le 2A_* \quad \text{and} \quad R_n \le \varrho\big(R_{n-1} + R_{n-2}\big) \quad \text{with} \quad \varrho < \frac{1}{2}$$

from which it follows that

$$\|\mathbf{v}_m - \mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_m - q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)}$$

$$+\|\Upsilon_m - \Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + |\chi_m - \chi_n| \to 0 \quad \text{as} \quad n, m \to \infty.$$

Hence the sequence

$$\big\{\mathbf{v}_n, q_n, \Upsilon_n, \chi_n\big\} \mapsto \big\{\mathbf{v}, q, \Upsilon, \chi_*\big\}$$

in $C^{l+2+\delta}(\Omega_0; \beta) \times C^{l+1+\delta}(\Omega_0; \beta) \times C^{l+3+\delta}(\mathbb{R}; \beta) \times \mathbb{R}$.

Existence of the solution

Obviously,

$$\mathbf{u}(x) = \mathbf{v}^0(x) + \mathbf{v}(x),\ p(x) = p^0(x) + q(x),\ \psi(x_1) = 1 + \chi_* H_0(x_1) + \Upsilon(x_1)$$

is the solution of problem (FB).

THEOREM. *Let* $\varphi_0 \in C^{l+3+\delta}(-1,1),\ \operatorname{supp}\varphi_0 \subset (-1,1)$,

$$\varepsilon|\sin\alpha|^{-1} \ll 1 \ \ as \ \ \varepsilon,\ \alpha \to 0.$$

Then for sufficiently small ε *and* $|\alpha|$ *problem (FB) has a unique solution* $(\mathbf{u}, p, \psi)$*. This solution admits a representation*

$$\mathbf{u}(x) = \mathbf{v}^0 + \mathbf{v}, \quad p(x) = p^0(x) + q(x), \quad \psi(x_1) = 1 + \chi_* H_0(x_1) + \Upsilon(x_1),$$

where

$$v_1^0(x) = \frac{g\sin\alpha}{2\nu} x_2(2 - x_2),\ v_2^0(x) = 0,\ p^0(x) = g\cos\alpha(1 - x_2),$$

χ_* *is a constant,*

$$\mathbf{v} \in C^{l+2+\delta}(\Omega_0;\beta), \quad q \in C^{l+1+\delta}(\Omega_0;\beta), \quad H_0,\ \Upsilon \in C^{l+3+\delta}(\mathbb{R};\beta).$$