ON THE REGULARITY AND STABILITY OF THE FREE BOUNDARY OF OBSTACLE TYPE HETEROGENEOUS PROBLEMS

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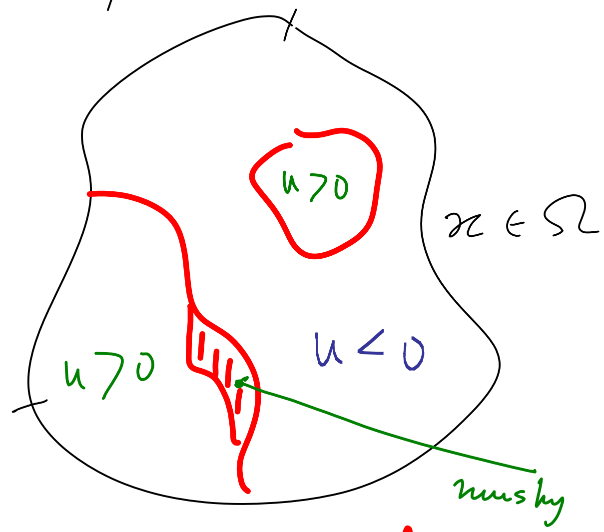


(*) $$Au + \zeta = f \quad \text{in } \Omega \subset \mathbb{R}^n, \text{ bd.}$$

$$Au = -\operatorname{div}(a(x, Du)), \quad f = f(x)$$

$$\lambda_\pm = \lambda_\pm(x) \geq 0$$

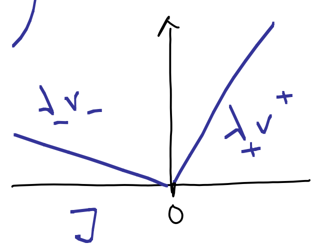
(ζ) a.e. in Ω
$$\begin{cases} \zeta = \lambda_+ \chi_+ - \lambda_- \chi_- \in \partial J(u) \\ 0 \leq \chi_{\{u \underset{<}{>} 0\}} \leq \chi_{\underline{+}} \leq 1 - \chi_{\{u \underset{>}{<} 0\}} \end{cases}$$

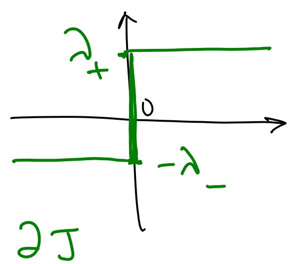
$\Phi_0 = \{u = 0\}$ free boundary

$\Phi = \partial\{u > 0\} \cup \partial\{u < 0\}$

$$J(v) = \int_\Omega \lambda_+ v^+ + \int_\Omega \lambda_- v^- \quad (v = v^+ - v^-)$$

$$\partial J(u) = \{\zeta : J(v) - J(u) \geq \langle \zeta, v - u \rangle, \ \forall v\}$$





Quasilinear 2nd order elliptic operator of Δ_p-type

$$\begin{cases} Au \simeq -\operatorname{div}\Big(\underbrace{M(x)\left(\kappa+|Du|^2\right)^{\frac{p-2}{2}}Du}_{a(x,Du)}\Big) \\ \dfrac{\partial u}{\partial \nu_A} = a(Du)\cdot\vec{n}\Big|_{\Gamma_N} = g\,, \quad u\Big|_{\Gamma_D} = h \end{cases}$$

$\kappa \in [0,1]$

- $a(x,0) = 0$

$\cdot\cdot$ $\displaystyle\sum_{i,j} \frac{\partial a_i}{\partial \eta_j}\xi_i\xi_j \geq c\left(\kappa+|\eta|^2\right)^{\frac{p-2}{2}}|\xi|^2$

$\cdot\cdot\cdot$ $\displaystyle\sum_{i,j}\left|\frac{\partial a_i}{\partial \eta_j}\right| \leq c\left(\kappa+|\eta|^2\right)^{\frac{p-2}{2}}$

$\cdot\cdot\cdot\cdot$ $|a(x,\eta)-a(y,\eta)| \leq C|x-y||\eta|^{p-1}$

$$\langle Au, v\rangle = \int_\Omega a(Du)\cdot Dv .$$

$$\langle L, v\rangle = \int_\Omega fv + \int_{\Gamma_N} gv\,; \quad V_h = \{v \in W^{1,p}(\Omega) : v = h \text{ on } \Gamma_D\}, \quad 1<p<\infty$$

Thm ($\exists!$) $\cdot$ For $h \in W^{1,p}(\Omega)$, $f \in L^\infty(\Omega)$, $g \in L^p(\Gamma_N)$ there exists a unique solution to the variational inequality problem

$$u \in V_h : \quad \langle Au, v-u\rangle + J(v) - J(u) \geq \langle L, v-u\rangle, \quad \forall v \in V_h$$

$\cdot\cdot$ If $h_k \xrightarrow{W^{1,p}} h$, $g_k \xrightarrow{L^p} g$, $f_k \xrightarrow{L^p} f$ then $u_k \to u$ in $W^{1,p}(\Omega)$.

NOTE. $Au = f - \lambda_+\chi_+ + \lambda_-\chi_- \in L^\infty(\Omega)$; $u \in C^{1,\alpha}_{loc}$ and solves **(*) (S)**

Other choices of the operator A

$$Au = -\Delta_{p(x)} u = -\nabla\cdot\left(|\nabla u|^{p(x)-2}\nabla u\right)$$

$p(x)$-Laplacian

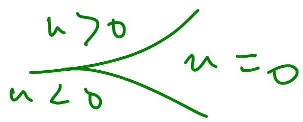
$1 < \underline{p} \le p(x) \le \overline{p} < \infty$

p Lipschitz continuous

$$A = -\Delta \quad (\text{Laplacian})$$

- If $A = -\Delta$ and $f = 0$ it has shown that $u \in C^{1,1}(\Omega)$ for $\lambda_\pm$ constants (Uraltseva, 2001) and $\lambda_\pm \in C^{0,1}$ (Shahgholian, etc.)

- • $A = -\Delta$, $f = 0$, $\lambda_\pm \in \mathbb{R}$ then $\mathcal{H}^{n-1}(\partial\{u>0\} \cup \partial\{u<0\}) < \infty$ (Weiss - 2001) and (!).

When $f \equiv 0$ we may have a situation 

(Uraltseva-Shahgholian-Weiss)

- • • $-\Delta_p$, $\lambda_\pm \in \mathbb{R}$ Edquist & Lindgren 2009

2012 Book by Petrosyan, Shahgholian, Uraltseva

Nondegeneracy of the free boundary (null mushy region)

(*) $\quad Au + \lambda_+ \chi_+ - \lambda_- \chi_- = f$ a.e. in Ω $\quad \left(0 \le \chi_{\{u>0\}} \le \chi_+ \le 1 - \chi_{\{u<0\}}\right)$

(#) $\quad \operatorname{meas}(\{u=0\}) = 0$ $\quad$ (Lebesgue measure in Ω or Q)

Note that $\{0<\chi_\pm<1\} \subset \{u=0\}$. So (#) implies $\operatorname{meas}(\{0<\chi_+<1\} \cup \{0<\chi_-<1\}) = 0$, which is sufficient to

$$\chi_+ = \chi_{\{u>0\}} \quad \text{and} \quad \chi_- = \chi_{\{u<0\}} \quad (!)$$

Thm 2 If $A0=0$ and $f>\lambda_+$ or $f<-\lambda_-$ a.e. in Ω then (#) holds!

When we have $Au=0$ a.e. in $\{u=0\}$, then from (*) we have

$$-\lambda_- \le \lambda_+ \chi_+ - \chi_- \lambda_- = f \le \lambda_+ \quad \text{a.e. impossible!}$$

NOTE When $f \equiv 0$ it is well-known that (#) may in general be violated.

Continuous Dependence of the Phases +/−

Thm 3 ($\lambda_\pm > 0$) Let $f_k \xrightarrow[k]{} f$, $g_k \xrightarrow[k]{} g$ and $h_k \xrightarrow[k]{} h$ s.t. $u_k \to u$ in $W^{1,q}$. Then

$$\chi_{\{u_k \gtrless 0\}} \xrightarrow[k]{} \chi_{\{u \gtrless 0\}} \quad \text{in } L^q(\Omega),\ 1 \le q < \infty,$$

provided **(#)** (nondegeneracy of Φ_0) holds, i.e. we have for u_k and u

(*) $$Au + \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} = f \quad \text{a.e. in } \Omega.$$

Proof: Let $\chi_{\{u^k \gtrless 0\}} \xrightarrow[k]{} \chi^*_\pm$ in $L^\infty(\Omega)$-weakly*. We have

$$0 = \int_\Omega u_k^\mp \chi_{\{u_k \gtrless 0\}} \to \int_\Omega u^\mp \chi^*_\pm = 0, \quad \text{and } \chi^*_\pm = 0 \text{ if } u \lessgtr 0 \quad \text{and}$$

we find $0 \le \chi^*_\pm \le 1 - \chi_{\{u \lessgtr 0\}}$. Since $\int_\mathcal{O} u_k^\pm \chi_{\{u_k \gtrless 0\}} \to \int_\mathcal{O} u^\pm \chi^*_\pm = \int_\mathcal{O} u^\pm$

for arbitrary $\mathcal{O} \subset \Omega$, we obtain $\chi_{\{u \gtrless 0\}} \le \chi^*_\pm$. But (#) implies $\chi_{\{u \gtrless 0\}} = 1 - \chi_{\{u \lessgtr 0\}}$

we conclude $\chi^*_\pm = \chi_{\{u \gtrless 0\}}$ and the convergence follows (strongly)

L^1- estimates on the two phases

For our class of operators A, we may extend a lemma by Brezis-Strauss

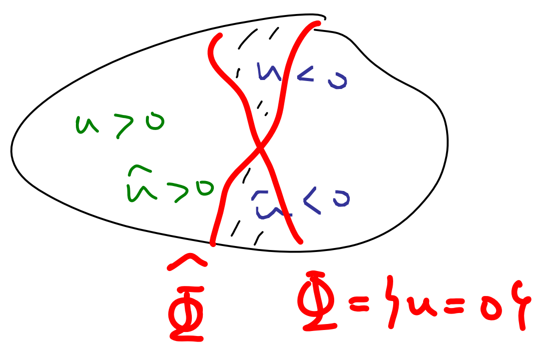
(BS) $$\|\zeta-\hat\zeta\|_{L^1(\Omega)} \le \|f-\hat f\|_{L^1(\Omega)} + \|g-\hat g\|_{L^1(\Gamma_N)} \qquad \begin{array}{l}\zeta\in\partial J(u)\\ \hat\zeta\in\partial J(\hat u)\end{array} \qquad h=\hat h$$

Thm 3 If nondegeneracy **(#)** holds (in particular, $\hat\chi_\pm = \chi_{\{\hat u \gtrless 0\}} = 1-\chi_{\{u\lessgtr 0\}}$) and $\lambda_+,\lambda_- \geq \lambda > 0$ in Ω, then

$$\operatorname{meas}(\{u>0\}\div\{\hat u>0\}) = 2\lambda\|\chi_\pm-\hat\chi_\pm\|_{L^1(\Omega)} \le \|f-\hat f\|_{L^1(\Omega)} + \|g-\hat g\|_{L^1(\Gamma_N)} \qquad (h=\hat h)$$

to **Observation:** $\zeta = \lambda_+\chi_+ - \lambda_-\chi_-$ and $\hat\zeta = \lambda_+\hat\chi_+ - \lambda_-\hat\chi_-$. Apply

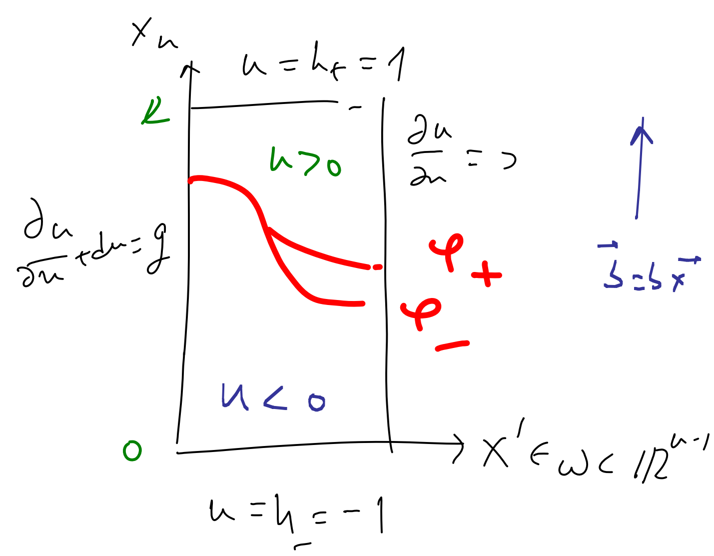
$$\begin{aligned}|\zeta-\hat\zeta| &= |\lambda_+(\chi_+-\hat\chi_+) - \lambda_-(\chi_--\hat\chi_-)|\\ &= |(\lambda_++\lambda_-)(\chi_+-\hat\chi_+)|\\ &\geq 2\lambda|\chi_+-\hat\chi_+|\end{aligned}$$



SPECIAL GEOMETRIES

i) Monotone in x_n: $\nabla_n u \geq 0$

$Au = -\Delta u + b\,\nabla_n u + cu$



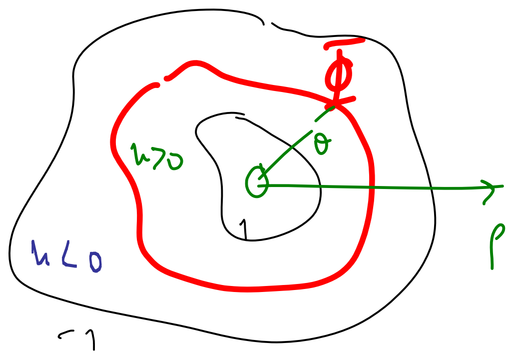
$\nabla_n f \geq 0$, $\nabla_n g \geq 0$ $\quad \Omega = \omega \times (0,\ell)$

$\varphi_{\pm}(x') = \inf_{\sup} \{x_n : u(x', x_n) >_{<} 0\}$

if $\varphi_+ = \varphi_- = \varphi$ $\quad \int_\Omega |\chi_+ - \hat{\chi}_+| = \int_\omega |\varphi - \hat{\varphi}| \leq \frac{1}{2\lambda}\{\int_\Omega |f - \hat{f}| + \int_\Gamma |g - \hat{g}|\}$

if also $\|\nabla\varphi\|_{L^\infty} \leq C$ we can estimate the Hölder norm $\|\varphi - \hat{\varphi}\|_{C^{0,\alpha}}$

ii) Starshaped in $\mathbb{R}^n$
with respect to a small ball $B_\delta(0)$:



$\rho = \varphi(\theta)$, $\theta \in \Theta_{n-1}$

$\partial_\rho u \leq 0$ yields $\varphi \in C^{0,1}_\Theta$

REGULARITY OF THE FREE BOUNDARY

$$f, \lambda_\pm \in C^0(\Omega) \cap W^{1,1}_{loc}(\Omega).$$

$$\sum_{i,j} \left| \frac{\partial^2 a_k}{\partial x_i \partial x_j}(x,\eta) \right| \le c \left(k + |\eta|^2\right)^{\frac{p-1}{2}}$$

$$\sum_{i,j} \left| \frac{\partial^2 a_k}{\partial \eta_i \partial x_j}(x,\eta) \right| \le c \left(k + |\eta|^2\right)^{\frac{p-2}{2}}$$

Thm 4 Suppose that $u \in C^{1,\alpha}(\Omega) \cap W^{2,1}_{loc}(\Omega)$ is a solution of $Au + \lambda_+ \chi_+ - \lambda_- \chi_- = f$ a.e. in Ω

Then $Au \in BV_{loc}(\Omega)$.

NOTE: The proof uses a remark of Brezis-Kinderlehrer for the obstacle problem. But here $0 \le \chi_{\{u>0\}} \le \chi_+ \le 1 - \chi_{\{u<0\}}$

Thm 5 If $(\lambda_+ + \lambda_-) \geq \lambda > 0$, and the problem is nondegenerate, i.e.

$$(\#) \quad \text{meas}(\{u=0\}) = 0 \qquad (\text{Lebesgue measure in } \Omega \text{ or } Q)$$

Both $\chi_\pm = \chi_{\{u \gtrless 0\}} \in BV_{loc}$ and the free boundary is, up to a set of null perimeter, the union of at most a countable family of C^1 hypersurfaces.

NOTE: Since $\chi_- = 1 - \chi_+$ a.e., we have

$$\chi_+ = \frac{f - Au + \lambda_-}{\lambda_+ + \lambda_-}$$