L_p-estimates of solutions of some problems of hydrodynamics and magnetohydrodynamics

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We consider the following free boundary problems related to evolution of isolated mass of a viscous capillary fluid:

1. *Incompressible fluid*

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\ \nabla\cdot\boldsymbol{v}(x,t) = 0, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad x\in\Omega_t, \quad t>0, \\ S(\boldsymbol{v})\boldsymbol{n} - p\boldsymbol{n} = \sigma\mathcal{H}\boldsymbol{n}, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad x\in\Gamma_t, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x\in\Omega_0, \end{cases} \tag{.1}$$

where Ω_t is a bounded variable domain in $\mathbb{R}^3$ filled with the fluid, $\Gamma = \partial\Omega_t$, V_n is the velocity of evolution of Γ_t in the direction of the exterior normal $\boldsymbol{n}$, $\mathcal{H}$ is the doubled mean curvature of Γ_t, σ is a positive coefficient of the surface tension, $S(\boldsymbol{v}) = (\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})_{i,j=1,2,3}$ is the doubled rate-of-strain tensor;

2. *Compressible fluid*

$$\begin{cases} \rho(\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) - \nabla\cdot T(\boldsymbol{v}) + p(\rho) = 0, \\ \rho_t + \nabla\cdot(\rho\boldsymbol{v}) = 0, \quad x \in \Omega_t, \\ T(\boldsymbol{v})\boldsymbol{n} - p(\rho) = \sigma\mathcal{H}\boldsymbol{n}, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad z \in \Gamma_t, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad \rho(x,0) = \rho_0(x), \quad x \in \Omega_0, \end{cases} \tag{.2}$$

where $\rho(x,t)$ is the positive density of the fluid, $p(\rho)$ is the positive strictly increasing pressure function, $T(\boldsymbol{v}) = \mu S(\boldsymbol{v}) + \mu_1 I\nabla\boldsymbol{v}$, μ, μ_1 are positive viscosity coefficients;

3. *MHD problem for the incompressible fluid.*

Navier-Stokes equations:

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla\cdot S(\boldsymbol{v}) + \nabla p - \nabla\cdot T_M(\boldsymbol{H}) = 0, \\ \nabla\cdot\boldsymbol{v}(x,t) = 0, \quad x\in\Omega_{1t}, \;\; t>0, \end{cases} \tag{.3}$$

Maxwell equations:

$$\begin{cases} \mu\boldsymbol{H}_t = -rot\boldsymbol{E}, \quad \nabla\cdot\boldsymbol{H} = 0, \quad x\in\Omega_{1t}\cup\Omega_{2t}, \\ rot\boldsymbol{H} = \alpha(\boldsymbol{E} + \mu(\boldsymbol{v}\times\boldsymbol{H}), \quad x\in\Omega_{1t}, \\ rot\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{E} = 0, \quad x\in\Omega_{2t}, \end{cases} \tag{.4}$$

boundary, jump and initial conditions:

$$\begin{cases} [\mu\boldsymbol{H}\cdot\boldsymbol{n}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \\ S(\boldsymbol{v})\boldsymbol{n} - p\boldsymbol{n} + [T_M(\boldsymbol{H})\boldsymbol{n}] = \sigma\mathcal{H}\boldsymbol{n}, \quad x\in\Gamma_t, \\ \boldsymbol{H}\cdot\boldsymbol{n} = 0, \quad x\in S, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x\in\Omega_{10}, \\ \boldsymbol{H}(x,0) = \boldsymbol{H}_0(x), \quad x\in\Omega_{10}\cup\Omega_{20}. \end{cases} \tag{.5}$$

where $\boldsymbol{H}(x,t)$ and $\boldsymbol{E}(x,t)$ are magnetic and electric fields, $\boldsymbol{H}_\tau = \boldsymbol{H} - \boldsymbol{n}(\boldsymbol{H}\cdot\boldsymbol{n})$ is the tangential component of $\boldsymbol{H}$, Ω_{2t} is a variable bounded vacuum region surrounding Ω_{1t}, $\Omega = \Omega_{1t}\cup\Gamma_t\cup\Omega_{2t}$, $S = \partial\Omega$, μ is a positive piecewise constant function equal to μ_i in Ω_{it}, $\alpha = const > 0$, $T_M = \mu(\boldsymbol{H}\otimes\boldsymbol{H} + \frac{1}{2}I|\boldsymbol{H}|^2)$ is the magnetic stress tensor, $[u] = u^{(1)} - u^{(2)}$, is the jump of the function $u(x,t)$ given in Ω on Γ_t; $u^{(i)} = u(x,t)$, $x\in\overline{\Omega}_{it}$.

We work in anisotropic Sobolev spaces

$$W_p^{l,l/2}(\mathcal{D}_T) = L_p(0,T;W_p^l(\mathcal{D}))\cap L_p(\mathcal{D};W_p^{l/2}(0,T)),$$

$p>3$, $l>0$, $\mathcal{D}_T = \mathcal{D}\times(0,T)$, $\mathcal{D}\subset\mathbb{R}^n$, with the norm given by

$$\begin{aligned}\|u\|^p_{W_p^{l,l/2}(Q_T)} &= \int_0^T \|u\|^p_{W_p^l(\mathcal{D})}dt + \int_{\mathcal{D}}\|u\|^p_{W_p^{l/2}(0,T)}dx\\ &= \|u\|^p_{W_p^{l,0}(D_T)} + \|\boldsymbol{u}\|^p_{W_p^{0,l/2}(0,T)}.\end{aligned}$$

By $W_p^l(\mathcal{D})$ we mean (isotropic) space with the norm defined by

$$\|u\|^p_{W_p^l(\mathcal{D})} = \sum_{|j|=l}\int_{\mathcal{D}}|D^j u(x)|^p dx,$$

if l is an integer, and

$$\|u\|^p_{W^l_p(\mathcal{D})} = \|u\|^p_{W^{[l]}_p(\mathcal{D})} + \sum_{|j|=[l]} \int_{\mathcal{D}} \int_{\mathcal{D}} \frac{|D^j u(x) - D^j u(y)|^p}{|x-y|^{n+p\lambda}} dxdy,$$

if $l = [l] + \lambda$, $0 < \lambda < 1$.

Transformation of free boundary problems to problems in fixed domains.

Equations of the free boundary:

$$\Gamma_t : x = y + \boldsymbol{N}(y) r(y,t), \quad y \in \mathcal{G}, \tag{.6}$$

where $\mathcal{G}$ is a smooth surface close to Γ_0;

Hanzawa coordinate transformation:

$$x = y + \boldsymbol{N}^*(y) r^*(y,t), \quad y \in \Omega, \tag{.7}$$

$\boldsymbol{N}^*, r^*$: extensions of $\boldsymbol{N}, r$ from $\mathcal{G}$ into Ω.

1. *Incompressible fluid; linearized problem (0.1) written in the coordinates* y:

Making the coordinate transformation (0.7) and omitting the nonlinear terms we arrive at the linear problem

$$\begin{cases} \boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \\ \nabla\cdot\boldsymbol{v} = f(y,t), \quad y\in\mathcal{F}, \ t>0, \\ S(\boldsymbol{v})\boldsymbol{N}(x) - p\boldsymbol{N} + \sigma\boldsymbol{N}(y)\mathfrak{L}r = \boldsymbol{d}(y,t), \\ r_t(y,t) - \boldsymbol{v}(y,t)\cdot\boldsymbol{N}(y) = g(y,t), \quad y\in\mathcal{G}, \\ \boldsymbol{v}(y,0) = \boldsymbol{v}_0(y), \quad y\in\mathcal{F}_1, \quad r(y,0) = r_0(y), \\ y\in\mathcal{G}, \end{cases} \tag{.8}$$

$\mathfrak{L}r = -(\Delta_{\mathcal{G}} + b(x))r$, $\Delta_{\mathcal{G}}$ is the Laplace-Beltrami operator on $\mathcal{G}$, $\mathcal{F}$ is the domain bounded by $\mathcal{G}$. We set $Q_T = \mathcal{F}\times(0,T)$, $G_T = \mathcal{G}\times(0,T)$.

Theorem 1. *Assume that* $\boldsymbol{f}\in L_p(Q_T)$, $f\in W_p^{1,0}(Q_T)$, $f = \nabla\boldsymbol{F} + f'$, $\boldsymbol{F}, f' \in W_p^{0,1}(Q_T)$, $\boldsymbol{d}_\tau \in W_p^{1-1/p,1/2-1/(2p)}(G_T)$, $\boldsymbol{d}\cdot\boldsymbol{N} \in W_p^{1-1/2p,0}(G_T)$,

$\boldsymbol{g} \in W_p^{2-1/p,1-1/2p}(G_T)$, $\boldsymbol{v}_0 \in W_p^{2-2/p}(\mathcal{F})$, $r_0 \in W_p^{3-2/p}(\mathcal{G})$, *and let the compatibility conditions*

$$\begin{aligned} &\nabla \cdot \boldsymbol{v}_0(x) = f(x,0), \quad x \in \mathcal{F}, \\ &\nu(S(\boldsymbol{v}_0)\boldsymbol{N})_\tau = \boldsymbol{d}_\tau(x,0), \quad x \in \mathcal{G}, \quad \text{if} \quad p \geq 3 \end{aligned}$$

be satisfied. Then the problem (.8) has a unique solution $\boldsymbol{v}, p, \rho$ *such that* $\boldsymbol{v} \in W_p^{2,1}(Q_T)$, $\nabla p \in L_p(Q_T)$, $p \in W_p^{1-1/p,0}(G_T)$, $r \in W_p^{3-1/p,0}(G_T)$, $r_t \in W_p^{2-1/p,1-1/2p}(G_T)$, $r(\cdot,t) \in W_p^{3-2/p}(\mathcal{G})$, $\forall t \in (0,T)$, *and the solution satisfies the inequality*

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_p^{2,1}(Q_T)} + \|\nabla p\|_{L_p(Q_T)} \\ &+ \|r\|_{W_p^{3-1/p,0}(G_T)} + \|r_t\|_{W_2^{2-1/p,1-1/2p}(G_T)} \\ &\leq c\Big(\|\boldsymbol{f}\|_{L_p(Q_T)} + \|f\|_{W_p^{1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_p^{0,1}(Q_T)} \\ &+ \|f'\|_{W_p^{0,1}(Q_T)} + \|\boldsymbol{d}_\tau\|_{W_p^{1-1/p,l/2-1/(2p)}(G_T)} \\ &+ \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_p^{1-1/p,0}(G_T)} + \|g\|_{W_p^{2-1/p,1-1/2p}(G_T)} \\ &+ \|\boldsymbol{v}_0\|_{W_p^{2-2/p}(\mathcal{F})} + \|r_0\|_{W_p^{3-2/p}(\mathcal{G})}\Big), \end{aligned} \tag{.9}$$

Theorem 1 implies the solvability of the problem (.1) and "maximal regularity" estimates of the solution.

The proof of (.9) is based on the analysis of the model problem in the half-space $\mathbb{R}^3_+ = \{x_3 > 0\}$ (we restrict ourselves to the case $\boldsymbol{f} = 0$, $f = 0$, $\boldsymbol{v}_0 = 0$, $r_0 = 0$):

$$\begin{cases} \boldsymbol{v}_t(x) - \nu\nabla^2\boldsymbol{v}(x,t) + \nabla p(x,t) = 0, \\ \nabla\cdot\boldsymbol{v}(x,t) = 0, \quad x_3 > 0, \quad t > 0, \\ \nu\Big(\dfrac{\partial v_3}{\partial x_j} + \dfrac{\partial v_j}{\partial x_3}\Big) = b_j(x',t), \quad j = 1,2, \\ -p + 2\nu\dfrac{\partial v_3}{\partial x_3} - \sigma\nabla^{2'} r = b_3(x',t), \\ r_t + v_3(x,t) = g(x,t), \quad x_3 = 0, \quad x' = (x_1,x_2), \\ \boldsymbol{v}(x,0) = 0, \quad \rho(x,0) = 0. \end{cases} \tag{.10}$$

We solve this problem in anisotropic Sobolev space with exponential weight $e^{-\gamma t}$, $\gamma > 0$.

Theorem 2. *Let* $\mathcal{R}_+ = \mathbb{R}^3_+ \times \mathbb{R}_+$ ($x \in \mathbb{R}^3_+ = \{x_3 > 0\}, t > 0$), $\mathcal{R}'_+ = \mathbb{R}^2 \times \mathbb{R}_+$. *Assume that* $e^{-\gamma t} b_j \in W_p^{1-1/p,1/2-1/(2p)}(\mathcal{R}'_+)$, $j = 1, 2$, $e^{-\gamma t} b_3 \in W_p^{1-1/p,0}(\mathcal{R}'_+)$, $e^{-\gamma t} g \in W_p^{2-1/p,1-1/2p}(\mathcal{R}'_+)$ *with* $\gamma > 0$ *and that*

$$b_1(x', 0) = b_2(x', 0) = 0, \quad \text{if} \quad p \geq 3.$$

Then the problem (.10) has a unique solution $(\boldsymbol{v}, p, r)$ *such that* $e^{-\gamma t}\boldsymbol{v} \in W_p^{2,1}(\mathcal{R}_+)$, $e^{-\gamma t}\nabla p \in L_p(\mathcal{R}_+)$, $e^{-\gamma t} p \in W_p^{1-1/p,0}(\mathcal{R}'_+)$, $e^{-\gamma t} r \in W_p^{3-1/p,0}(\mathcal{R}'_+)$, $e^{-\gamma t} r_t \in W_p^{2-1/p,0}(\mathcal{R}'_+)$, *and*

$$\begin{aligned}
&\|e^{-\gamma t}\boldsymbol{v}\|_{W_p^{2,1}(\mathcal{R}_+)} + \|e^{-\gamma t}\nabla p\|_{L_p(\mathcal{R}_+)} \\
&+ \|e^{-\gamma t} r\|_{W_p^{3-1/p,0}(\mathcal{R}'_+)} + \|e^{-\gamma t} r_t\|_{W_2^{2-1/p,1-1/2p}(\mathcal{R}'_+)} \\
&\leq c\Big(\sum_{i=1}^{2} \|e^{-\gamma t} b_i^{(0)}\|_{W_p^{1-1/p,l/2-1/(2p)}(\mathbb{R}^3)} \\
&+ \|e^{-\gamma t} b_3\|_{W_p^{1-1/p,0}(\mathcal{R}'_+)} + \|e^{-\gamma t} g\|_{W_p^{2-1/p,1-1/2p}(\mathcal{R}'_+)}\Big),
\end{aligned} \tag{.11}$$

where $b_i^{(0)}(x', t) = b_i(x', t)$ for $t > 0$, $b_i^{(0)}(x', t) = 0$ for $t < 0$.

Using the Fourier transform in x_1, x_2 and the Laplace transform with respect to t,

$$\widetilde{u}(\xi, s) \equiv Fu(x', t) = \int_0^\infty e^{-st} dt \int_{\mathbb{R}^2} e^{-i\xi\cdot x'} u(\xi, t) dx', \tag{.12}$$

where $s = \gamma + i\xi_0$, $\gamma > 0$, $\xi_0 \in \mathbb{R}$, we reduce (.10) to the boundary value problem on the half-axis $\mathbb{R}_+ = \{x_3 > 0\}$:

$$\begin{cases} \nu\Big(r_1^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j + i\xi_j \widetilde{p} = 0 \quad j = 1, 2, \\ \nu\Big(r_1^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_3 + \dfrac{d\widetilde{p}}{dx_3} = 0, \\ i\xi_1 \widetilde{v}_1 + i\xi_2 \widetilde{v}_2 + \dfrac{d\widetilde{v}_3}{dx_3} = 0, \quad x_3 > 0, \\ \nu\Big(\dfrac{d\widetilde{v}_j}{dx_3} + i\xi_j \widetilde{v}_3\Big) = \widetilde{b}_j, \quad j = 1, 2, \\ -\widetilde{p} + 2\nu \dfrac{d\widetilde{v}_3}{dx_3} + \sigma|\xi|^2 \widetilde{r} = \widetilde{b}_3, \\ s\widetilde{r} + \widetilde{v}_3 = \widetilde{g}, \quad x_3 = 0, \\ \widetilde{\boldsymbol{v}} \to 0, \quad \widetilde{p} \to 0, \quad (x_3 \to \infty), \end{cases} \tag{.13}$$

where $\xi = (\xi_1, \xi_2)$, $r_1 = r_1(s,\xi) = \sqrt{s\nu^{-1} + |\xi|^2}$, $-\pi \le arg\ r_1 < \pi$, $\mathrm{Re} s = \gamma > 0$. Eliminating $\widetilde{r}$ we obtain

$$\begin{cases} \nu\Big(r_1^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j + i\xi_j\widetilde{p} = 0 \quad j = 1,2, \\ \nu\Big(r_1^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j + \dfrac{d\widetilde{p}}{dx_3} = 0, \\ i\xi_1\widetilde{v}_1 + i\xi_2\widetilde{v}_2 + \dfrac{d\widetilde{v}_3}{dx_3} = 0, \quad x_3 > 0, \\ \nu\Big(\dfrac{d\widetilde{v}_j}{dx_3} + i\xi_j\widetilde{v}_3\Big) = \widetilde{b}_j, \quad j = 1,2, \\ -\widetilde{p} + 2\nu\dfrac{d\widetilde{v}_3}{dx_3} - \dfrac{\sigma}{s}|\xi|^2\widetilde{v}_3 = \widetilde{b}_3 - \dfrac{\sigma}{s}|\xi|^2\widetilde{g}, \quad x_3 = 0, \\ \widetilde{\boldsymbol{v}} \to 0, \quad \widetilde{p} \to 0, \quad (x_3 \to \infty). \end{cases} \tag{.14}$$

The solution of (.14) is given by the formula

$$\begin{aligned}
\widetilde{v}_i &= -\frac{1-\delta_{i3}}{\nu r_1} e_0(x_3)\widetilde{b}_i \\
&+ \frac{e_0(x_3)}{\nu^2 r_1 (r_1+|\xi|)P} \sum_{j=1}^{3} U_{ij}\widetilde{d}_j \\
&+ \frac{e_1(x_3)}{\nu^2 (r_1+|\xi|)P} \sum_{j=1}^{3} V_{ij}\widetilde{d}_j, \quad i=1,2,3, \\
\widetilde{p} &= \frac{r_1 s}{\nu P}\Big[\Big(2\nu + \frac{\sigma\xi^2}{s r_1}\Big)(i\xi_1\widetilde{b}_1 + i\xi_2\widetilde{b}_2) \\
&- \nu\Big(r_1 + \frac{\xi^2}{r_1}\Big)\widetilde{d}_3 e^{-|\xi|x_3},
\end{aligned} \tag{.15}$$

where $\widetilde{d}_\alpha = \widetilde{b}_\alpha$, $\alpha = 1,2$, $\widetilde{d}_3 = \widetilde{b}_3 - \frac{\sigma}{s}|\xi|^2\widetilde{g}$,

$$e_0(x_3) = e^{-r_1 x_3}, \quad e_1(x_3) = \frac{e^{-r_1 x_3} - e^{-|\xi|x_3}}{r_1 - |\xi|},$$

$$P = \frac{s}{\nu}\Big(\frac{s}{\nu} + 4|\xi|^2 \frac{r_1}{r_1+|\xi|} + \frac{\sigma|\xi|^3}{\nu s}\Big).$$

If Re$s \geq \gamma > 0$, then

$$\frac{\gamma^2}{\nu^2} + |s||\xi|^2 + |s|^2 + \sigma|\xi|^3 \leq c(\gamma)|P|. \tag{.16}$$

The elements U_{ij}, V_{ij} are given by

$$\begin{aligned}
U_{\alpha\beta} &= \xi_\alpha\xi_\beta((3r_1 - |\xi|)s + \frac{\sigma}{\nu}|\xi|^2),\\
U_{\alpha 3} &= i\xi_\alpha r_1 s(r_1 - |\xi|),\\
U_{3\beta} &= -i\xi_\beta r_1 s(r_1 - |\xi|), \quad \alpha, \beta = 1, 2,\\
U_{33} &= -|\xi| r_1 s(r_1 + |\xi|),
\end{aligned}$$

$$\begin{aligned}
V_{\alpha\beta} &= -\xi_\alpha\xi_\beta(2r_1 s + \frac{\sigma}{\nu|}\xi|^2),\\
V_{3\beta} &= -i\xi_\beta s(r_1^2 + |\xi|^2),\\
V_{\alpha 3} &= -i\xi_\alpha|\xi|(r_1^2 + |\xi|^2),\\
V_{33} &= |\xi| s(r_1^2 + |\xi|^2),
\end{aligned}$$

Our aim is to estimate the derivatives $D_t^k D_x^j \boldsymbol{v}$, $2k + |j| \le 2$ and ∇p. They can be written as linear combinations of $D_t^k D_x^j F^{-1} \frac{e_0(x_3)}{r} d_\alpha$, $\alpha = 1, 2$, and

$$\begin{aligned} &F^{-1} Q_0(\xi, s) e_0(x_3) \tilde{d}_0(\xi, s), \\ &F^{-1} Q_1(\xi, s) e_1(x_3) \tilde{d}_1(\xi, s), \\ &F^{-1} Q_2(\xi, s) e^{-|\xi| x_3} \tilde{d}_2(\xi, s). \end{aligned} \tag{.17}$$

We use the Marcinkiewicz-Mihlin-Lizorkin (MML) theorem and the results of I.Mogilevskii and L.Volevich.

The MML theorem: *Let*

$$v(x', t) = F^{-1} m(\xi, s) \tilde{f}(\xi, s),$$

where $Re s = \gamma > 0$ *and* $\widetilde{f} = Ff$*. If*

$$M_p(m) = \sup_{\xi,s} |m(\xi, s)| + \sum_{\alpha=1}^{2} \sup_{\xi,s} |\xi_\alpha \frac{\partial m}{\partial \xi_\alpha}|$$
$$+ \sup_{\xi,s} |\xi_1 \xi_2 \frac{\partial^2 m}{\partial \xi_1 \partial \xi_2}| + \sup_{\xi,s} |s \frac{\partial m}{\partial s}| + \sum_{\alpha=1}^{2} \sup_{\xi,s} |s \xi_\alpha \frac{\partial^2 m}{\partial s \partial \xi_\alpha}| \qquad (.18)$$
$$+ \sup_{\xi,s} |s \xi_1 \xi_2 \frac{\partial^3 m}{\partial s \partial \xi_1 \partial \xi_2}| < \infty,$$

then

$$\|e^{-\gamma t} \boldsymbol{v}\|_{L_p(\mathbb{R}^3)} \leq c M_p(m) \|e^{-\gamma t} \boldsymbol{f}\|_{L_p(\mathbb{R}^3)}.$$

The set of m satisfying (.18) is denoted by $\mathcal{M}$.

Proposition 1. *If* $Q_0(\xi,s)|\xi|^{-1}\in\mathcal{M}$, *then*

$$\|e^{-\gamma t}w_0\|_{L_p(\mathcal{R}_+)}\le cM_p(Q_0|\xi|^{-1})\|d_0\|_{W_p^{1-1/p,0}(\mathcal{R}'_+)}. \tag{.19}$$

If $Q_1(\xi,s)|\xi|^{-1}r_1^{-1}\in\mathcal{M}$, *then*

$$\|e^{-\gamma t}w_1\|_{L_p(\mathcal{R}_+)}\le cM_p(Q_1|r_1^{-1}\xi|^{-1})\|d_1\|_{W_p^{1-1/p,0}(\mathcal{R}'_+)}. \tag{.20}$$

If $Q_2(\xi,s)|\xi|^{-1}\in\mathcal{M}$, *then*

$$\|e^{-\gamma t}w_2\|_{L_p(\mathcal{R}_+)}\le cM_p(Q_2|\xi|^{-1})\|d_2\|_{W_p^{1-1/p,0}(\mathcal{R}'_+)}. \tag{.21}$$

We consider w_0. Following L.Volevich, we extend d_0 into the half-space $\mathbb{R}^3_+=\{x_3>0\}$ so that

$$\|\nabla d_0\|_{L_p(\mathbb{R}^3_+)}\le c\|d_0\|_{W_p^{1-1/p,0}(\mathbb{R}^2)}$$

and write w_0 as

$$w_0(x,t)=-\int_0^\infty\frac{d}{dx_3}(F^{-1}Q_0e_0(x_3)Fd_0(\xi,s,y_3))dy_3$$

$$= \int_0^\infty e_0(x_3+y_3)Q_0F^{-1}(r\widetilde{d}_0(\xi,s,y_3) - \frac{d\widetilde{d}_0}{dy_3})dy_3.$$

As shown by I.Mogilevskii,

$$\begin{aligned} &M_p(r_1e_0(x_3+y_3)) + M_p(r_1|\xi|e_1(x_3+y_3)) \\ &+ M_p(|\xi|e^{-|\xi|(x_3+y_3)}) \le \frac{c}{x_3+y_3}, \end{aligned}$$

hence, by the MML theorem,

$$\begin{aligned} &\|e^{-\gamma t}w_0\|_{L_p(\mathcal{R}'_+)} \\ &\le c\int_0^\infty \|e^{-\gamma t}F^{-1}Q_0\widetilde{d}_0(\xi,s,y_3)\|_{L_p(\mathcal{R}'_+)}\frac{dy_3}{x_3+y_3}. \end{aligned}$$

Using the L_p-continuity of the Hilbert transform, we obtain

$$\begin{aligned} &\Big(\int_0^\infty \|e^{-\gamma t}w_0\|^p_{L_p(\mathcal{R}'_+)}dx_3\Big)^{1/p} \\ &\le c\Big(\int_0^\infty \|e^{-\gamma t}F^{-1}Q_0\widetilde{d}_0(\xi,s,y_3)\|^p_{L_p(\mathcal{R}'_+)}dy_3\Big)^{1/p}. \end{aligned}$$

Since

$$F^{-1}Q_0\widetilde{d}_0(\xi,s,y_3) = -F^{-1}Q_0\frac{i\xi}{|\xi|^2}\cdot F\nabla' d_0(t,y_3)$$

and $\frac{i\xi_\alpha}{|\xi|}$ are L_p-multipliers, we have

$$\|e^{-\gamma t}F^{-1}Q_0\widetilde{d}_0(\xi,s,y_3)\|_{L_p(\mathcal{R}_+)}$$
$$\le c\|e^{-\gamma t}\nabla' d_0\|_{L_p(\mathcal{R}_+)} \le c\|e^{-\gamma t}d_0\|_{W_p^{1-1/p,0}(\mathcal{R}'_+)}.$$

The functions w_1 and w_2 are estimated in a similar way.

The estimates of Q_i in (.19), (.20), (.21) follow from the formula (.15) and inequality (.16). We estimate the derivatives of

$$F^{-1}\frac{U_{ij}\widetilde{d}_j}{r_1(r_1+|\xi|)P}.$$

Proposition 2. *If* $Re s = \gamma > 0$*, then*

$$M_p(P^{-1}) \le c|P^{-1}|. \tag{.22}$$

It follows that

$$M_p\left(\frac{s^k(i\xi)^{j'}r_1^{j_3}}{r_1(r_1+|\xi|)}\frac{U_{ij}}{|\xi|P}\right)$$
$$\le cM_p\left(\frac{s^k(i\xi)^{j'}r_1^{j_3}}{r_1(r_1+|\xi|)}\right)M_p\left(\frac{U_{ij}}{|\xi|}\right)M_p\left(\frac{1}{P}\right)\le c,$$

which implies

$$\|D_t^kD_x^jF^{-1}\frac{U_{ij}\tilde{d}_j}{r_1(r_1+|\xi|)P}\|_{L_p(\mathcal{R}_+)}$$

$$\le c\|e^{-\gamma t}d_j\|_{W_p^{1-1/p,0}(\mathcal{R}'_+)}.$$

The estimate of

$$E^{-1}\frac{e_1(x_3)}{(r_1+|\xi|)P}V_{ij}\tilde{d}_j$$

is obtained by similar arguments, and the term $w(x,t) = F^{-1}\frac{e^{-r_1 x_3}}{r_1}\tilde{b}_\alpha$ represents the solution of the Neumann problem for the heat equation in the half-space with the boundary condition $w_{x_3}|_{x_3=0} = b_\alpha(x',t)$.

This proves the inequality (.11).

2. *Compressible fluid.*

We describe the linearization procedure of the problem (.2). We set

$$\rho(x,t) = \overline{\rho}(t) + \theta(x,t),$$

where $\overline{\rho}(t) = M/|\Omega_t|$ and $M = \int_{\Omega_t} \rho(x,t)dx$ is the mass of the fluid. Hence $\int_{\Omega_t} \theta(x,t)dx = 0$ Since

$$\frac{d}{dt}|\Omega_t| = \int_{\Omega_t} \nabla \cdot \boldsymbol{v}(x,t)dx,$$

the continuity equation takes the form

$$\theta_t + \overline{\rho}\Big(\nabla \cdot \boldsymbol{v} - \frac{1}{|\Omega_t|}\int_{\Omega_t} \nabla \cdot \boldsymbol{v}(x,t)dx\Big) = 0.$$

Omitting the nonlinear terms, we easily arrive at the linearized problem

$$\begin{cases} \boldsymbol{v}_t - \nabla \cdot T'(\boldsymbol{v}) + \pi_M \nabla\theta = \boldsymbol{f}, \\ \theta_t + \rho_M(\nabla \cdot \boldsymbol{v} - \dfrac{1}{|\mathcal{F}|}\displaystyle\int_{\mathcal{F}} \nabla \cdot \boldsymbol{v} dy) = h(x,t), \quad x \in \mathcal{F}, \\ r_t - \boldsymbol{v} \cdot \boldsymbol{N} = g(x,t), \\ T'(\boldsymbol{v})\boldsymbol{N} - (\pi_M\theta + \sigma\mathfrak{L}r)\boldsymbol{N} = \boldsymbol{d}(x,t), \quad x \in \mathcal{G}, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \mathcal{F}, \\ r(x,0) = r_0(x), \quad x \in \mathcal{G}, \end{cases} \tag{.23}$$

where $\rho_M = M/|\mathcal{F}|$, $\pi_M = p_M/\rho_M$, $p_M = p'(\rho_M)$, $T'(\boldsymbol{v}) = \nu S(\boldsymbol{v}) + \nu' I \nabla \cdot \boldsymbol{v}$, $\nu = \mu/\rho_M$, $\nu' = \mu'/\rho_M$.

Theorem 3. *Let* $p > 3$, $\boldsymbol{f} \in L_p(Q_T)$, $h \in W_p^{1,0}(Q_T)$, $g \in W_p^{1-1/p,1/2-1/2p}(G_T)$, $\boldsymbol{d} \in W_p^{1-1/p,1/2-1/2p}(G_T)$, $\boldsymbol{v}_0 \in W_p^{2-2/p}(\mathcal{F})$, $\theta_0 \in W_p^1(\mathcal{F})$, $r_0 \in W_p^{3-2/p}(\mathcal{G})$, *and*

$$\boldsymbol{d}(x,0) = T'(\boldsymbol{v}_0)\boldsymbol{N} - \pi_M\sigma_0\boldsymbol{N} + \sigma\mathfrak{L}r_0\boldsymbol{N}.$$

Then the problem (.23) is uniquely solvable in an arbitrary finite time interval $(0,T)$, *and*

$$\begin{aligned}&\|\boldsymbol{v}\|_{W_p^{2,1}(Q_T)}+\|\theta\|_{W_p^{0,1}(Q_T)}+\|\theta\|_{W_p^{1,0}(Q_T)}\\&+\|r\|_{W_p^{3-1/p,0}(G_T)}+\|r_t\|_{W_p^{2-1/p,1-1/2p}(G_T)}\\&\leq c(T)(\|\boldsymbol{f}|_{L_p(Q_T)}+\|h\|_{W_p^{1,0}(Q_T)}\\&+\|\boldsymbol{d}\|_{W_p^{1-1/p.1-1/2p}(G_T)}+\|g\|_{W_p^{2-1/p,1-2/2p}(G_T)}\\&+\|\boldsymbol{v}_0\|_{W_p^{2-2/p}(\mathcal{F})}+\|\theta_0\|_{W_p^1(\mathcal{F})}+\|r_0\|_{W_p^{3-2/p}(\mathcal{G})}.)\end{aligned}\tag{.24}$$

Remark. In the proof of local solvability of the problem (.23), it is convenient to eliminate θ with the help of

$$\begin{aligned}&\theta(x,t)=\theta_0(x)+\int_0^t\theta_\tau d\tau=\theta_0\\&+\rho_M\int_0^t(\nabla\cdot\boldsymbol{v}-|\mathcal{F}|^{-1}\int_{\mathcal{F}}\nabla\cdot\boldsymbol{v}(y,\tau)dy+h(x,\tau))d\tau\end{aligned}\tag{.25}$$

and to reduce (.23) to a similar problem for the Lame equations

$$\begin{cases} \boldsymbol{v}_t - \nabla \cdot T'(\boldsymbol{v}) = \boldsymbol{f}', \quad x \in \mathcal{F}, \\ r_t - \boldsymbol{v} \cdot \boldsymbol{N} = g(x,t), \\ T'(\boldsymbol{v})\boldsymbol{N}(x) + \sigma \mathfrak{L} r \boldsymbol{N} = \boldsymbol{d}', \quad x \in \mathcal{G}, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \mathcal{F}, \\ r(x,0) = r_0(x), \quad x \in \mathcal{G}. \end{cases} \tag{.26}$$

The estimate (.24) for this problem is simpler:

$$\|\boldsymbol{v}\|_{W_p^{2,1}(Q_T)} + \|r\|_{W_p^{3-1/p,0}(G_T)} + \|r_t\|_{W_p^{2-1/p,1-1/2p}(G_T)}$$
$$\leq c(\|\boldsymbol{f}'|_{L_p(Q_T)} + \|\boldsymbol{d}'\|_{W_p^{1-1/p.1-1/2p}(G_T)}$$
$$+ \|g\|_{W_p^{2-1/p.1-2/2p}(G_T)} + \|\boldsymbol{v}_0\|_{W_p^{2-2/p}(\mathcal{F})}$$
$$+ \|r_0\|_{W_p^{3-2/p}(\mathcal{G})})$$

It implies (.24) for the problem (.23), because (.25) contains the Volterra operators with respect to the derivatives of $\boldsymbol{v}$.

Model problem in $\mathbb{R}^3_+$ for (.26).

$$
\begin{cases}
\boldsymbol{w}_t - \Big((\nu+\nu')\nabla(\nabla\cdot\boldsymbol{w}) + \nu\nabla^2\boldsymbol{w}\Big) = 0, \\
x_3 > 0, \quad t > 0, \\
\nu\Big(\dfrac{\partial w_\alpha}{\partial x_3} + \dfrac{\partial w_3}{\partial x_\alpha}\Big) = b_\alpha(x',t), \quad x_3 = 0, \\
\nu'\nabla\cdot\boldsymbol{w} + 2\nu\dfrac{\partial w_3}{\partial x_3} - \sigma\nabla'^2\displaystyle\int_0^t w_3 d\tau = b_3(x',t), \\
\boldsymbol{w}(x,0) = 0, \quad x_3 > 0,
\end{cases}
$$

The Fourier-Laplace transform (.15) converts this problem in

$$
\begin{cases}
s\widetilde{\boldsymbol{w}}_\alpha - (\nu+\nu')i\xi_\alpha\Big(\sum_{\alpha=1}^{2} i\xi_\alpha\widetilde{w}_1 + \frac{d\widetilde{w}_3}{dx_3}\Big) \\
-\nu\Big(\frac{d^2}{dx_3^2}\widetilde{w}_3 - \xi^2\widetilde{w}_\alpha\Big) = 0, \\
s\widetilde{\boldsymbol{w}}_3 - (\nu+\nu')\frac{d}{dx_3}\Big(\sum_{\alpha=1}^{2} i\xi_1\widetilde{w}_\alpha + \frac{d\widetilde{w}_3}{dx_3}\Big) \\
-\nu\Big(\frac{d^2}{dx_3^2}\widetilde{w}_3 - \xi^2\widetilde{w}_\alpha\Big) = 0, \quad x_3 > 0, \\
\mu\Big(\frac{\partial\widetilde{w}_\alpha}{\partial x_3} + i\xi_\alpha\widetilde{w}_3\Big) = \widetilde{b}_\alpha(x', t), \\
\quad x_3 = 0, \quad \alpha = 1, 2, \\
\mu'\Big(\sum_{\alpha=1}^{2} i\xi_\alpha\widetilde{w}_\alpha + \frac{d\widetilde{w}_3}{dx_3}\Big) + 2\mu\frac{d\widetilde{w}_3}{dx_3} - \sigma\frac{|\xi|^2}{s}\widetilde{w}_3 = \widetilde{b}_3, \\
x_3 = 0, \\
\widetilde{\boldsymbol{w}} \to 0 \quad (x_3 \to \infty).
\end{cases}
\tag{.27}
$$

The solution of (.27) has the form

$$
\begin{aligned}
\widetilde{\boldsymbol{w}}(\xi, s, x_3) &= -\frac{\widetilde{\boldsymbol{b}}'}{\nu r} e_0(x_3) \\
&+ \begin{pmatrix} \frac{i\xi_1}{r_1} \sum_{i=1}^{3} (U_i(\xi, s) + V_i(\xi, s)) \widetilde{b}_i \\ \frac{i\xi_2}{r_1} \sum_{i=1}^{3} (U_i(\xi, s) + V_i(\xi, s)) \widetilde{b}_i \\ \sum_{i=1}^{3} U_i(\xi, s) \widetilde{b}_i \end{pmatrix} e_0(x_3) \\
&+ \begin{pmatrix} \frac{i\xi_1}{r_1} \\ \frac{i\xi_2}{r_1} \\ -1 \end{pmatrix} \sum_{i=1}^{3} r_1 V_i \widetilde{b}_i e_1(x_3),
\end{aligned}
$$

where

$$
\begin{aligned}
& e_0(x_3) = e^{-r_1 x_3}, \\
& e_1(x_3) = (e^{-r_1 x_3} - e^{-r_2 x_3})/(r_1 - r_2), \\
& r_1 = \sqrt{\frac{s}{\nu} + |\xi|^2}, \quad r_2 = \sqrt{\frac{s}{2\nu + \nu'} + |\xi|^2}, \\
& \widetilde{b}' = (\widetilde{b}_1, \widetilde{b}_2, 0).
\end{aligned}
$$

$$U_\alpha = -\frac{i\xi_\alpha}{\rho_0\mathcal{P}}[\nu((r_2-r_1)^2-(r_1^2-\xi^2))+\frac{\nu+\nu'}{2\nu+\nu'}\rho_0 s]$$
$$= -\frac{i\xi_\alpha s}{\mathcal{P}}\Big(\frac{2r_1}{r_1+r_2}\frac{\nu+\nu'}{2\nu+\nu'}-1\Big), \quad \alpha=1,2,$$
$$U_3 = -\frac{sr_2}{\mathcal{P}},$$
$$V_\alpha = \frac{i\xi_\alpha}{r_1+r_2}\frac{\nu+\nu'}{(2\nu+\nu')\nu}\frac{1}{\mathcal{P}}(2\nu rs+\sigma|\xi|^2), \quad \alpha=1,2,$$
$$V_3 = -\frac{s}{r_1+r_2}\frac{\nu+\nu'}{2\nu+\nu'}\frac{r^2+|\xi|^2}{\mathcal{P}},$$
$$\mathcal{P} = \rho_0 s^2 + \tfrac{4\nu(\nu+\nu')}{2\nu+\nu'}s\xi^2\tfrac{r_1}{r_1+r_2} + \tfrac{\sigma\rho_0}{2\nu+\nu'}\tfrac{s\xi^2}{r'+\xi} + \sigma|\xi|^3.$$

The function $\mathcal{P}$ satisfies the estimates (.16), i.e.,

$$|s||\xi|^2+|s|^2+\sigma|\xi|^3 \leq c|\mathcal{P}|,$$

and, as a consequence, (.22).

We show that

$$\|e^{-\gamma t}\boldsymbol{w}\|_{W_p^{2,1}(\mathcal{R}_+)} \leq c\|e^{-\gamma t}\boldsymbol{b}^{(0)}\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^3)}, \qquad (.28)$$

using the following analog of Proposition 1.

Proposition 3. *If*

$$Q_0(\xi,s)r_1^{-1} \in \mathcal{M},$$

$$Q_1(\xi,s)r_1^{-2} \in \mathcal{M},$$

then the functions $w_0(x,t) = F^{-1}Q_0e_0(x_3)Fd_0$ *and* $w_1(x,t) = F^{-1}Q_1e_1(x_3)Fd_1$ *satisfy*

$$\|e^{-\gamma t}w_i\|_{L_p(\mathcal{R}_+)} \leq c\|e^{-\gamma t}d_0^{(0)}\|_{W_p^{1-1/p,1/2-1/(2p)}(\mathbb{R}^3)}, \qquad (.29)$$

$$i = 0,1.$$

The proof is the same as that of Proposition 1: we write $\widetilde{w}_0$ as

$$\widetilde{w}(\xi,s,x_3) = \int_0^\infty e_0(x_3+y_3)Q_0(r_1\widetilde{d}_3(\xi,s,y_3) - \frac{d}{dy_3}\widetilde{d}_0)dy_3,$$

where

$$\widetilde{d}_0(\xi, s, y_3) = e^{-r_1 y_3}\widetilde{d}_0(\xi, s)$$

and apply the MML theorem:

$$\begin{aligned}
&\|e^{-\gamma t}w_0\|_{L_p(\mathcal{R}'_+)} \\
&\leq c\int_0^\infty \|e^{-\gamma t}F^{-1}Q_0\widetilde{d}_0(\xi, s, y_3)\|_{L_p(\mathcal{R}'_+)}\frac{dy_3}{x_3+y_3}, \\
&\|e^{-\gamma t}w_0\|_{L_p(\mathcal{R}_+)} \\
&\leq c\Big(\int_0^\infty \|e^{-\gamma t}F^{-1}Q_0\widetilde{d}_0(\xi, s, y_3)\|^p_{L_p(\mathcal{R}'_+)}dy_3\Big)^{1/p} \\
&\leq c\|e^{-\gamma t}F^{-1}r_1\widetilde{d}_0(\xi, s, y_3)\|_{L_p(\mathcal{R}_+)}.
\end{aligned}$$

Since

$$F^{-1}r_1\widetilde{d}_0(\xi, s, y_3) = F^{-1}\frac{r_1^2}{r_1}\widetilde{d}_0 = \Big(\frac{1}{\nu}\frac{\partial}{\partial t} - \nabla^2\Big)F^{-1}\frac{1}{r_1}\widetilde{d}_0,$$

it holds

$$
\begin{aligned}
&\|e^{-\gamma t}F^{-1}r_1\widetilde{d}_0(\xi,s,y_3)\|_{L_p(\mathcal{R}_+)}\\
&\quad\le c\|e^{-\gamma t}F^{-1}\frac{1}{r_1}\widetilde{d}_0(\xi,s,y_3)\|_{W_p^{2,1}(\mathcal{R}_+)}\\
&\quad\le c\|e^{-\gamma t}d_0^{(0)}\|_{W_p^{1-1/p,1/2-1/(2p)}(\mathbb{R}^3)},\\
&\|e^{-\gamma t}w_0\|_{L_p(\mathcal{R}_+)}\le c\|e^{-\gamma t}d_0^{(0)}\|_{W_p^{1-1/p,1-1/(2p)}(\mathbb{R}^3)}.
\end{aligned}
$$

The function w_1 is estimated in the same way, and we easily obtain (.28).

3. *MHD linearized problem.*

It consists of two separate problems: (.8) and another problem for $\boldsymbol{H}$, $\boldsymbol{E}$. After elimination of $\boldsymbol{E}$ the second problem takes the form

$$\begin{cases} \mu_1 \boldsymbol{H}_t + \alpha^{-1} rotrot \boldsymbol{H} = \boldsymbol{G}(x,t), \\ \nabla \cdot \boldsymbol{H}(x,t) = 0, \quad x \in \Omega_1, \quad t > 0, \\ rot \boldsymbol{H} = 0, \quad \nabla \cdot \boldsymbol{H}(x,t) = 0, \quad x \in \Omega_2, \\ [\mu \boldsymbol{H} \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \quad x \in S_1, \\ H_n = 0, \quad x \in S, \\ \boldsymbol{H}(x,0) = \boldsymbol{H}_0(x), \quad x \in \Omega_1 \cup \Omega_2, \end{cases} \tag{.30}$$

where $S_1 = \partial\Omega_1$, $S = \partial\Omega$, $\Omega = \Omega_1 \cup S_1 \cup \Omega_2$. We assume that Ω_1 and Ω are simply connected.

Theorem 1. *Assume that* $\boldsymbol{G} \in L_p(Q_T^1)$, $\boldsymbol{H}_0 \in W_p^{2-2/p}(\Omega_i)$, $i = 1, 2$, *where* $Q_T^1 = \Omega_1 \times (0, T)$, *and that the conditions*

$$\begin{aligned} &\nabla \cdot \boldsymbol{G}(x,t) = 0, \quad \nabla \cdot \boldsymbol{H}_0(x) = 0, \quad x \in \Omega_{10} \cup \Omega_{20}, \\ &[\boldsymbol{H}_{0\tau}] = 0, \quad [\mu \boldsymbol{H} \cdot \boldsymbol{n}] = 0, \quad x \in \Gamma_t, \\ &\boldsymbol{H} \cdot \boldsymbol{n} = 0, \quad x \in S \end{aligned} \tag{.31}$$

are satisfied. Then the problem (.30) has a unique solution $\boldsymbol{H} \in W_p^{2,1}(Q_T^1) \cap$

$W_p^{2,1}(Q_T^2)$, $Q_T^i = \Omega_i \times (0,T)$, *and this solution satisfies the inequality*

$$\sum_{i=1,2} \|\boldsymbol{H}\|_{W_p^{2,1}(Q_T^i)} \\ \leq c(\|\boldsymbol{G}\|_{L_p(Q_T^1)} + \sum_{i=1,2} \|\boldsymbol{H}_0\|_{W_p^{2-2/p}(\Omega_i)}).$$

The equations $rot\boldsymbol{H} = 0$, $\nabla \cdot \boldsymbol{H} = 0$ in Ω_2 imply $\boldsymbol{H}(x,t) = \nabla \cdot \varphi(x,t)$, $\nabla^2\varphi = 0, \quad x \in \Omega_2$. Hence (0.30) is equivalent to

$$\begin{aligned}
&\mu_1 \boldsymbol{H}_t + \alpha^{-1} rotrot\boldsymbol{H} = \boldsymbol{G}(x,t), \\
&\nabla \cdot \boldsymbol{H}(x,t) = 0, \quad x \in \Omega_1, \quad t > 0, \\
&\nabla^2\varphi(x,t) = 0, \quad x \in \Omega_2, \quad \frac{\partial\varphi}{\partial n}\Big|_{x\in S} = 0, \\
&\mu_2 \frac{\partial\varphi}{\partial n} - \mu_1 \boldsymbol{H} \cdot \boldsymbol{n} = 0, \\
&\boldsymbol{H}_\tau = \nabla_\tau \varphi(x,t), \quad x \in S_1, \\
&\boldsymbol{H}(x,0) = \boldsymbol{H}_0, \quad x \in \Omega_1.
\end{aligned}$$

and to

$$\begin{cases} \mu_1 \boldsymbol{H}_t - \alpha^{-1}\nabla^2 \boldsymbol{H} = \boldsymbol{G}(x,t), \quad x \in \Omega_1, \quad t > 0, \\ \nabla^2 \varphi(x,t) = 0, \quad x \in \Omega_2, \quad \left.\dfrac{\partial \varphi}{\partial n}\right|_{x \in S} = 0, \\ \mu_2 \dfrac{\partial \varphi}{\partial n} - \mu_1 \boldsymbol{H} \cdot \boldsymbol{n} = 0, \\ \nabla \cdot \boldsymbol{H}(x,t) = 0, \quad x \in S_1, \\ \boldsymbol{H}_\tau = \nabla_\tau \varphi(x,t), \quad x \in S_1, \\ \boldsymbol{H}(x,0) = \boldsymbol{H}_0, \quad x \in \Omega_1, \end{cases}$$

if $\nabla \cdot \boldsymbol{H}_0(x) = 0$.

We consider the model problem:

$$
\begin{cases}
\mu_1 \boldsymbol{h}_t - \alpha^{-1}\nabla^2 \boldsymbol{h} = 0, \quad x_3 > 0, \quad t > 0, \\
\nabla^2 \varphi(x,t) = 0, \quad x_3 < 0, \quad t > 0, \\
\nabla \cdot \boldsymbol{h} = d(x',t), \quad x_3 = 0, \quad x' = (x_1, x_2), \\
h_\alpha = \dfrac{\partial \varphi}{\partial x_\alpha} + b_\alpha(x',t), \quad \alpha = 1,2, \\
\mu_1 h_3 = \mu_2 \dfrac{\partial \varphi}{\partial x_3} + b_3, \quad x_3 = 0, \\
\boldsymbol{h}(x,0) = 0
\end{cases}
\tag{.32}
$$

The Fourier-Laplace transform converts (.32) into

$$\begin{cases} r_3^2(\xi,s)\widetilde{\boldsymbol{h}}(\xi,s,x_3) - \dfrac{d^2\widetilde{\boldsymbol{h}}}{dx_3^2} = 0, \quad x_3 > 0, \\ |\xi|^2\widetilde{\varphi} - \dfrac{d^2\widetilde{\varphi}}{dx_3^2} = 0, \quad x_3 < 0, \\ \dfrac{d\widetilde{h}_3}{dx_3} + \sum\limits_{\beta=1}^{2} i\xi_\beta \widetilde{h}_\beta = \widetilde{d}(\xi,s), \quad x_3 = 0, \\ \widetilde{h}_\alpha = i\xi_\alpha\widetilde{\varphi} + \widetilde{b}_\alpha, \quad \alpha = 1,2, \\ \mu_1\widetilde{h}_3 = \mu_2\dfrac{d\widetilde{\varphi}}{dx_3} + \widetilde{b}_3, \quad x_3 = 0, \\ \widetilde{\boldsymbol{h}}(\xi,s,x_3) \xrightarrow[x_3\to+\infty]{} 0, \quad \widetilde{\varphi}(\xi,s,x_3) \xrightarrow[x_3\to-\infty]{} 0, \end{cases}$$

where $r_3^2 = \mu_1\alpha s + |\xi|^2$, $-\pi < arg\ r_3 < \pi$, Re$s = \gamma > 0$.

This problem is easily solved. We have

$$
\begin{aligned}
&\widetilde{\boldsymbol{h}}(\xi, s, x_3) = \widetilde{\boldsymbol{h}}(\xi, s, 0)e^{-rx_3},\\
&\widetilde{\varphi}(\xi, s, x_3) = \widetilde{\varphi}(\xi, s, 0)e^{|\xi|x_3},\\
&\widetilde{h}_\alpha(\xi, s, 0) = i\xi_\alpha\widetilde{\varphi}(\xi, s, 0) + \widetilde{b}_\alpha, \quad \alpha = 1, 2,\\
&\mu_1\widetilde{h}_3(\xi, s, 0) = \mu_2|\xi|\widetilde{\varphi}(\xi, s, 0) + \widetilde{b}_3\\
&-r_3\widetilde{h}_3(\xi, s, 0) + \sum_{\beta=1}^{2} i\xi_\beta\widetilde{h}_\beta = \tilde{d},
\end{aligned}
$$

which implies

$$\begin{aligned}
&\widetilde{h}_\alpha(\xi, s, 0) = \widetilde{b}_\alpha \\
&- \frac{i\xi_\alpha}{|\xi|}\frac{1}{mr_3 + |\xi|}(\widetilde{d} - \sum_{\beta=1}^{2} i\xi_\beta\widetilde{b}_\beta + \mu_1^{-1}r\widetilde{b}_3), \qquad \alpha = 1, 2, \\
&\widetilde{h}_3(\xi, s, 0) = \mu_1^{-1}\widetilde{b}_3 \\
&- \frac{m}{mr + |\xi|}(\widetilde{d} - \sum_{\beta=1}^{2} i\xi_\beta\widetilde{b}_\beta + \mu_1^{-1}r\widetilde{b}_3), \\
&\widetilde{\varphi}(\xi, s, 0) = -\frac{1}{mr_3 + |\xi|}(\widetilde{d} - \sum_{\beta=1}^{2} i\xi_\beta\widetilde{b}_\beta + \mu_1^{-1}r\widetilde{b}_3), \\
&m = \mu_2\mu_1^{-1}.
\end{aligned}$$

The expressions $i\xi_\alpha/|\xi|$, $\alpha = 1, 2$,

$$\widetilde{M}_0 = \frac{r_3(\xi, s)}{mr_3(\xi, s) + |\xi|}, \qquad \widetilde{M}_\alpha = \frac{i\xi_\alpha}{mr_3 + |\xi|}$$

are the Fourier L_p-multipliers, hence

$$\|\boldsymbol{h}(\cdot,0)\|_{L_p(\mathbb{R}^2\times(-\infty,T))} \leq c(T)\Big(\|\boldsymbol{b}(\cdot,0)\|_{L_p(\mathbb{R}^2\times(-\infty,T))} + \|D(\cdot,0)\|_{L_p(\mathbb{R}^2\times(-\infty,T))}\Big),$$

moreover,

$$\|\boldsymbol{h}(\cdot,0)\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^2\times(-\infty,T))} \leq c\Big(\|\boldsymbol{b}(\cdot,0)\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^2\times(-\infty,T))} + \|D\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^2\times(-\infty,T))}\Big),$$

where $D(x',t) = (FL)^{-1}r^{-1}\tilde{d}e^{-rx_3}|_{x_3=0} =$ $w(x,t)|_{x_3=0}$, $w(x,t)$ being the solution of the problem

$$\mu_1 w_t - \alpha^{-1}\nabla^2 w = 0, \quad x_3 > 0,$$
$$\frac{\partial w}{\partial x_3}\Big|_{x_3=0} = -d(x',t), \quad w|_{t=0} = 0,$$
$$\lim_{x_3\to\infty} w(x,t) = 0,$$

hence

$$\begin{aligned}
&\|D\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^2\times(-\infty,T))} \\
&\leq c\|w\|_{W_p^{2,1}(\mathbb{R}^3_+\times(0,T))} \\
&\leq c\|\boldsymbol{d}\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^2\times(-\infty,T))},
\end{aligned}$$

which implies

$$\begin{aligned}
&\|\boldsymbol{h}\|_{W_p^{2,1}(\mathbb{R}^3_+\times(0,T))} \\
&\leq c\Big(\|\boldsymbol{h}(\cdot,0)\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^2\times(-\infty,T))} \\
&\leq c\Big(\|\boldsymbol{b}\|_{W_p^{2-1/p,1-1/2p}(\mathbb{R}^2\times(-\infty,T))} \\
&+ \|d\|_{W_p^{1-1/p,1/2-1/2p}(\mathbb{R}^2\times(-\infty,T))}\Big),
\end{aligned}$$

q.e.d.