# Existence and regularity results for solutions of linear parabolic systems with non smooth data

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## System of equations

$$\frac{\partial u}{\partial t} - \quad \mathrm{div} \quad (ADu) \;=\; f \quad \text{on} \quad Q = \Omega \times (0, T)$$

$$u \;=\; 0 \quad \text{on} \quad \partial\Omega \times (0, T)$$

$$u \;=\; 0 \quad \text{on} \quad \Omega \times \{0\}$$

## Assumptions

$\Omega \subset \mathbb{R}^n$ bounded domain with $C^1$ boundary,
$T > 0, Q = \Omega \times (0, T)$,

$u : Q \to \mathbb{R}^N$,

$Du$ gradient of $u$ with respect to space variables,

$A \in L^\infty(Q) \cap VMO(Q)$ symmetric, elliptic matrix,

$f \in L^1(\Omega)$

## Questions

1. Existence of very weak solutions for $f \in L^1(Q)$ (or $f$ bounded Borel measure)

2. Morrey type regularity for $f \in L^{1,\lambda}(Q)$

3. Fractional differentiability of $Du$ for better coefficients.

## Measure valued right hand sides

Q. Stampacchia (1965)(Ann. Inst. Fourier)- elliptic equations with discontinuous coefficients

L.Boccardo, T.Gallöuet (1989)(J.Funct. Anal.)- nonlinear elliptic and parabolic equations with measure data

L. Boccardo, A, Dall'Aglio, T.Gallöuet, L. Orsina (1997)(J. Func. Anal.)- nonlinear parabolic equations with measure data

P. Baras, M. Pierre (1984)(Appl.Anal.)- semilinear parabolic equations with measure data

P. Baroni, J. Habermann (2012)(J. Diff. Equations) nonlinear parabolic equations with measure data

Mingione, Vazquez, Bénilan, Gariepy, Lions, Murat, Maly, Ziemer, Amann-Quittner, Dolzman-Hungermühler-Müller

## Weak solution

$u \in L^2(0, T; W_0^{1,2}(\Omega))$, $u(x,0) = 0$ solves the problem

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \quad \mathrm{div} \quad (ADu) &= g \quad \text{on} \quad Q \\
u &= 0 \quad \text{on} \quad \partial\Omega \times (0, T) \\
u &= 0 \quad \text{on} \quad \Omega \times \{0\}
\end{aligned}
$$

if for each sufficiently smooth $\varphi$ such that $\varphi(x, T) = 0$ it holds

$$\int_Q -u(z)\varphi_t(z) + A(z)Du(z)D\varphi(z)dz =$$

$$\int_Q g(z)\varphi(z)dz.$$

Thus also

$$\int_Q u(z)\left(-\varphi_t(z) - \operatorname{div}(A(z)D\varphi(z))\right) dz =$$

$$\int_Q g(z)\varphi(z)dz.$$

Set $H(\varphi) = -\varphi_t(z) - \mathrm{div}(A(z)D\varphi(z)$ and rewrite the equation as

$$\int_Q u(z)H(\varphi)(z)dz = \int_Q g(z)\varphi(z)dz.$$

**Definition** ( Very weak solution for $f \in L^1(Q)$)
$u \in L^1(0, T; W_0^{1,1}(\Omega))$ is a very weak solution to our problem iff for all "sufficiently smooth" $\varphi$ such that $\varphi(x, T) = 0, x \in \Omega$ and $H(\varphi) \in C(\overline{Q})$ it holds

$$\int_Q u(z)H(\varphi)(z)dz = \int_Q f(z)\varphi(z)dz,$$

and

$$\lim_{t\to 0} \int_\Omega u(x, t)\psi(x)dx = 0 \quad \text{for} \quad \text{all} \quad \psi \in C_0(\Omega).$$

Important question: How good is $\varphi$ for $H(\varphi) \in C(\overline{Q})$?
How smooth is the solution with regular right hand side?

**Theorem 1**

Let $\Omega$ be a bounded domain with $C^1$ boundary, $A$ elliptic, symmetric with entries in $L^\infty(Q) \cap VMO(Q), g \in L^{2,\theta}(Q)$ for $\theta \in [0,1)$. Then space gradient $Dw$ of weak solution $w$ of the problem

$$H(w) = g, \qquad (1)$$

$$w = 0 \quad \text{on} \quad \partial_p Q \qquad (2)$$

belongs to $L^{2,\theta}(Q)$ and

$$||Dw||_{L^{2,\theta}} \leq C||g||_{L^{2,\theta}}.$$

Proof is based on $A$ caloric approximation lemma (F. Duzaar, G. Mingione) which is very useful for systems with coefficients in VMO, continuous only in an integral sense.

Moreover, it is easy to see from classical existence estimates that an inverse mapping $G$ to $H$ , $G : L^2(0, T; W^{-1,2}(\Omega)) \to L^2(0, T; W_0^{1,2}(\Omega))$ is a continuous mapping.

For any right hand side $\tilde{g} \in L^2(0, T; W^{-1,2}(\Omega))$ there is a function $g \in L^2(Q)$ so that

$$\tilde{g} = \quad \mathrm{div} \quad g$$

and

$$||g||_{L^2(Q)} \leq C||\tilde{g}||_{L^2(0,T;W^{-1,2}(\Omega))}.$$

Consider $g \in L^r(0, T; L^q(\Omega))$ for $2/r + n/q < 1; r, q > 2$. Then $g \in L^{2,\theta}$ with $\theta = \frac{n(1-2/q)+2(1-2/r)}{n+2}$. For $\theta \in (\frac{n}{n+2}, 1)$ we obtain that $w \in C^{0,\alpha}(\overline{Q})$.

**Theorem 2**

Let $\Omega$ be a bounded domain with $C^1$ boundary, $A$ an elliptic, symmetric matrix with entries in $L^\infty(Q) \cap VMO(Q)$ . Then there exists unique very weak solution $u$ to problem (1) such that $u \in L^{r'}(0, T; W_0^{1,q'}(\Omega))$ for any $r', q' > 2; 2/r' + n/q' > n + 1$. Moreover, $u$ satisfies the inequality

$$||u||_{L^{r'}(0,T;W_0^{1,q'}(\Omega))} \leq C||f||_{L^1(Q)}.$$

Thus $G$ maps $L^r(0, T; L^q(\Omega))$ into $C(\overline{Q})$ and its dual operator maps continuously $L^1(Q)$ to $L^{r'}(0, T; W_0^{1,q'}(\Omega))$ with $r'$and $q'$ dual exponents to $r$ and $q$.

**Remark** By rescaling we obtain that the corresponding norms on cylinders with radius $R$ can be estimated better, namely

$$||u||_{L^{r'}(t_0,t_0+R^2;W_0^{1,q'}(\Omega_R))} \leq CR^{2/r'+n/q'-(n+1)}||f||_{L^1(Q_R)}$$

for $r', q' \geq 1$ and $2/r' + n/q' > n + 1$. Thus $u$ belongs to Morrey space on $Q$.

Baroni-Habermann results for fractional differentiability of $Du$ can be generalized for systems under stronger conditions on coefficients

**Theorem 3**

Let assumptions of Theorem 2 be satisfied and coefficients $A$ are $\alpha$ Hölder continuous functions of space variables. Let $u$ be very weak solutions to problem (1) with right hand side $f \in L^1(Q)$. Then $Du \in W^{\eta,\eta/2,q}_{loc}(Q)$ for $q \in [1, \frac{n+2}{n+1})$ and $\eta \in [0, \alpha\frac{n+2-q(n+1)}{q})$.

Thank you for your attention