A Chemotaxis-Model with Non-Diffusing Attractor

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The classical Keller-Segel model for chemotaxis:

u density of cells, v concentration of chemo-attractant

$$\begin{aligned}\partial_t u &= \Delta u - \chi \nabla \cdot (u \nabla v)\\ \partial_t v &= \eta \Delta v + \alpha u - \beta v\end{aligned}$$

with Neumann boundary conditions on $\partial\Omega$.

$\chi > 0$ is the chemotactic sensitivity.

Define $\bar{w} = \frac{1}{|\Omega|}\int_\Omega w\,dx$. Then

$$\bar{u}(t) = \bar{u}_0 \quad , \quad \frac{1}{\eta}(\partial_t + \beta)\bar{v} = \frac{\alpha}{\eta}\bar{u} = \frac{\alpha}{\eta}\bar{u}_0$$

Consider $\tilde{v} := v - \bar{v}$, then

$$\frac{1}{\eta}(\partial_t + \beta)\tilde{v} = \Delta\tilde{v} + \frac{\alpha}{\eta}(u - \bar{u}_0)$$

For large diffusion of the chemo-attractant we obtain the approximating system

$$\begin{aligned} \partial_t u &= \Delta u - \chi\nabla\cdot(u\nabla v) \\ 0 &= \Delta\tilde{v} + \tfrac{\alpha}{\eta}u - \tfrac{\alpha}{\eta}\bar{u}_0 \end{aligned}$$

The last equation can be further rescaled:

$$0 = \Delta v + u - 1$$

Jäger/Luckhaus, '92, proved rigorously:

For $\Omega \subset \mathbb{R}^2$ open and bounded there exist a critical number $c(\Omega)$ such that for $\alpha \bar{u}_0 \chi < c(\Omega)$
there exists a unique, smooth, positive solution for all times.

For a disk Ω there exist $c^* > 0$ such that for $\alpha \bar{u}_0 \chi > c^*$
radially symmetric positive initial data can be constructed such that blow-up happens in the center of the disc in finite time.

Haptotaxis and Trail Following:

Consider a Keller-Segel model with non-diffusive memory, namely

$$\partial_t u = \Delta u - \nabla \cdot (u \nabla \log(v)) \quad , \quad \partial_t v = u v^{\lambda} \ .$$

Earlier results:

$\lambda = 0$ global solutions (Chen Hua et al),

$\lambda = 1$ blow-up for specific initial data (Levine and Sleeman).

For $w = \log v$ we obtain:

$$\partial_t u = \Delta u - \nabla \cdot (u \nabla w) \quad , \quad \partial_t w = u w^{\lambda - 1}$$

Let $\theta = \frac{1}{1-\lambda}$ and $z = \frac{1}{1-\lambda} v^{(1-\lambda)} = \theta v^{\frac{1}{\theta}}$, then

$$\partial_t u = \Delta u - \theta \nabla \cdot (u \nabla \log(z)) \quad , \quad \partial_t z = u \quad , \quad \theta \in (0, \infty)$$

[Kang - S.- Velázquez]:

Space Dimension 1: periodic boundary conditions.

For $\theta = 1$, i.e. $\lambda = 0$ formally every space dependent function is asymptotically a steady state for $t \to \infty$.
It was shown, that the long time dynamics are strongly dependent on the initial data.

For $1 < \theta < 3$, i.e. $0 < \lambda < \frac{2}{3}$ it was rigorously proved that solutions converge to a Dirac mass for $t \to \infty$.

First we give a heuristic argument:

Let $I = [-1, 1]$, $\int_I u\, dx = m$. Consider

$$\bar{z}_t = \frac{\bar{z}^\theta}{\int_I \bar{z}^\theta dx},$$

which results from the quasisteady approximation

$$\begin{aligned} 0 &= \nabla \cdot (\nabla u - \theta u \nabla \log(z)) \\ \partial_t z &= u \end{aligned}$$

Assume that this is a good approximation
for the original problem for $t \to \infty$.

For $\bar{z}(0,0) > \bar{z}(0,x)$ we obtain

$\bar{z}^{1-\theta}(t,x) = \bar{z}^{1-\theta}(0,x) - (\theta - 1)\int_0^t \frac{ds}{\int_I \bar{z}^{\theta}(s,x)dx}$

Assume the following expansion:

$\bar{z}^{1-\theta}(0,x) = \bar{z}^{1-\theta}(0,0) + Bx^2 + h.o.t.$ for $x \to 0$.

Thus $\bar{z}^{1-\theta}(t,x) \approx \bar{z}^{1-\theta}(0,0) + Bx^2 - (\theta - 1)\int_0^t \frac{ds}{\int_I \bar{z}^{\theta}(s,x)dx}$.

So $\bar{z}^{1-\theta}(t,x) :\approx Bx^2 + \psi(t)$, therefore $\bar{z}(t,x) \approx (Bx^2 + \psi(t))^{\frac{1}{1-\theta}}$.

Explicit calculations show that

$\frac{1-\theta}{\psi'(t)} \approx \int_I \frac{dx}{(Bx^2+\psi(t))^{\frac{\theta}{\theta-1}}}$, and

$\psi'(t) \approx -K\psi^{\frac{\theta+1}{2(\theta-1)}}(t)$, so $\psi(t) \approx At^{\frac{2(1-\theta)}{3-\theta}}$.

With this we can calculate, that

$$\bar{z}(t,x) \approx \frac{t^{\frac{2}{3-\theta}}}{\left(Bx^2 t^{\frac{2(\theta-1)}{3-\theta}} + A\right)^{\frac{1}{\theta-1}}}$$

Theorem:

There exist a family of initial data $(u_0, z_0) \in C^{2,\alpha}$ such that the corresponding solutions (u, z) of our system satisfy $u(t,x) \to m\delta(x)$ and

$$\bar{z}(t,x) \quad \approx \quad \frac{t^{\frac{2}{3-\theta}}}{\left(Bx^2 t^{\frac{2(\theta-1)}{3-\theta}} + A\right)^{\frac{1}{\theta-1}}}$$

for $t \to \infty$, and where A, B are constants, which depend on the initial data.

Proof:

- Eigenvalue problem for the steady state eqn. in u.

$$A_z(f) = f_{xx} - \theta(f z_x/z)_x = \tilde{\lambda} f$$

- This operator is self-adjoined w.r.t. the weighted integral with weight dx/z^θ.
- Negative upper bound for second eigenvalue of $A_z(.)$ for all t.
- Sobolev inequality with weighted norm by adaptation of a result by Maz'ja (using the reformulation by Horiuchi) for $\tilde{z}$ behaving like a power law similar to the law in our Theorem.

$$\left(\int_0^1 \tilde{z}^{(p-1)\theta}|\zeta|^p dx\right)^{1/p} \leq C\left(\int_0^1 \tilde{z}^\theta|\zeta_x|^2 dx\right)^{1/2} \text{ for } p = \frac{6\theta - 2}{\theta + 1}$$

- Since z does not behave like a power law everywhere,
  the estimate can not be used directly.
  Additionally a boundary layer estimate has to be introduced.
- A priori estimates for z and $v = u - z^\theta / \int_I z^\theta dx$.
- Existence of u, z which fulfill assumptions
  by methods of Herrero/Velazquez.

Asymptotic Results (S.-Velázquez)

$$\begin{aligned} \partial_t u &= \Delta u - \nabla \left(u \frac{\nabla v}{v} \right) \\ \partial_t v &= u v^{\lambda} \end{aligned}$$

for $x \in \mathbb{R}^n$, $t > 0$, and suitable initial conditions for u and v.

Depending on the space dimension n, the growth exponent λ and the regularity properties of the initial conditions, blow-up in finite time, mass aggregation in infinite time, or mass spreading can be observed.

1 Intuitive Understanding of the Model in $\mathbb{R}^d$

The exponent λ measures the strength of the localized reinforcement, thus the tendency for aggregation increases with larger values of λ, respectively larger values of θ.

The dynamics of the cells are described by random motility and by chemotactic drift towards higher concentrations of v. The number of times that a brownian particle approaches a given point in space depends very strongly on the space dimension. Thus the environment, where the cells move, is modified stronger in lower dimensions than it is in higher dimensions. So in this model the tendency to aggregate increases for smaller spatial dimension.

In contrast to this, in the original Keller-Segel model with diffusion finite time blow-up is more likely in higher dimensions.

Regular initial data, $n = 1$:

- For $\lambda > \frac{2}{3}$ we observe blow-up in finite time.
- For $0 < \lambda < \frac{2}{3}$ we observe blow-up in infinite time.
  The rate of growth is a power law.
- For $\lambda = \frac{2}{3}$ also blow-up in infinite time can be observed.
  The rate of growth is exponential.
- For $\lambda = 0$ the solution is highly sensitive on the initial data.
  These play an important role for the diffusive tails of the solution.
- For $\lambda < 0$ self-similar behavior can be observed.
  The reinforcement plays a non-trivial role.
  The solution behaves non-diffusive.

Regular initial data, $n = 2$:

- For $\lambda > 1 - \frac{1}{1+\frac{2}{n}} = \frac{1}{2}$ we observe blow-up in finite time.
- For $0 < \lambda \leq 1 - \frac{1}{1+\frac{2}{n}} = \frac{1}{2}$ we observe blow-up in infinite time.
- For $\lambda \leq 0$ non-diffusive self-similar behavior without mass aggregation can be observed.

Regular initial data, $n \geq 3$

- For $\lambda \geq \frac{2}{n}$ both, finite time blow-up without mass aggregation and diffusive self-similar behavior without mass aggregation can be observed.
- For $1 - \frac{1}{1+\frac{2}{n}} < \lambda < \frac{2}{n}$ blow-up in infinite time and diffusive self-similar behavior without mass aggregation can be observed.
- For $\lambda \leq 1 - \frac{1}{1+\frac{2}{n}}$ diffusive self-similar behavior without mass aggregation can be observed.

Further, the size of the region w.r.t. time was calculated, where an amount of mass of order one is distributed during the aggregation process.

The classical Keller-Segel model.

$d = 1$	$d = 2$	$d \geq 3$
No singularities	Mass aggregation in finite time for $M > M_{crit}$	Mass aggregation in finite time with arbitrary mass
	Non-diffusive self-similar behavior for $M < M_{crit}$	Singularity formation without mass aggregation in finite time
		Diffusive self-similar behavior without mass aggregation

- The classical Keller-Segel system for chemotaxis behaves in a different way than the PDE-ODE-system presented.
  The later one behaves ‘more hyperbolic’.
  The reaction to an attractive but non-diffusive signal creates a different long time behavior in comparison to attractive diffusible signals.

Dear Nina,

Happy Birthday!

And many more healthy
and happy years to come!