Homogenization of Periodic Elliptic Operators: Error Estimates in Dependence of the Spectral Parameter

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Plan

- Introduction
- Statement of the problem
- The effective operator
- Main results
- Survey
- Method of investigation
- Application to parabolic problems
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We study the *operator error estimates* in homogenization theory. Such estimates were intensively studied during the last 10–12 years by many authors: Birman, Suslina, Zhikov, Pastukhova, Griso, Kenig, Lin, Shen, others (more detailed survey will be given later on).
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In this talk we present some recent results on the operator error estimates.

- We study homogenization problems for a wide class of the second order *matrix elliptic operators* $A_\varepsilon$, both in $\mathbb{R}^d$ and in a bounded domain with the Dirichlet or Neumann boundary conditions. The coefficients of $A_\varepsilon$ are periodic and depend on $\mathbf{x}/\varepsilon$. 
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- We obtain approximations of the resolvent $(A_\varepsilon - \zeta I)^{-1}$ of an operator $A_\varepsilon$ in a regular point $\zeta$ in different operator norms.
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- We obtain approximations of the resolvent $(A_\varepsilon - \zeta I)^{-1}$ of an operator $A_\varepsilon$ in a regular point $\zeta$ in different operator norms.

- We find *twoparametric operator error estimates* in dependence of $\varepsilon$ and $\zeta$. 
Statement of the problem

Let \( \Gamma \) be a lattice in \( \mathbb{R}^d \), let \( \Omega \) be the cell of \( \Gamma \).

**Example:** \( \Gamma = \mathbb{Z}^d, \quad \Omega = (-\frac{1}{2}, \frac{1}{2})^d \).
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We study $(n \times n)$-matrix elliptic second order DO

$$b(D)^* g^\varepsilon(x) b(D)$$

in $\mathbb{R}^d$ and in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$. 
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The operator

$$b(D) = \sum_{j=1}^{d} b_j D_j$$

is a first order $(m \times n)$-matrix DO; $b_j$ are constant matrices, and $m \geq n$. 
Statement of the problem

The symbol \( b(\xi) = \sum_{j=1}^{d} b_j \xi_j \) is such that

\[
\text{rank } b(\xi) = n, \quad 0 \neq \xi \in \mathbb{R}^d, \quad (1_D)
\]

for the problem in \( \mathbb{R}^d \) and for the case of the Dirichlet condition, or

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2) The operator of elasticity theory.
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Examples. 1) The acoustics operator \( A_\varepsilon = -\nabla g_\varepsilon(x) \nabla = D^* g_\varepsilon(x) D \).
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Let \( \mathcal{O} \subset \mathbb{R}^d \) be a bounded domain with the boundary \( \partial \mathcal{O} \in C^{1,1} \).
Statement of the problem

Main objects

- **By** $A_\varepsilon$ **we denote the operator in** $L_2(\mathbb{R}^d; \mathbb{C}^n)$ **given by**
  \[ b(D)^* g^\varepsilon(x) b(D). \]
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1. By $A_\varepsilon$ we denote the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ given by $b(D)^*g^\varepsilon(x)b(D)$.

2. By $A_{D,\varepsilon}$ we denote the operator in $L_2(\mathcal{O}; \mathbb{C}^n)$ given by $b(D)^*g^\varepsilon(x)b(D)$ with the Dirichlet boundary condition.

Precise definitions are given in terms of the corresponding quadratic forms. Under our assumptions, the operators $A_\varepsilon$, $A_{D,\varepsilon}$, $A_{N,\varepsilon}$ are strongly elliptic; the corresponding quadratic forms satisfy the coercivity conditions.
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The problem is to study the behavior of the resolvents

\[(A_\varepsilon - \zeta I)^{-1}, \ (A_{D,\varepsilon} - \zeta I)^{-1}, \ (A_{N,\varepsilon} - \zeta I)^{-1}\]

in dependence of \(\varepsilon\) and \(\zeta\). We wish to find two-parametric approximations for these resolvents in the \(L_2 \to L_2\) and \(L_2 \to H^1\) operator norms.
In order to formulate the results, we define the effective operator with constant coefficients:

\[ b(D)^* g^0 b(D), \]

where \( g^0 \) is a constant positive matrix called the effective matrix.
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**Definition of the effective matrix:**

Let \( \Lambda(x) \) be the \((n \times m)\)-matrix-valued \( \Gamma \)-periodic solution of the equation

\[ b(\mathbf{D})^* g(x)(b(\mathbf{D})\Lambda(x) + \mathbf{1}_m) = 0, \quad \int_{\Omega} \Lambda(x) \, dx = 0. \]
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Then \( g^0 \) is an \((m \times m)\)-matrix given by

\[
g^0 = |\Omega|^{-1} \int_{\Omega} g(x)(b(D)\Lambda(x) + 1_m) \, dx.
\]
By $A^0$ we denote the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ given by $b(D)^* g^0 b(D)$. Its domain is $H^2(\mathbb{R}^d; \mathbb{C}^n)$. 
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By $A^0_D$ we denote the operator in $L_2(\mathcal{O}; \mathbb{C}^n)$ given by $b(D)^*g^0b(D)$ with the Dirichlet boundary condition. Its domain is $H^2(\mathcal{O}; \mathbb{C}^n) \cap H^1_0(\mathcal{O}; \mathbb{C}^n)$. 

By $A^0_N$ we denote the operator in $L_2(\mathcal{O}; \mathbb{C}^n)$ given by $b(D)^*g^0b(D)$ with the Neumann boundary condition. Its domain is $H^2(\mathcal{O}; \mathbb{C}^n) \cap H^1_0(\mathcal{O}; \mathbb{C}^n)$. 

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Main results in the whole space

We start with the results in $\mathbb{R}^d$. A point $\zeta \in \mathbb{C} \setminus [0, \infty)$ is regular both for $A_\epsilon$ and $A^0$. 

Theorem 1 [T. Suslina, 2014]

Let $\zeta = |\zeta| e^{i\phi} \in \mathbb{C} \setminus [0, \infty)$. Then for $\epsilon > 0$ we have

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\| (A_\epsilon - \zeta I) - 1 - (A^0 - \zeta I) - 1 \|_{L^2(\mathbb{R}^d)} \leq C(\phi) \epsilon |\zeta|^{1/2}.
$$

(2)

Inequality (2) is uniform with respect to $\phi$ in any sector $\phi \in [\phi_0, 2\pi - \phi_0]$, i.e., $C(\phi) \leq C(\phi_0)$ in this sector.
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In order to approximate the resolvent in the $L_2 \rightarrow H^1$ norm, we need to introduce a corrector

$$K(\varepsilon; \zeta) = \Lambda^\varepsilon S_\varepsilon b(D)(A^0 - \zeta I)^{-1}.$$
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Here $S_\varepsilon$ is the Steklov smoothing operator defined by

$$(S_\varepsilon v)(x) = |\Omega|^{-1} \int_\Omega v(x - \varepsilon z) \, dz.$$ 

The operator $K(\varepsilon; \zeta)$ is continuous from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$. 

Remark. In some cases $\Lambda \in L_\infty$ automatically. In particular, 1) if $d = 2$; 2) for the acoustics operator $-\text{div} g(\varepsilon)\nabla$ (and any $d$).
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$$K^0(\varepsilon; \zeta) = \Lambda^\varepsilon b(D)(A^0 - \zeta I)^{-1},$$

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$$

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Estimate (3) is uniform with respect to $\varphi$ in any sector $\varphi \in [\varphi_0, 2\pi - \varphi_0]$, i.e., $C(\varphi) \leq C(\varphi_0)$ in this sector.
Main results in a bounded domain

Now we formulate the results for $A_{D,\varepsilon}$ and $A_{N,\varepsilon}$. First, we assume in addition that $|\zeta| \geq 1$. 

Theorem 3 \cite{T. Suslina, 2014} 

Let $\zeta = |\zeta| e^{i\phi} \in \mathbb{C} \setminus [0, \infty)$ and $|\zeta| \geq 1$. Then there exists a number $\varepsilon_0$ (depending only on $O$ and $\Gamma$) such that for $0 < \varepsilon \leq \varepsilon_0$ we have

$$
\| (A_{D,\varepsilon} - \zeta I) - 1 - (A_{0,\varepsilon} - \zeta I) - 1 \|_{L^2(O) \to L^2(O)} \leq C(\phi)^{(\varepsilon |\zeta| - 1/2 + \varepsilon^2)}.
$$

Here $\flat = D, N$. Estimate (4) is uniform with respect to $\phi$ in any domain \{$\zeta \in \mathbb{C}: |\zeta| \geq 1, \phi \in [\phi_0, 2\pi - \phi_0]$\}, i.e., $C(\phi) \leq C(\phi_0)$ in this domain.
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$$
\|(A_{b,\varepsilon} - \zeta I)^{-1} - (A_{b}^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O}) \rightarrow L_2(\mathcal{O})} \leq C(\varphi) \left( \frac{\varepsilon}{|\zeta|^{1/2}} + \varepsilon^2 \right). \quad (4)
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In order to approximate the resolvent in the $L_2 \rightarrow H^1$ norm, we need to introduce a corrector

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Here

$$P_O : H^s(O; \mathbb{C}^n) \to H^s(\mathbb{R}^d; \mathbb{C}^n), \quad s = 0, 1, 2,$$

is a linear continuous extension operator.
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is a linear continuous extension operator. The operator $K_b(\varepsilon; \zeta)$ is continuous from $L_2(\mathcal{O}; \mathbb{C}^n)$ to $H^1(\mathcal{O}; \mathbb{C}^n)$. 
Main results in a bounded domain

In order to approximate the resolvent in the $L_2 \rightarrow H^1$ norm, we need to introduce a corrector

$$K_b(\varepsilon; \zeta) = \Lambda^\varepsilon S_\varepsilon b(D)P_{\mathcal{O}}(A^0_b - \zeta I)^{-1}, \ b = D, N.$$ 

Here

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In the case where $\Lambda \in L_\infty$ we can use the standard corrector

$$K^0_b(\varepsilon; \zeta) = \Lambda^\varepsilon b(D)(A^0_b - \zeta I)^{-1}, \ b = D, N,$$

which in this case is a continuous mapping of $L_2(\mathcal{O}; \mathbb{C}^n)$ to $H^1(\mathcal{O}; \mathbb{C}^n)$.
Main results in a bounded domain

**Theorem 4 [T. Suslina, 2014]**

Let \( \zeta = |\zeta|e^{i\varphi} \in \mathbb{C} \setminus [0, \infty) \) and \( |\zeta| \geq 1 \). Then there exists a number \( \varepsilon_0 \) (depending only on \( O \) and \( \Gamma \)) such that for \( 0 < \varepsilon \leq \varepsilon_0 \) we have

\[
\| (A_b, \varepsilon - \zeta I)^{-1} - (A^0_b - \zeta I)^{-1} - \varepsilon K_b(\varepsilon; \zeta) \|_{L^2(O) \to H^1(O)} \leq C(\varphi) \left( \frac{\varepsilon^{1/2}}{|\zeta|^{1/4}} + \varepsilon \right).
\]

(5)

Here \( b = D, N \). In the case where \( \Lambda \in L_\infty \) the same is true with \( K_b(\varepsilon; \zeta) \) replaced by \( K^0_b(\varepsilon; \zeta) \).

Estimate (5) is uniform with respect to \( \varphi \) in any domain \( \{ \zeta \in \mathbb{C} : |\zeta| \geq 1, \ \varphi \in [\varphi_0, 2\pi - \varphi_0] \} \), i.e., \( C(\varphi) \leq C(\varphi_0) \) in this domain.
Main results in a bounded domain: a different approximation of the resolvent

Also, we find a different approximation of the resolvent for a wider domain of $\zeta$. Let us formulate this result for the Dirichlet case.

**Theorem 5 [T. Suslina, 2014]**

Let $c_* > 0$ be a common lower bound of the operators $A_{D, \varepsilon}$ and $A^0_D$. Let $\zeta \in \mathbb{C} \setminus [c_*, \infty)$. We put $\zeta - c_* = |\zeta - c_*| e^{i\psi}$. There exists a number $\varepsilon_0$ such that for $0 < \varepsilon \leq \varepsilon_0$ we have

$$\| (A_{D, \varepsilon} - \zeta I)^{-1} - (A^0_D - \zeta I)^{-1} \|_{L_2(\mathcal{O}) \rightarrow L_2(\mathcal{O})} \leq \rho(\zeta) \varepsilon,$$

(6)

$$\rho(\zeta) = \begin{cases} c(\psi)|\zeta - c_*|^{-2}, & |\zeta - c_*| < 1, \\ c(\psi), & |\zeta - c_*| \geq 1. \end{cases}$$
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**Theorem 5 [T. Suslina, 2014]**

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$$\| (A_{D,\varepsilon} - \zeta I)^{-1} - (A^0_D - \zeta I)^{-1} \|_{L^2(\Omega) \to L^2(\Omega)} \leq \rho(\zeta) \varepsilon, \quad (6)$$

$$\rho(\zeta) = \begin{cases} 
  c(\psi) |\zeta - c_*|^{-2}, & |\zeta - c_*| < 1, \\
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\end{cases}$$

Inequality (6) is uniform with respect to $\psi$ in any sector $\psi \in [\psi_0, 2\pi - \psi_0]$. 
Main results in a bounded domain: a different approximation of the resolvent

Theorem 6 [T. Suslina, 2014]

Under the assumptions of Theorem 5 for $0 < \varepsilon \leq \varepsilon_0$ we have

$$
\|(A_{D,\varepsilon} - \zeta I)^{-1} - (A_D^0 - \zeta I)^{-1} - \varepsilon K_D(\varepsilon; \zeta)\|_{L_2(O) \to H^1(O)} \leq \rho(\zeta)\varepsilon^{1/2}. \quad (7)
$$

Here

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\rho(\zeta) = \begin{cases} 
c(\psi)|\zeta - c_*|^{-2}, & |\zeta - c_*| < 1, 
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In the case where $\Lambda \in L_\infty$ the same is true with $K_D(\varepsilon; \zeta)$ replaced by $K_D^0(\varepsilon; \zeta)$.

Estimate (7) is uniform with respect to $\psi$ in any sector $\psi \in [\psi_0, 2\pi - \psi_0]$, i.e., $c(\psi) \leq c(\psi_0)$ in this sector.
Main results in a bounded domain: a different approximation of the resolvent

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*In the case where $\Lambda \in L_\infty$ the same is true with $K_D(\varepsilon; \zeta)$ replaced by $K_D^0(\varepsilon; \zeta)$.*

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Analogs of Theorems 5 and 6 are obtained also for $A_{N,\varepsilon}$:
Estimates of the form (2)–(7) are called the *operator error estimates in homogenization theory*. Before such estimates were studied for a fixed regular point $\zeta$. 
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Estimates of the form (2)–(7) are called the operator error estimates in homogenization theory. Before such estimates were studied for a fixed regular point $\zeta$.

In a series of papers by M. Birman and T. Suslina (2001–2006), operator error estimates for homogenization problems in $\mathbb{R}^d$ were obtained by the operator-theoretic method. In particular, for the operator $A_\varepsilon$ estimates (2) and (3) were proved in the case $\zeta = -1$:

$$
\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C\varepsilon, \quad (8)
$$

$$
\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq \tilde{C}\varepsilon. \quad (9)
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A different approach. V. V. Zhikov and S. E. Pastukhova (2005, 2006) studied the acoustics operator \( A_\varepsilon = -\text{div} \, g(\varepsilon) \nabla \) and the operator of elasticity theory. The results similar to (8) and (9) in \( \mathbb{R}^d \) were obtained by "the modified method of the first approximation". Also, they studied the Dirichlet and Neumann problems for the same operators in a bounded domain. They obtained estimate

\[
\|A_\varepsilon - 1 - (A_0 - 1 - \varepsilon K)(\varepsilon)\|_{L^2(O)} \rightarrow H^1(O) \leq C \varepsilon^{1/2}.
\]

The error becomes worse because of the boundary influence. As a consequence, they obtained estimate

\[
\|A_\varepsilon - 1 - (A_0 - 1)\|_{L^2(O)} \leq C \varepsilon^{1/2}.
\] (10)

The improvement of estimate (10) was a natural problem. In the Dirichlet problem for the acoustics equation, using the maximum principle, Zhikov and Pastukhova obtained estimate of order \( O(\varepsilon^{d/(2d-2)}) \) for \( d \geq 3 \) and \( O(\varepsilon |\log \varepsilon|) \) for \( d = 2 \).
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$$\|A_{b,\varepsilon}^{-1} - (A_0^{-1} - \varepsilon K_b(\varepsilon))\|_{L^2(\Omega) \to H^1(\Omega)} \leq C\varepsilon^{1/2}.$$ 

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In the recent paper by C. Kenig, F. Lin and Z. Shen (2012) a sharp order estimate (11) was obtained for uniformly elliptic systems. However, they assume that the coefficients are real-valued and Hölder continuous.
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Note that the class of operators $A_\varepsilon$ that we consider is wider than the class studied by Kenig, Lin and Shen. Also, we do not impose any smoothness conditions on coefficients.
The operators $A_{D,\varepsilon}$ and $A_{N,\varepsilon}$ that we consider have been studied in recent papers (2012–2014) by the author; one paper (about $L_2 \rightarrow H^1$ approximation of $A_{D,\varepsilon}^{-1}$) is joint with M. Pakhnin. First, the results for a fixed $\zeta$ were obtained, and in 2014 the two-parametric estimates presented above have been proved.
Let us discuss the method of investigation. Theorems 1 and 2 (about the operator $A_\varepsilon$ in $L_2(\mathbb{R}^d; \mathbb{C}^n)$) can be easily deduced from the known results of Birman and Suslina for $\zeta = -1$ by appropriate resolvent identities and the scaling transformation.
For the problems in a bounded domain, it is impossible to deduce the results of Theorems 3 and 4 for any $\zeta$ from the results for $\zeta = -1$. 

The proof is based on using the results for the problem in $\mathbb{R}^d$, introduction of the boundary layer correction term and a careful analysis of this term. Some technical tricks, in particular, using the extension to $\mathbb{R}^d$ and the Steklov smoothing operator, are borrowed from the papers by Zhikov and Pastukhova.
Method of the proof. Theorems 3 and 4

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$$u_\varepsilon = (A_{D,\varepsilon} - \zeta I)^{-1} F, \quad u_0 = (A_D^0 - \zeta I)^{-1} F,$$

where $F \in L_2(\mathcal{O}; \mathbb{C}^n)$. 
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where \( F \in L_2(O; \mathbb{C}^n) \). It means that \( u_\varepsilon \in H^1_0(O; \mathbb{C}^n) \) is the generalized solution of the Dirichlet problem

\[ b(D)^* g^\varepsilon b(D)u_\varepsilon - \zeta u_\varepsilon = F \quad \text{in} \; O, \quad u_\varepsilon |_{\partial O} = 0, \]

and \( u_0 \in H^2(O; \mathbb{C}^n) \cap H^1_0(O; \mathbb{C}^n) \) is the solution of the ”homogenized” Dirichlet problem

\[ b(D)^* g^0 b(D)u_0 - \zeta u_0 = F \quad \text{in} \; O, \quad u_0 |_{\partial O} = 0. \]
Method of the proof. Theorems 3 and 4

Let $\tilde{u}_0 = P_O u_0 \in H^2(\mathbb{R}^d; \mathbb{C}^n)$ is the extension of $u_0$ to $\mathbb{R}^d$. Denote by $v_\varepsilon = u_0 + \varepsilon K_D(\varepsilon; \zeta) F$ the first order approximation to the solution $u_\varepsilon$:

$$v_\varepsilon = u_0 + \varepsilon \Lambda^\varepsilon S_\varepsilon b(D) \tilde{u}_0.$$
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In order to prove Theorem 3, we have to estimate \( \|u_\varepsilon - u_0\|_{L^2(O)} \).
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In order to prove Theorem 3, we have to estimate $\|u_\varepsilon - u_0\|_{L^2(O)}$. In order to prove Theorem 4, we have to estimate $\|u_\varepsilon - v_\varepsilon\|_{H^1(O)}$.

The difference $u_\varepsilon - v_\varepsilon$ does not satisfy the Dirichlet condition on $\partial O$. 
Method of the proof. Theorems 3 and 4

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In order to prove Theorem 3, we have to estimate \( \| u_\varepsilon - u_0 \|_{L^2(\Omega)} \). In order to prove Theorem 4, we have to estimate \( \| u_\varepsilon - v_\varepsilon \|_{H^1(\Omega)} \).

The difference \( u_\varepsilon - v_\varepsilon \) does not satisfy the Dirichlet condition on \( \partial \Omega \). We consider the ”discrepancy” \( w_\varepsilon \), which is the solution of the problem

\[
b(D)^* g^\varepsilon b(D) w_\varepsilon - \zeta w_\varepsilon = 0 \text{ in } \Omega; \quad w_\varepsilon_{|\partial \Omega} = \varepsilon \Lambda^\varepsilon (S_\varepsilon b(D) u_0)_{|\partial \Omega}.
\]

This \( w_\varepsilon \) is also called ”the boundary layer correction term”.

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Method of the proof. Theorems 3 and 4

Using Theorems 1 and 2 (for the problem in $\mathbb{R}^d$), it is easy to prove that

$$\|u_\varepsilon - v_\varepsilon + w_\varepsilon\|_{H^1(\Omega)} \leq C(\varphi)\varepsilon \|F\|_{L^2(\Omega)},$$

(12)

$$\|u_\varepsilon - u_0 + w_\varepsilon\|_{L^2(\Omega)} \leq C(\varphi)\frac{\varepsilon}{|\zeta|^{1/2}} \|F\|_{L^2(\Omega)}.$$  

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Method of the proof. Theorems 3 and 4

Using Theorems 1 and 2 (for the problem in $\mathbb{R}^d$), it is easy to prove that

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- In order to prove Theorem 4, we have to obtain appropriate estimate for $\|w_\varepsilon\|_{H^1(\Omega)}$.
- In order to prove Theorem 3, we have to obtain appropriate estimate for $\|w_\varepsilon\|_{L^2(\Omega)}$. 
Lemma 1

\[ \| \mathbf{w}_\varepsilon \|_{H^1(\Omega)} \leq C(\varphi) \left( \frac{\varepsilon^{1/2}}{|\zeta|^{1/4}} + \varepsilon \right) \| \mathbf{F} \|_{L^2(\Omega)}. \]
Method of the proof. Theorems 3 and 4

Lemma 1

\[ \| w_\varepsilon \|_{H^1(\mathcal{O})} \leq C(\varphi) \left( \frac{\varepsilon^{1/2}}{|\zeta|^{1/4}} + \varepsilon \right) \| F \|_{L^2(\mathcal{O})}. \]

Lemma 2

\[ \| w_\varepsilon \|_{L^2(\mathcal{O})} \leq C(\varphi) \left( \frac{\varepsilon}{|\zeta|^{1/2}} + \varepsilon^2 \right) \| F \|_{L^2(\mathcal{O})}. \]
Method of the proof. Theorems 3 and 4

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$$\|w_\varepsilon\|_{L^2(O)} \leq C(\varphi) \left( \frac{\varepsilon}{|\zeta|^{1/2}} + \varepsilon^2 \right) \|F\|_{L^2(O)}.$$

Lemma 1 and estimate (12) imply Theorem 4.
Lemma 2 and estimate (13) imply Theorem 3.
Method of the proof. Theorems 3 and 4

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Lemma 1 and estimate (12) imply Theorem 4.
Lemma 2 and estimate (13) imply Theorem 3.
Main technical work is the proof of Lemmas 1 and 2. Main technical difficulties are related to estimates in the \( \varepsilon \)-neighborhood of \( \partial O \).
Theorems 5 and 6 are deduced from the already proved estimates with $\zeta = -1$ by appropriate resolvent identities.
Application to parabolic problems

The results of Theorems 3–6 can be applied to the study of the parabolic initial boundary-value problems in the domain $O$. Such problems are reduced to the study of the operator exponential $\exp(-A^{\flat,\varepsilon}t)$. It is natural to use representation

$$\exp(-A^{\flat,\varepsilon}t) = \frac{1}{2\pi i} \int_{\gamma} e^{-\zeta t} (A^{\flat,\varepsilon} - \zeta I)^{-1} d\zeta,$$

where $\gamma$ is a suitable contour in the complex plane. In order to find two-parameter approximations of the exponential of right order (with respect to $\varepsilon$ and $t$), two-parameter approximations of the resolvent (with respect to $\varepsilon$ and $\zeta$) found in Theorems 3–6 are needed. The corresponding results for parabolic problems were obtained in 2014 jointly with Y. Meshkova.
Application to parabolic problems

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where $\gamma$ is a suitable contour in the complex plane. In order to find twoparametric approximations of the exponential of right order (with respect to $\varepsilon$ and $t$), twoparametric approximations of the resolvent (with respect to $\varepsilon$ and $\zeta$) found in Theorems 3–6 are needed.
The results of Theorems 3–6 can be applied to the study of the parabolic initial boundary-value problems in the domain $\mathcal{O}$. Such problems are reduced to the study of the operator exponential $\exp(-\mathcal{A}_{b,\varepsilon}t)$. It is natural to use representation

$$\exp(-\mathcal{A}_{b,\varepsilon}t) = \frac{1}{2\pi i} \int_{\gamma} e^{-\zeta t}(\mathcal{A}_{b,\varepsilon} - \zeta I)^{-1} d\zeta,$$

where $\gamma$ is a suitable contour in the complex plane. In order to find twoparametric approximations of the exponential of right order (with respect to $\varepsilon$ and $t$), twoparametric approximations of the resolvent (with respect to $\varepsilon$ and $\zeta$) found in Theorems 3–6 are needed. The corresponding results for parabolic problems were obtained in 2014 jointly with Y. Meshkova.
References


References

References
