Homogenization of Periodic Elliptic Operators: Error Estimates in Dependence of the Spectral Parameter

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Plan

- Introduction
- Statement of the problem
- The effective operator
- Main results
- Survey
- Method of investigation
- Application to parabolic problems

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Books on homogenization theory:

- A. Bensoussan, J.-L. Lions, G. Papanicolaou. Asymptotic analysis for periodic structures, 1978.
- N. S. Bakhvalov, G. P. Panasenko. Homogenization: averaging of processes in periodic media, 1984.
- V. V. Zhikov, S. M. Kozlov, O. A. Oleinik. Homogenization of differential operators, 1993.

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We study the *operator error estimates* in homogenization theory. Such estimates were intensively studied during the last 10–12 years by many authors: Birman, Suslina, Zhikov, Pastukhova, Griso, Kenig, Lin, Shen, others (more detailed survey will be given later on).

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In this talk we present some recent results on the operator error estimates.

- We study homogenization problems for a wide class of the second order *matrix elliptic operators* A_ε, both in $\mathbb{R}^d$ and in a bounded domain with the Dirichlet or Neumann boundary conditions. The coefficients of A_ε are periodic and depend on $\mathbf{x}/\varepsilon$.

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- We obtain approximations of the resolvent $(A_\varepsilon - \zeta I)^{-1}$ of an operator A_ε in a regular point ζ in different operator norms.
- We find *twoparametric operator error estimates* in dependence of ε and ζ.

Statement of the problem

Let Γ be a lattice in $\mathbb{R}^d$, let Ω be the cell of Γ.
Example: $\Gamma = \mathbb{Z}^d, \quad \Omega = \left(-\frac{1}{2}, \frac{1}{2}\right)^d.$

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We study $(n \times n)$*-matrix elliptic second order DO*

$$b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$$

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The operator

$$b(\mathbf{D}) = \sum_{j=1}^{d} b_j D_j$$

is a first order $(m \times n)$-matrix DO; b_j are constant matrices, and $m \geq n$.

Statement of the problem

The symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^{d} b_j \xi_j$ is such that

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d, \qquad (1_D)$$

for the problem in $\mathbb{R}^d$ and for the case of the Dirichlet condition, or

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Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain with the boundary $\partial\mathcal{O} \in C^{1,1}$.

Statement of the problem

Main objects

- *By* $\mathcal{A}_\varepsilon$ *we denote the operator in* $L_2(\mathbb{R}^d;\mathbb{C}^n)$ *given by* $b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$.

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Precise definitions are given in terms of the corresponding quadratic forms. Under our assumptions, the operators $\mathcal{A}_\varepsilon$, $\mathcal{A}_{D,\varepsilon}$, $\mathcal{A}_{N,\varepsilon}$ are strongly elliptic; the corresponding quadratic forms satisfy the coercivity conditions.

Statement of the problem

Problem

The problem is to study the behavior of the resolvents

$$(\mathcal{A}_\varepsilon - \zeta I)^{-1}, \ (\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1}, \ (\mathcal{A}_{N,\varepsilon} - \zeta I)^{-1}$$

in dependence of ε *and* ζ. *We wish to find two-parametric approximations for these resolvents in the* $L_2 \to L_2$ *and* $L_2 \to H^1$ *operator norms.*

The effective operator

In order to formulate the results, we define the *effective operator* with constant coefficients:

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Definition of the effective matrix:

Let $\Lambda(\mathbf{x})$ be the $(n \times m)$-matrix-valued Γ-periodic solution of the equation

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0.$$

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Then g^0 is an $(m \times m)$-matrix given by

$$g^0 = |\Omega|^{-1} \int_\Omega g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m)\, d\mathbf{x}.$$

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- *By* $\mathcal{A}_D^0$ *we denote the operator in* $L_2(\mathcal{O}; \mathbb{C}^n)$ *given by* $b(\mathbf{D})^* g^0 b(\mathbf{D})$ *with the Dirichlet boundary condition. Its domain is* $H^2(\mathcal{O}; \mathbb{C}^n) \cap H_0^1(\mathcal{O}; \mathbb{C}^n)$.

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- *By* $\mathcal{A}_N^0$ *we denote the operator in* $L_2(\mathcal{O};\mathbb{C}^n)$ *given by* $b(\mathbf{D})^* g^0 b(\mathbf{D})$ *with the Neumann boundary condition. Its domain is* $\{\mathbf{u} \in H^2(\mathcal{O};\mathbb{C}^n) : \ \partial_{\boldsymbol{\nu}}^0 \mathbf{u}|_{\partial\mathcal{O}} = 0\}$*.*

Main results in the whole space

We start with the results in $\mathbb{R}^d$. A point $\zeta \in \mathbb{C} \setminus [0, \infty)$ is regular both for $\mathcal{A}_\varepsilon$ and $\mathcal{A}^0$.

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Theorem 1 [T. Suslina, 2014]

Let $\zeta = |\zeta| e^{i\varphi} \in \mathbb{C} \setminus [0, \infty)$. *Then for* $\varepsilon > 0$ *we have*

$$\|(\mathcal{A}_\varepsilon - \zeta I)^{-1} - (\mathcal{A}^0 - \zeta I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(\varphi) \frac{\varepsilon}{|\zeta|^{1/2}}. \qquad (2)$$

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Inequality (2) is uniform with respect to φ *in any sector* $\varphi \in [\varphi_0, 2\pi - \varphi_0]$, *i. e.,* $C(\varphi) \leq C(\varphi_0)$ *in this sector.*

Main results in the whole space

In order to approximate the resolvent in the $L_2 \to H^1$ norm, we need to introduce a corrector

$$K(\varepsilon;\zeta) = \Lambda^{\varepsilon} S_{\varepsilon} b(\mathbf{D})(\mathcal{A}^0 - \zeta I)^{-1}.$$

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The operator $K(\varepsilon;\zeta)$ is continuous from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$. In the case where $\Lambda \in L_\infty$ we can use the standard corrector

$$K^0(\varepsilon;\zeta) = \Lambda^\varepsilon b(\mathbf{D})(\mathcal{A}^0 - \zeta I)^{-1},$$

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which in this case is a continuous mapping of $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$.
Remark. In some cases $\Lambda \in L_\infty$ automatically. In particular, 1) if $d = 2$; 2) for the acoustics operator $-\mathrm{div}\, g^\varepsilon \nabla$ (and any d).

Main results in the whole space

Theorem 2 [T. Suslina, 2014]

For $\zeta \in \mathbb{C} \setminus [0, \infty)$ *and* $\varepsilon > 0$ *we have*

$$\|(\mathcal{A}_\varepsilon - \zeta I)^{-1} - (\mathcal{A}^0 - \zeta I)^{-1} - \varepsilon K(\varepsilon;\zeta)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le C(\varphi)(1+|\zeta|^{-1/2})\varepsilon. \quad (3)$$

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Estimate (3) is uniform with respect to φ *in any sector* $\varphi \in [\varphi_0, 2\pi - \varphi_0]$, *i. e.,* $C(\varphi) \leq C(\varphi_0)$ *in this sector.*

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Now we formulate the results for $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}_{N,\varepsilon}$. First, we assume in addition that $|\zeta| \geq 1$.

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Theorem 3 [T. Suslina, 2014]

Let $\zeta = |\zeta| e^{i\varphi} \in \mathbb{C} \setminus [0,\infty)$ and $|\zeta| \geq 1$. Then there exists a number ε_0 (depending only on $\mathcal{O}$ and Γ) such that for $0 < \varepsilon \leq \varepsilon_0$ we have

$$\|(\mathcal{A}_{\flat,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_{\flat}^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq C(\varphi)\left(\frac{\varepsilon}{|\zeta|^{1/2}} + \varepsilon^2\right). \quad (4)$$

Here $\flat = D, N$. Estimate (4) is uniform with respect to φ in any domain $\{\zeta \in \mathbb{C} : \ |\zeta| \geq 1, \ \varphi \in [\varphi_0, 2\pi - \varphi_0]\}$, i. e., $C(\varphi) \leq C(\varphi_0)$ in this domain.

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In order to approximate the resolvent in the $L_2 \to H^1$ norm, we need to introduce a corrector

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$$P_{\mathcal{O}} : H^s(\mathcal{O};\mathbb{C}^n) \to H^s(\mathbb{R}^d;\mathbb{C}^n), \quad s = 0, 1, 2,$$

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In the case where $\Lambda \in L_\infty$ we can use the standard corrector

$$K_\flat^0(\varepsilon;\zeta) = \Lambda^\varepsilon b(\mathbf{D}) (\mathcal{A}_\flat^0 - \zeta I)^{-1}, \quad \flat = D, N,$$

which in this case is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$.

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Theorem 4 [T. Suslina, 2014]

Let $\zeta = |\zeta| e^{i\varphi} \in \mathbb{C} \setminus [0, \infty)$ *and* $|\zeta| \geq 1$. *Then there exists a number* ε_0 *(depending only on* $\mathcal{O}$ *and* Γ*) such that for* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$\|(\mathcal{A}_{\flat,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_\flat^0 - \zeta I)^{-1} - \varepsilon K_\flat(\varepsilon;\zeta)\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \leq C(\varphi) \left(\frac{\varepsilon^{1/2}}{|\zeta|^{1/4}} + \varepsilon \right). \qquad (5)$$

Here $\flat = D, N$. *In the case where* $\Lambda \in L_\infty$ *the same is true with* $K_\flat(\varepsilon;\zeta)$ *replaced by* $K_\flat^0(\varepsilon;\zeta)$.
Estimate (5) is uniform with respect to φ *in any domain* $\{\zeta \in \mathbb{C} : \ |\zeta| \geq 1, \ \varphi \in [\varphi_0, 2\pi - \varphi_0]\}$, *i. e.,* $C(\varphi) \leq C(\varphi_0)$ *in this domain.*

Main results in a bounded domain: a different approximation of the resolvent

Also, we find a different approximation of the resolvent for a wider domain of ζ. Let us formulate this result for the Dirichlet case.

Theorem 5 [T. Suslina, 2014]

Let $c_* > 0$ *be a common lower bound of the operators* $\mathcal{A}_{D,\varepsilon}$ *and* $\mathcal{A}_D^0$. *Let* $\zeta \in \mathbb{C} \setminus [c_*, \infty)$. *We put* $\zeta - c_* = |\zeta - c_*| e^{i\psi}$. *There exists a number* ε_0 *such that for* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_D^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq \rho(\zeta)\varepsilon, \tag{6}$$

$$\rho(\zeta) = \begin{cases} c(\psi)|\zeta - c_*|^{-2}, & |\zeta - c_*| < 1, \\ c(\psi), & |\zeta - c_*| \geq 1. \end{cases}$$

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Also, we find a different approximation of the resolvent for a wider domain of ζ. Let us formulate this result for the Dirichlet case.

Theorem 5 [T. Suslina, 2014]

Let $c_* > 0$ be a common lower bound of the operators $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}_D^0$. Let $\zeta \in \mathbb{C} \setminus [c_*, \infty)$. We put $\zeta - c_* = |\zeta - c_*| e^{i\psi}$. There exists a number ε_0 such that for $0 < \varepsilon \le \varepsilon_0$ we have

$$\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_D^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le \rho(\zeta)\varepsilon, \qquad (6)$$

$$\rho(\zeta) = \begin{cases} c(\psi)|\zeta - c_*|^{-2}, & |\zeta - c_*| < 1, \\ c(\psi), & |\zeta - c_*| \ge 1. \end{cases}$$

Inequality (6) is uniform with respect to ψ in any sector $\psi \in [\psi_0, 2\pi - \psi_0]$.

Main results in a bounded domain: a different approximation of the resolvent

Theorem 6 [T. Suslina, 2014]

Under the assumptions of Theorem 5 for $0 < \varepsilon \le \varepsilon_0$ *we have*

$$\|(\mathcal{A}_{D,\varepsilon} - \zeta I)^{-1} - (\mathcal{A}_D^0 - \zeta I)^{-1} - \varepsilon K_D(\varepsilon;\zeta)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le \rho(\zeta)\varepsilon^{1/2}. \quad (7)$$

Here

$$\rho(\zeta) = \begin{cases} c(\psi)|\zeta - c_*|^{-2}, & |\zeta - c_*| < 1, \\ c(\psi), & |\zeta - c_*| \ge 1. \end{cases}$$

In the case where $\Lambda \in L_\infty$ *the same is true with* $K_D(\varepsilon;\zeta)$ *replaced by* $K_D^0(\varepsilon;\zeta)$.
Estimate (7) is uniform with respect to ψ *in any sector* $\psi \in [\psi_0, 2\pi - \psi_0]$, *i. e.,* $c(\psi) \le c(\psi_0)$ *in this sector.*

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Analogs of Theorems 5 and 6 are obtained also for $\mathcal{A}_{N,\varepsilon}$.

Survey

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- In a series of papers by M. Birman and T. Suslina (2001–2006), operator error estimates for homogenization problems in $\mathbb{R}^d$ were obtained by the *operator-theoretic method*. In particular, for the operator $\mathcal{A}_\varepsilon$ estimates (2) and (3) were proved in the case $\zeta = -1$:

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leq C\varepsilon, \tag{8}$$

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leq \widetilde{C}\varepsilon. \tag{9}$$

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$$\|\mathcal{A}_{\flat,\varepsilon}^{-1} - (\mathcal{A}_\flat^0)^{-1} - \varepsilon K_\flat(\varepsilon)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leq C\varepsilon^{1/2}.$$

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The improvement of estimate (10) was a natural problem. In the Dirichlet problem for the acoustics equation, using the maximum principle, Zhikov and Pastukhova obtained estimate of order $O(\varepsilon^{d/(2d-2)})$ for $d \ge 3$ and $O(\varepsilon|\log\varepsilon|)$ for $d = 2$.

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- In the recent paper by **C. Kenig, F. Lin and Z. Shen** (2012) a sharp order estimate (11) was obtained for uniformly elliptic systems. However, they assume that the coefficients are real-valued and Hölder continuous.

  Note that the class of operators $\mathcal{A}_\varepsilon$ that we consider is wider than the class studied by Kenig, Lin and Shen. Also, we do not impose any smoothness conditions on coefficients.

Survey

- The operators $\mathcal{A}_{D,\varepsilon}$ and $\mathcal{A}_{N,\varepsilon}$ that we consider have been studied in recent papers (2012–2014) by the author; one paper (about $L_2 \to H^1$ approximation of $\mathcal{A}_{D,\varepsilon}^{-1}$) is joint with M. Pakhnin. First, the results for a fixed ζ were obtained, and in 2014 the two-parametric estimates presented above have been proved.

Method of the proof

Let us discuss the method of investigation. Theorems 1 and 2 (about the operator $\mathcal{A}_\varepsilon$ in $L_2(\mathbb{R}^d; \mathbb{C}^n)$) can be easily deduced from the known results of Birman and Suslina for $\zeta = -1$ by appropriate *resolvent identities* and the *scaling transformation*.

Method of the proof. Theorems 3 and 4

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The proof is based on using the results for the problem in $\mathbb{R}^d$, introduction of the boundary layer correction term and a careful analysis of this term.
Some technical tricks, in particular, using the extension to $\mathbb{R}^d$ and the Steklov smoothing operator, are borrowed from the papers by Zhikov and Pastukhova.

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where $\mathbf{F} \in L_2(\mathcal{O}; \mathbb{C}^n)$.

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where $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. It means that $\mathbf{u}_\varepsilon \in H_0^1(\mathcal{O};\mathbb{C}^n)$ is the generalized solution of the Dirichlet problem

$$b(\mathbf{D})^* g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon - \zeta \mathbf{u}_\varepsilon = \mathbf{F} \ \text{ in } \mathcal{O}, \quad \mathbf{u}_\varepsilon|_{\partial\mathcal{O}} = 0,$$

and $\mathbf{u}_0 \in H^2(\mathcal{O};\mathbb{C}^n) \cap H_0^1(\mathcal{O};\mathbb{C}^n)$ is the solution of the "homogenized" Dirichlet problem

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Let $\widetilde{\mathbf{u}}_0 = P_{\mathcal{O}}\mathbf{u}_0 \in H^2(\mathbb{R}^d;\mathbb{C}^n)$ is the extension of $\mathbf{u}_0$ to $\mathbb{R}^d$. Denote by $\mathbf{v}_\varepsilon = \mathbf{u}_0 + \varepsilon K_D(\varepsilon;\zeta)\mathbf{F}$ the first order approximation to the solution $\mathbf{u}_\varepsilon$:

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The difference $\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon$ does not satisfy the Dirichlet condition on $\partial\mathcal{O}$.

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The difference $\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon$ does not satisfy the Dirichlet condition on $\partial\mathcal{O}$. We consider the "discrepancy" $\mathbf{w}_\varepsilon$, which is the solution of the problem

$$b(\mathbf{D})^* g^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon - \zeta\mathbf{w}_\varepsilon = 0 \ \text{ in } \mathcal{O}; \quad \mathbf{w}_\varepsilon|_{\partial\mathcal{O}} = \varepsilon\Lambda^\varepsilon(S_\varepsilon b(\mathbf{D})\mathbf{u}_0)|_{\partial\mathcal{O}}.$$

This $\mathbf{w}_\varepsilon$ is also called "the boundary layer correction term".

Method of the proof. Theorems 3 and 4

Using Theorems 1 and 2 (for the problem in $\mathbb{R}^d$), it is easy to prove that

$$\|\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon + \mathbf{w}_\varepsilon\|_{H^1(\mathcal{O})} \leq C(\varphi)\varepsilon\|\mathbf{F}\|_{L_2(\mathcal{O})}, \tag{12}$$

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 + \mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})} \leq C(\varphi)\frac{\varepsilon}{|\zeta|^{1/2}}\|\mathbf{F}\|_{L_2(\mathcal{O})}. \tag{13}$$

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- In order to prove Theorem 4, we have to obtain appropriate estimate for $\|\mathbf{w}_\varepsilon\|_{H^1(\mathcal{O})}$.
- In order to prove Theorem 3, we have to obtain appropriate estimate for $\|\mathbf{w}_\varepsilon\|_{L_2(\mathcal{O})}$.

Method of the proof. Theorems 3 and 4

Lemma 1

$$\|\mathbf{w}_\varepsilon\|_{H^1(\mathcal{O})} \leq C(\varphi) \left(\frac{\varepsilon^{1/2}}{|\zeta|^{1/4}} + \varepsilon \right) \|\mathbf{F}\|_{L_2(\mathcal{O})}.$$

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Lemma 2

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Lemma 1 and estimate (12) imply Theorem 4.
Lemma 2 and estimate (13) imply Theorem 3.

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Lemma 1 and estimate (12) imply Theorem 4.
Lemma 2 and estimate (13) imply Theorem 3.
Main technical work is the proof of Lemmas 1 and 2. Main technical difficulties are related to estimates in the ε-neighborhood of $\partial\mathcal{O}$.

Method of the proof. Theorems 5 and 6

Theorems 5 and 6 are deduced from the already proved estimates with $\zeta = -1$ by appropriate resolvent identities.

Application to parabolic problems

The results of Theorems 3–6 can be applied to the study of the parabolic initial boundary-value problems in the domain $\mathcal{O}$.

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$$\exp(-\mathcal{A}_{\flat,\varepsilon} t) = \frac{1}{2\pi i} \int_{\gamma} e^{-\zeta t} (\mathcal{A}_{\flat,\varepsilon} - \zeta I)^{-1} \, d\zeta,$$

where γ is a suitable contour in the complex plane.

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In order to find twoparametric approximations of the exponential of right order (with respect to ε and t), twoparametric approximations of the resolvent (with respect to ε and ζ) found in Theorems 3–6 are needed.

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The corresponding results for parabolic problems were obtained in 2014 jointly with Y. Meshkova.

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