OPTIMAL HARDY-TYPE INEQUALITY FOR SECOND-ORDER ELLIPTIC OPERATOR: AN ANSWER TO A PROBLEM OF SHMUEL AGMON

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We give a general answer to the following fundamental problem posed by Shmuel Agmon 30 years ago:

Given a (symmetric) linear elliptic operator P of second-order in $\mathbb{R}^n$, find a nonnegative weight function W which is "as large as possible", such that for some neighborhood of infinity G the following inequality holds

$$(P - W) \geq 0 \qquad \text{in the sense of the associated quadratic form on } C_0^\infty(G).$$

We construct, for a general subcritical second-order elliptic operator P on a domain M in $\mathbb{R}^n$ (or on a noncompact manifold M), a Hardy-type weight W which is optimal in the following natural sense:

- $(P - \lambda W) \geq 0$ *on* $C_0^\infty(M)$ *for all* $\lambda \leq 1$,
- *For* $\lambda = 1$*, the operator* $(P - \lambda W)$ *is null-critical in* M,
- *For any* $\lambda > 1$*, and any neighborhood of infinity* $G \subset M$*, the operator* $(P - \lambda W)$ *is not nonnegative on* $C_0^\infty(G)$.
- *If* P *is symmetric and* $W > 0$*, then the spectrum and the essential spectrum of the operator* $W^{-1}P$ *are equal to* $[1, \infty)$.

Our method is based on the theory of positive solutions and applies to both symmetric and nonsymmetric operators on noncompact manifolds. Moreover, the results can be generalized to certain p-Laplacian type operators and to Schrödinger operators on graphs. The constructed weight W is given by an explicit simple formula involving two positive solutions of the equation $Pu = 0$.

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