BEURLING'S THEOREM, DAVENPORT'S FORMULA, AND THE RIEMANN HYPOTHESIS

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One of versions of A. Beurling's theorem about shift-invariant subspaces in the Hardy space on the half-plane $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ is as follows:

A closed subspace of $H^2(\mathbb{C}_+)$ is invariant under the operators of multiplication by $\exp(itx)$ for all non-negative real t if and only if it has the form $\theta H^2(\mathbb{C}_+)$ for some inner function θ.

A bounded analytic function θ in $\mathbb{C}_+$ is *inner* if its boundary values are unimodular almost everywhere on the boundary of $\mathbb{C}_+$.
A function $f \in H^2(\mathbb{C}_+)$ is *outer* if the closed linear span of the functions $\exp(itx)f(x)$, $t \in \mathbb{R}$, $t \geqslant 0$, coincides with $H^2(\mathbb{C}_+)$.

The function $\frac{s-1}{s^2}\zeta(s)$, where ζ is the Riemann zeta function, belongs to the Hardy class H^2 on the half-plane $\{s : \operatorname{Re} s > \frac{1}{2}\}$. The famous Riemann Hypothesis about the zeros of ζ is equivalent to the statement that this function is outer.
An application of the inverse Mellin transform gives us the criterion of A. Beurling and B. Nyman:

The Riemann Hypothesis is equivalent to the assertion that the indicator of the interval $(0,1)$ belongs to the closed linear span in $L^2(0,+\infty)$ of the functions $\rho(\frac{1}{ax})$, $a \in \mathbb{R}$, $a \geqslant 1$,
where $\rho(\cdot)$ denotes the fractional part of a real number.

L. Báez-Duarte proved that if the Riemann Hypothesis is true, then the indicator of $(0,1)$ can be approximated by finite linear combinations of $\rho(\frac{1}{nx})$ with positive integers n.

We use the Mellin transform modified by multiplication operators to obtain several criteria for the Riemann Hypothesis connected with H. Davenport's formula

$$-\frac{\sin 2\pi x}{\pi} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\, \rho(nx),$$

where μ is the Möbius function.