Euler International Mathematical Institute
St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences

8th St. Petersburg Conference in Spectral Theory

3 – 6 July 2016

Dedicated to the memory of M. Sh. Birman (1928–2009)

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Chebyshev Laboratory (St. Petersburg State University)

Programme and abstracts

St. Petersburg, 2016

Organizers:

Alexandre Fedotov, Nikolai Filonov, Alexander Sobolev, Tatyana Suslina, Dmitri Yafaev

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Conference website: http://www.pdmi.ras.ru/EIMI/2016/ST/index.html

Speakers

- Rafael BENGURIA, *Catholic University, Santiago, Chile*
- Dmitry CHELKAK, *University of Geneva, Switzerland*
- Alexander FEDOTOV, *St. Petersburg State University*
- Vladimir KAPUSTIN, *Steklov Institute, St. Petersburg*
- Frdric KLOPP, *Université Pierre et Marie Curie, Paris*
- Nikolay KOPACHEVSKY, *Federal University of Crimea*
- Vladislav KRAVCHENKO, *CINVESTAV del IPN - Querétaro, Mexico*
- Hajo LESCHKE, *Universität Erlangen-Nüremberg, Germany*
- Peter MUELLER, *Technische Universitt Mnchen, Germany*
- Sergei NAZAROV, *St. Petersburg State University*
- Yehuda PINCHOVER, *Technion, Israel*
- Oleg SARAFANOV, *St. Petersburg State University*
- Sergey SIMONOV, *Steklov Institute, St. Petersburg*
- Wolfgang SPITZER, *FU Hagen, Germany*
- Vasiliy VASYUNIN, *Steklov Institute, St.Petersburg*
- Timo WEIDL, *Universität Stuttgart, Germany*

Young Researchers

- Andre HÄNEL, *Universität Stuttgart, Germany*
- Dmitry KORIKOV, *St. Petersburg State University*
- Nikita RASTEGAEV, *Steklov Institute and Chebyshev Laboratory, St. Petersburg*
- Ekaterina SHCHETKA, *St. Petersburg State University*

Conference programme

SUNDAY, 3 July

9:30 – 10:00: REGISTRATION

10:00 – 10:50: Rafael Benguria (Catholic University, Santiago, Chile)
A Generalized Brezis–Nirenberg Problem on the unit ball of spaces of fractional dimension

COFFEE BREAK

11:20 – 12:10: Dmitry Chelkak (University of Geneva, Switzerland)
2D Ising Model: Correlation functions at criticality via Riemann-type boundary value problems

12:20 – 13:10: Wolfgang Spitzer (FU Hagen, Germany)
Localization for transversally periodic random potentials on binary trees

LUNCH

15:10 – 16:00: Nikolay Kopachevsky (Federal University of Crimea)
Spectral and initial boundary value problems generated by sesquilinear forms

COFFEE BREAK

16:30 – 17:20: Sergei Nazarov (St. Petersburg State University)
Wandering eigenvalues of the Laplacian with an improper Robin condition

18:00 BOAT TRIP

MONDAY, 4 July

10:00 – 10:50: Sergey Simonov (Steklov Institute, St. Petersburg)
Spectral properties of the half-line Schrödinger operator with slowly decaying Wigner-von Neumann potential

COFFEE BREAK

11:20 – 12:10: Vladislav Kravchenko (CINVESTAV del IPN - Querétaro, Mexico)
Construction of transmutations and applications to spectral problems

12:20 – 13:10: Hajo Leschke (Universität Erlangen-Nüremberg, Germany)
Local and entanglement entropy of the ideal Fermi gas

LUNCH

14:30 – 15:20: Peter Mueller (Technische Universität München, Germany)
Bounds on the averaged spectral shift function for random Schrödinger operators with applications

COFFEE BREAK

15:50 – 16:50: Vladimir Kapustin (Steklov Institute, St. Petersburg)
Beurling's theorem, Davenport's formula, and the Riemann hypothesis

TUESDAY, 5 July

10:00 – 10:50: Oleg Sarafanov (St. Petersburg State University)
Asymptotic and numerical study of resonsant tunneling in quantum waveguides of variable cross-section

COFFEE BREAK

11:20 – 12:10: Frédéric Klopp (Université Pierre et Marie Curie, Paris)
Interacting electrons in a random background

12:20 – 13:10: Alexander Fedotov (St. Petersburg State University)
Stark-Wannier resonances and cubic exponential sums

LUNCH

15:10 – 15:35: Andre Hänel (Universität Stuttgart, Germany)
Spectral asymptotics for an elastic strip with an interior crack

15:40 – 16:05: Dmitry Korikov (St. Petersburg State University)
Asymptotics of solutions to nonstationary Maxwell system in domain with small holes

COFFEE BREAK

16:40 – 17:05: Nikita Rastegaev (Steklov Institute and Chebyshev Laboratory, St. Petersburg)
Spectral asymptotics of operators of the tensor product type with almost regular marginal asymptotics

17:10 – 17:35: Ekaterina Shchetka (St. Petersburg State University)
Complex WKB method for difference equations in the complex plane

18:00 CONFERENCE PARTY

WEDNESDAY, 6 July

10:00 – 10:50: Vasiliy Vasyunin (Steklov Institute, St.Petersburg)
Bellman function method in analysis

COFFEE BREAK

11:20 – 12:10: Yehuda Pinchover (Technion, Israel)
Optimal Hardy-type inequality for second-order elliptic operator: an answer to a problem of Shmuel Agmon

12:20 – 13:10: Timo Weidl (Universität Stuttgart, Germany)
Semiclassical spectral bounds with remainder terms

END OF CONFERENCE AND LUNCH

Abstracts of the talks

A Generalized Brezis–Nirenberg Problem on the unit ball of spaces of fractional dimension

Rafael BENGURIA

Catholic University, Santiago, Chile

Joint work with Soledad Benguria. Supported by Fondecyt (Chiile) Project # 112–0836 and by the Nucleo Milenio en “Fisica Matemática”, RC–12–0002 (ICM, Chile).

In 1983 Brezis and Nirenberg, considered the nonlinear eigenvalue problem,

$$-\Delta u = \lambda u + |u|^{4/(n-2)} u,$$

with $u \in H_0^1(\Omega)$, where Ω is bounded smooth domain in $\mathbb{R}^n$, with $n \geq 3$. Among other results, they proved that if $n \geq 4$, there is a positive solution of this problem for all $\lambda \in (0, \lambda_1)$ where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Ω. They also proved that if $n = 3$, there is a $\mu(\Omega) > 0$ such that for any $\lambda \in (\mu, \lambda_1)$, the nonlinear eigenvalue problem has a positive solution. Moreover, if Ω is a ball, $\mu = \lambda_1/4$. For positive radial solutions of this problem in a (unit) ball, one is led to an ODE that still makes sense when n is a real number rather than a natural number.

In this talk I will consider the family of “radial” problems in *dimension* n, where $2 \leq n \leq 4$, with critical Sobolev exponent, given by,

$$-u''(x) - (n-1)\frac{a'(x)}{a(x)} u'(x) = \lambda\, u(x) + |u(x)|^{p-1}\, u(x),$$

in $(0, R)$ with $u'(0) = 0$ and $u(R) = 0$, for some appropriate class of functions $a(x)$ that include the Euclidean and the Hyperbolic case. We determine the range of values for existence and nonexistence of positive solutions, generalizing the classical Brezis–Nirenberg problem.

2D Ising model: correlation functions at criticality via Riemann-type boundary value problems

Dmitry CHELKAK

University of Geneva, Switzerland

In this talk we give a survey of convergence results for correlation functions in the critical 2D Ising model obtained during the last several years. In particular, it includes the convergence (as the mesh size tends to zero, in arbitrary planar domains) of properly rescaled spin expectations to the conformal covariant limits predicted by Conformal Field Theory. We start with reviewing the combinatorics of the nearest-neighbor Ising model considered on a general planar graph and the existence of discrete holomorphic observables in the critical model on a regular grid, which solve some special Riemann-type boundary value problems. Though spin correlations cannot be directly obtained as the values of these discrete holomorphic functions, one can express their spatial derivatives via discrete holomorphic spinors defined in a similar manner. Analyzing the convergence of these spinors as the mesh size tends to zero, one can reconstruct the scaling limits of spin correlations from the asymptotic behaviour at singularities of their continuous counterparts. Interestingly, one can use the same approach to give a short proof of some classical results about the diagonal spin-spin expectations in the full plane via orthogonal polynomials techniques. The core part of the talk talk is based on a joint work with Clément Hongler (Lausanne) and Konstantin Iyzurov (Helsinki).

Localization for transversally periodic random potentials on binary trees

Wolfgang SPITZER

FU Hagen, Germany

We consider a random Schrödinger operator on the binary tree with a random potential which is the sum of a random radially symmetric potential, Q_r, and a random transversally periodic potential, Q_t, with coupling constant κ. Using a new one-dimensional dynamical systems approach combined with Jensen's inequality in hyperbolic space (our key estimate) we obtain a fractional moment estimate proving localization for small and large κ. Together with a previous result we therefore obtain a model with two Anderson transitions, from localization to delocalization and back to localization, when increasing κ. As a by-product we also have a partially new proof of one-dimensional Anderson localization at any disorder. This is joint work with Richard Froese, Darrick Lee, Christian Sadel, and Günter Stolz.

Spectral and initial boundary value problems generated by sesquilinear forms

Nikolay KOPACHEVSKY

Federal University of Crimea

Joint work with K. Radomirskaya and A. Yakubova.

1. Abstract Green's Identity for a triple of Hilbert spaces and trace operator and its generalization for the case of boundary and uniformly accretive form is considered. We give also formulation of Abstract Greens Identity for mixed boundary value problems.

2. On example of generalized symmetric Greens Identity for Laplace operator we propose the general approach to transmission problems for some configurations of adjoined regions with Lipschitz boundaries. The necessary and sufficient conditions for solvability of these problems are formulated.

3. On the base of generalized Greens Identity for nonsymmetric form generated by Laplace operator we study spectral and initial boundary value problems. In particulary, these are: Dirichlet, Neumann, Steklov, Agranovich, S. Krein and Chueshov problems.We analyze connection between solutions of perturbed (nonsymmetrical forms) and unperturbed (symmetrical forms) problems. Spectral properties and basis ones of eigen- and associated elements of these problems are considered. The theorems on correct solvability of initial boundary value problems are proved.

Wandering eigenvalues of the Laplacian with an improper Robin conditions.

Sergei NAZAROV

St. Petersburg State University

The spectrum of the boundary value problem

$$-\Delta u(x) = \lambda u(x),\ x \in \Omega, \quad a(x)\partial_n u(x) + u(x) = 0,\ x \in \partial\Omega,$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^2$ will be considered under the condition

$$a(s) = a_0 s + O(s^2), \quad s \in \partial\Omega,\ a_0 \neq 0, \quad a(s) \neq 0 \text{ for } s \neq 0.$$

In other words, the coefficient a of the normal derivative ∂_n in the Robin condition changes sign on the boundary of the domain. It will be demonstrated that the spectrum of this problem is residual and covers the whole complex plane $\mathbb{C}$. All self-adjoint extensions of the operator A_0 with the domain

$$\mathcal{D}(A_0) = \{u \in H^1(\Omega) :\ \Delta u \in L^2(\Omega), |a|^{-1/2} u \in L^2(\partial\Omega)\}$$

will be described. They have the discrete spectrum but no appropriate choice of the extension is available. Also a skew-symmetric extension of A_0 can be constructed and it will be shown that this one describing wave processes in a finite volume has a clear physical sense. Namely, the problem

$$-\Delta u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x),\ x \in \Omega, \quad a^\varepsilon(x)\partial_n u^\varepsilon(x) + u^\varepsilon(x) = 0,\ x \in \partial\Omega,$$

with the perturbed coefficient

$$a^\varepsilon(s) = a(s) + \varepsilon \operatorname{sign} a(s) \neq 0 \text{ for all } s \in \partial\Omega$$

gets the discrete spectrum but its eigenvalues λ_k^ε depend periodically on $\ln \varepsilon$, i.e., wander (walk aimlessly like a drunkard or insane) when $\varepsilon \to +0$. Such kind of behavior is usually attributed to a skew-symmetric operator with some radiation conditions at the point $s = 0$ on the boundary $\partial\Omega$.

My work in this direction have been performing in cooperation with French mathematicians Lucas Chesnel, Xavier Clayes and Nicolas Popoff.

Spectral properties of the half-line Schrödinger operator with slowly decaying Wigner-von Neumann potential

Sergey SIMONOV

Steklov Institute, St. Petersburg

We will discuss new results [6] on asymptotic behavior of the spectral density of the operator $\mathcal{L}_\alpha$ in $L_2(\mathbb{R}_+)$,

$$\mathcal{L}_\alpha = -\frac{d^2}{dx^2} + q_{per}(x) + \frac{c\sin(2\omega x+\delta)}{x^\gamma} + q_1(x),$$

$$\operatorname{dom}\mathcal{L}_\alpha = \{u \in H^2_{loc}(\mathbb{R}_+) : u, \mathcal{L}_\alpha u \in L_2(\mathbb{R}_+), u(0)\cos\alpha = u'(0)\sin\alpha\},$$

near critical points which lie inside the absolutely continuous spectrum. Here q_{per} is a periodic background potential, $q_1 \in L_1(\mathbb{R}_+)$ and $\gamma \in (\frac{1}{2}, 1)$. Each spectral band of $\mathcal{L}_\alpha$ contains two critical points locations of which are determined by the frequency ω and the potential q_{per}. Spectral density ρ'_α of $\mathcal{L}_\alpha$ has exponential zeros at these critical points ν_{cr}:

$$\rho'_\alpha(\lambda) = \text{const}\cdot\exp\left(-\frac{2c_{cr}}{|\lambda-\nu_{cr}|^{\frac{1-\gamma}{\gamma}}}\right)(1+o(1)) \text{ as } \lambda\to\nu_{cr}$$

with

$$c_{cr} = \frac{(2\beta_{cr})^{\frac{1}{\gamma}}}{4\gamma} B\left(\frac{3}{2}, \frac{1-\gamma}{2\gamma}\right)\left(\frac{a}{2\pi k'(\nu_{cr})}\right)^{\frac{1-\gamma}{\gamma}},$$

$$\beta_{cr} = \frac{|c|}{2a|W\{\psi_+,\psi_-\}(\nu_{cr})|}\left|\int_0^a \psi_+^2(x,\nu_{cr})e^{2i\omega x}dx\right|,$$

where $\psi_\pm$ are Bloch solutions of the unperturbed periodic equation and k is the quasi-momentum. Earlier the case $\gamma = 1$ was studied for which power type zeros of ρ'_α take place.

[1] J. Janas and S. Simonov, *Weyl–Titchmarsh type formula for discrete Schrödinger operator with Wigner-von Neumann potential*, Studia Math. 201(2): 167–189, 2010.

[2] P. Kurasov and S. Simonov, *Weyl–Titchmarsh type formula for periodic Schrödinger operator with Wigner–von Neumann potential*, Proc. Roy. Soc. Edinburgh Sect. A, 143A:401–425, 2013.

[3] V. Lotoreichik and S. Simonov, *Spectral analysis of the Kronig–Penney model with Wigner-von Neumann perturbations.* Reports on Rep. Math. Phys, 74(1):45–72, 2014.

[4] S. Naboko and S. Simonov, *Zeroes of the spectral density of the periodic Schrödinger operator with Wigner–von Neumann potential*, Math. Proc. Cambridge Philos. Soc., 153(1):33–58, 2012.

[5] S. Simonov, *Zeroes of the spectral density of discrete Schrödinger operator with Wigner–von Neumann potential*, Integral Equations Operator Theory, 73(3):351–364, 2012.

[6] S. Simonov, *Zeroes of the spectral density of the Schrödinger operator with the slowly decaying Wigner–von Neumann potential*, Math. Z., 2016, DOI 10.1007/s00209-016-1659-0.

Construction of transmutations and applications to spectral problems

Vladislav KRAVCHENKO

CINVESTAV del IPN - Querétaro, Mexico

In the talk several new results concerning properties and construction of the transmutation (transformation) operator [1] relating the one-dimensional Schrödinger operator with the second derivative are presented. In particular, an exact representation for the integral transmutation kernel in the form of a Fourier-Legendre series with explicit formulas for the coefficients is obtained [2]. As a corollary, a new representation for solutions of the Sturm-Liouville equation is derived. For every x the solution is represented as a Neumann series of Bessel functions depending on the spectral parameter ω. Due to the fact that the representation is obtained using the corresponding transmutation operator, a partial sum of the series approximates the solution uniformly with respect to ω which makes it especially convenient for the approximate solution of spectral problems. The numerical method based on the proposed approach allows one to compute large sets of eigendata with a nondeteriorating accuracy. Similar results are valid for perturbed Bessel equations.

Additionally other applications of the main result are discussed such as construction of complete systems of solutions of partial differential equations including the extension of the method of fundamental solutions onto the PDEs with variable coefficients.

[1] V. A. Marchenko, *Sturm-Liouville Operators and Applications*. Birkhäuser, Basel, 1986.

[2] V. V. Kravchenko, L. J. Navarro and S. M. Torba, *Representation of solutions to the one-dimensional Schrödinger equation in terms of Neumann series of Bessel functions*. Submitted for publication, available at arxiv:1508.02738.

Local and entanglement entropy of the ideal Fermi gas

Hajo LESCHKE

Universität Erlangen-Nürnberg, Germany

The ideal Fermi gas was created in 1926 by Enrico Fermi (1901 – 1954) as a quantum version of the (ideal) Maxwell-Boltzmann gas of non-interacting particles in Euclidean space by taking into account the Pauli exclusion principle. This requires that two indistinguishable particles cannot be simultaneously in the same one-particle state. The emerging many-particle correlations or entanglement dominate at low temperatures and have led to the early spectacular succeses of the ideal Fermi gas in explaining properties of metals and white dwarfs. In recent years it has turned out that a simple but useful quantifier of the spatial entanglement present in a given many-partice state is related to the (von Neumann) entropies of the two local substates associated with a bounded spatial subregion and its complement. The resulting spatial entanglement entropy (EE) can then be used, for example, to detect the appearance of long-range correlations by enlarging the subregion. For the ideal Fermi gas in thermal equlibrium the leading asymptotic growth of the EE is presented and discussed. The talk is based on joint work with A. V. Sobolev and W. Spitzer.

Bounds on the averaged spectral shift function for random Schrödinger operators with applications

Peter MUELLER

Technische Universität München, Germany

We prove bounds on the expectation of the spectral shift function for continuum random Schrödinger operators that are locally uniform in energy. The bounds rely on a suitably adapted version of the Helffer-Sjöstrand formula which may be of independent interest.

Beurling's theorem, Davenport's formula, and the Riemann Hypothesis

Vladimir KAPUSTIN

Steklov Institute, St. Petersburg

One of versions of A. Beurling's theorem about shift-invariant subspaces in the Hardy space on the half-plane $\mathbb{C}_+ = \{z : \Im z > 0\}$ is as follows:

A closed subspace of $H^2(\mathbb{C}_+)$ *is invariant under the operators of multiplication by* $\exp(itx)$ *for all non-negative real* t *if and only if it has the form* $\theta H^2(\mathbb{C}_+)$ *for some inner function* θ*.*

A bounded analytic function θ in $\mathbb{C}_+$ is *inner* if its boundary values are unimodular almost everywhere on the boundary of $\mathbb{C}_+$.

A function $f \in H^2(\mathbb{C}_+)$ is *outer* if the closed linear span of the functions $\exp(itx)f(x)$, $t \in \mathbb{R}$, $t \geq 0$, coincides with $H^2(\mathbb{C}_+)$.

The function $\frac{s-1}{s^2}\zeta(s)$, where ζ is the Riemann zeta function, belongs to the Hardy class H^2 on the half-plane $\{s : \Re s > \frac{1}{2}\}$. The famous Riemann Hypothesis about the zeros of ζ is equivalent to the statement that this function is outer.

An application of the inverse Mellin transform gives us the criterion of A. Beurling and B. Nyman:

The Riemann Hypothesis is equivalent to the assertion that the indicator of the interval $(0, 1)$ *belongs to the closed linear span in* $L^2(0, +\infty)$ *of the functions* $\rho(\frac{1}{ax})$*,* $a \in \mathbb{R}$*,* $a \geq 1$*,*

where $\rho(\cdot)$ denotes the fractional part of a real number.

L. Báez-Duarte proved that if the Riemann Hypothesis is true, then the indicator of $(0, 1)$ can be approximated by finite linear combinations of $\rho(\frac{1}{nx})$ with positive integers n.

We use the Mellin transform modified by multiplication operators to obtain several criteria for the Riemann Hypothesis connected with H. Davenport's formula

$$-\frac{\sin 2\pi x}{\pi} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\,\rho(nx),$$

where μ is the Möbius function.

Asymptotic and numerical study of resonant tunneling in quantum waveguides of variable cross-section

Oleg SARAFANOV

St. Petersburg State University

We consider an infinite waveguide with two cylindric ends and two narrows of small diameter ε. The electron wave function satisfies the Helmholtz equation with the Dirichlet boundary condition. The narrows of the waveguide play the role of effective potential barriers for the longitudinal motion of electrons. Two narrows form a quantum resonator where a resonant tunneling may appear. It means that electrons with energy in a small range Δ pass through the resonator with probability near to 1. In other words, the transition coefficient $T(k, \varepsilon)$ has sharp peaks at some "resonant"energies k_{res}^2. Various electronic devices (resonant transistors, key devices etc.) can be based on this phenomenon.

To study the resonant tunneling, we calculate the scattering matrix numerically. As approximation for a row of the scattering matrix, we take the minimizer of a quadratic functional. To construct the functional, we solve an auxiliary boundary value problem in the bounded domain obtained by truncating the cylindrical ends of the waveguide at distance R. As $R \to \infty$, the minimizer $a(R)$ tends with exponential rate to the corresponding row of the scattering matrix uniformly on every finite closed interval of the continuous spectrum not containing the thresholds.

The width Δ of resonant peaks is rapidly decreasing as the diameter ε of narrows tends to zero, which presents difficulties for numerical modelling of the resonant tunneling. To give the qualitative picture of the phenomenon when the resonant peaks are "too sharp", we use asymptotic analysis. We give the asymptotics of the corresponding wave function as $\varepsilon \to 0$. Besides, the asymptotics of the resonant energies are presented and the behaviour of the transmission coefficient T(k) near a resonance is analysed. Asymptotic and numerical approaches complete each other and give the full description of the resonant tunneling.

[1] L. Baskin, P. Neittaanmäki, B. Plamenevskii, and O. Sarafanov, *Resonant Tunneling: Quantum Waveguides of Variable Cross-Section, Asymptotics, Numerics, and Applications* // Lecture Notes on Numerical Methods in Engineering and Sciences, Springer, 2015, 275 pp.

Interacting electrons in a random background.

Frédéric KLOPP

Université Pierre et Marie Curie, Paris

In this talk, we consider the d dimensional Schrdinger operator with a repulsive Poisson random potential. We consider n interacting electrons located in this random background and restricted to a cube of sidelength L. We study the limit of the ground state and of the ground state energy (per particle) of this quantum system when n and L go to infinity in such a way that n/L^d converges to a fixed positive density, say, ρ. The density of particles ρ is our main parameter to control the thermodynamic limit; it will be assumed to be small. The results are preliminary.

Stark-Wannier resonances and cubic exponential sums

Alexander FEDOTOV

St. Petersburg State University

The talk is based on a joint work with Frederic Klopp from the University of Pierre and Marie Curie (Paris VII). We discuss the Schrödinger operator $H = -\frac{\partial^2}{\partial x^2} + v(x) - \epsilon x$ acting in $L^2(\mathbb{R})$. Here, v is an entire 1-periodic function, and ϵ is a positive constant. This operator is a model describing an electron in a crystal placed in a constant electric field. The parameter ϵ is proportional to the value of the electric field. The spectrum of H is absolutely continuous and fills the real axis.

The operator attracted attention of both physicists and mathematicians after the discovery of the Stark-Wannier ladders. These are ϵ-periodic sequences of resonances, i.e., of the poles of the meromorphic continuation of the resolvent from the upper half-plane of the complex plane across the spectrum, see [1,3]. A series of papers was devoted to the description of the ladders for small ϵ, see, e.g., [2]. The complexity of this problem is related to the fact that, as $\epsilon \to 0$, there are ladders exponentially close to the real axis. Actually, only the case of finite gap potentials v was understood relatively well: for these potentials, there is only a finite number of ladders that are close to the real axis. It appeared that the ladders non-trivially "interact" as ϵ changes, and physicists conjectured that the behavior of the resonances strongly depends on the arithmetic nature of ϵ, see, for example, [3].

We assume that $v(x) = 2\cos(2\pi x)$ and study the reflection coefficient $r(E)$ in the lower half-plane of the complex plane of the spectral parameter E. There, the function $E \mapsto 1/r(E)$ is an analytic ϵ-periodic function. Its zeros are resonances (and vice versa). Represent $1/r$ by its Fourier series, $1/r(E) = \sum_{m\in\mathbb{Z}} p(m) e^{2\pi i m E/\epsilon}$. Let $a(\epsilon) = \sqrt{2/\epsilon}\,\pi e^{i\pi/4}$. We prove that, as $m \to \infty$,

$$p(m) = a(\epsilon)\sqrt{m}\, e^{-2\pi i \omega m^3 - 2m \log(2\pi m/e) + \delta(m)}, \quad \omega = \left\{ \frac{\pi^2}{3\epsilon} \right\}, \tag{1}$$

where $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$, and $\delta(m) = O(\log^2 m/m)$, the estimate being locally uniform in $\epsilon > 0$.

Obviously, the asymptotic behavior of $1/r(E)$ as $\operatorname{Im} E \to -\infty$, is determined by the Fourier series terms with large positive m, and so, roughly, as $\operatorname{Im} E \to -\infty$,

$$\frac{1}{r(E)} \approx a(\epsilon)\, \mathcal{P}(E/\epsilon), \quad \mathcal{P}(s) = \sum_{m\geq 1} \sqrt{m}\, e^{-2\pi i \omega m^3 - 2m\log(2\pi m/e) + 2\pi i m s}. \tag{2}$$

It is worth to compare the function $1/r$ with the cubic exponential sums $\sum_{n=1}^{N} e^{-2\pi i \omega n^3}$. They were extensively studied for large N in the analytic number theory, see [4], and proved to depend strongly on the arithmetic nature of ω. This appears to be true in our case too. In the talk, for rational ω, we describe in detail behavior of the resonances far from the real axis. In particular, we show that, if $\omega = p/q$ with coprime integers $0 < p < q$, then their asymptotics are determined by beautiful and nontrivial properties of the complete rational exponential sums $\sum_{l=0}^{q-1} e^{-2\pi i \frac{pl^3 - ml}{q}}$, $m \in \mathbb{Z}$.

[1] M. Gluck, A.R. Kolovsky and H.J. Korsch, *Physics Reports*, **366**, 103–182 (2002).

[2] V. Buslaev and A. Grigis, *Journal of mathematical physics*, **39**, 2520-2550 (1998).

[3] J.E. Avron *Annals of physics*, **143**, 33–53 (1982).

[4] H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, Cambridge University Press, 2004.

Bellman function method in analysis

Vasiliy VASYUNIN

Steklov Institute, St. Petersburg

I was invited to make a talk on this conference in spite of the fact that my field is far from the subject of the conference. Probably the idea of the organizers was to show how far from the original field of interest a former student of M. S. Birman can work. Since I understand that probably almost all participants of the conference hear the term *Bellman function* at the first time in their life, my goal will be not to present the most recent result, but to explain what the Bellman function is, to give some elementary examples, to show how this method works in analysis, etc.

My presentation will be based on examples of some classical inequalities for BMO functions (such as the well-known John–Nirenberg inequality). A prove of such inequality with sharp constants can be obtained by solving some boundary value problem for homogeneous Monge–Ampère equation. The graph of any solution of such equation is a so-called developable surface. Geometrical construction that foliate the domain of solution by special straight line segments is a corner stone of the method of constructing the required developable surface and therefore of finding a solution of the boundary value problem. The found Bellman function immediately supplies us with the desired inequality with the sharp constants.

Optimal Hardy-type inequality for second-order elliptic operator: an answer to a problem of Shmuel Agmon

Yehuda PINCHOVER

Technion, Israel

We give a general answer to the following fundamental problem posed by Shmuel Agmon 30 years ago:

Given a (symmetric) linear elliptic operator P of second-order in $\mathbb{R}^n$, find a nonnegative weight function W which is "as large as possible", such that for some neighborhood of infinity G the following inequality holds

$$(P - W) \geq 0 \qquad \text{in the sense of the associated quadratic form on } C_0^\infty(G).$$

We construct, for a general subcritical second-order elliptic operator P on a domain M in $\mathbb{R}^n$ (or on a noncompact manifold M), a Hardy-type weight W which is optimal in the following natural sense:

- $(P - \lambda W) \geq 0$ *on* $C_0^\infty(M)$ *for all* $\lambda \leq 1$,
- *For* $\lambda = 1$*, the operator* $(P - \lambda W)$ *is null-critical in* M*,*

- *For any $\lambda > 1$, and any neighborhood of infinity $G \subset M$, the operator $(P - \lambda W)$ is not nonnegative on $C_0^\infty(G)$.*

- *If P is symmetric and $W > 0$, then the spectrum and the essential spectrum of the operator $W^{-1}P$ are equal to $[1, \infty)$.*

Our method is based on the theory of positive solutions and applies to both symmetric and nonsymmetric operators on noncompact manifolds. Moreover, the results can be generalized to certain p-Laplacian type operators and to Schrödinger operators on graphs. The constructed weight W is given by an explicit simple formula involving two positive solutions of the equation $Pu = 0$.

Semiclassical spectral bounds with remainder terms

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The Berezin and the Li-Yau inequalities state that the first term in Weyl's asymptotic formula serves also as uniform spectral bound on partial eigenvalue sums of the Dirichlet Laplacian. I will report on various attempts to improve these bounds by taking terms of lower order into account. Moreover, I will sketch some recent results on the magnetic Laplacian and on the Heisenberg Laplacian, respectively.

[1] H. Kovarik, T. Weidl: *Improved Berezin-Li-Yau inequalities with magnetic fields.* Proceedings of the Royal Society of Edinburgh, 145A, 145 – 160, 2015

[2] H. Kovarik, B. Ruszkowski, T. Weidl: *Melas-type bounds for the Heisenberg Laplacian on bounded domains.* to appear in Journal of Spectral Theory.

[3] D. Barseghyan, P. Exner, H. Kovarik, T. Weidl: *Semiclassical bounds in magnetic bottles.* to appear in Reviews in Mathematical Physics, 28 (1), 2016.

Talks by young researchers

Complex WKB method for difference equations in the complex plane

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We consider the difference Schrödinger equation

$$\psi(z+h)+\psi(z-h)+v(z)\psi(z)=E\psi(z), \quad z\in\mathbb{C}, \tag{3}$$

where $h>0$ and $E\in\mathbb{C}$ are parameters, and v is a trigonometric polynomial, i.e., $v(z)=\sum_{k=-m}^{n} c_k e^{ikz}$, $m,n>0$, $c_n, c_{-m}\neq 0$. If $v=2\cos z$, equation (3) is called Harper equation. Harper equation with a small h is a model for an electron in a crystal placed in a weak constant magnetic field, see, e.g., [2]. V. Buslaev and A. Fedotov studied quasiclassical asymptotics of solutions of Harper equation in the complex plane, see [1]. It turned out that, as $h\to 0$, solutions have standard quasiclassical behavior in certain canonical domains in the complex plane. We generalized this result to the case where the potential v is a trigonometric polynomial and provide a new, relatively simple proof of it. The talk is based on a joint work with A. Fedotov.

[1] Buslaev V.S. and Fedotov A.A. The complex WKB method for Harpers equation. *St. Petersburg Math. J.* 6 (1995), No.3, 495-517.

[2] Guillement J.P., Helffer B. and Treton P. Walk inside Hofstadter's butterfly. J.Phys.France, 50 (1989), 2019-2058.

Spectral asymptotics for an elastic strip with an interior crack

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We consider an infinite elastic strip $\Omega := \mathbb{R}\times(-\frac{\pi}{2},\frac{\pi}{2})$ with zero Poisson ratio and a crack $\Gamma_\ell := [-\ell,\ell]\times\{0\}$. We impose traction-free boundary conditions and consider the existence of trapped modes, i.e., we search for square-integrable solutions $u:\Omega\backslash\Gamma_\ell\to\mathbb{C}^2$ of the eigenvalue problem

$$\begin{cases} (-\Delta - \text{ grad div})\,u &= \omega(\ell)u & \text{in } \Omega\backslash\Gamma_\ell,\\ 2\varepsilon(u)\cdot\mathbf{n} &= 0 & \text{on } \partial(\Omega\backslash\Gamma_\ell). \end{cases}$$

Here $\mathbf{n}$ is the outer normal unit vector, u is the displacement field of the elastic material and $\varepsilon(u)=\frac{1}{2}(\partial_i u_j+\partial_j u_i)_{i,j=1,2}$ the strain tensor.

In [1] the existence of two eigenvalues $\omega_1(\ell)$ and $\omega_2(\ell)$ embedded in the essential spectrum of the corresponding self-adjoint operator was proved. In the present talk we show that these eigenvalues satisfy the asymptotic estimates

$$\omega_1(\ell) = \Lambda - \nu_1 \ell^4 + \mathcal{O}(\ell^5) \qquad \text{as} \quad \ell \to 0,$$
$$\omega_2(\ell) = \Lambda - \nu_2 \ell^8 + \mathcal{O}(\ell^9) \qquad \text{as} \quad \ell \to 0,$$

where Λ is some spectral threshold and $\nu_1, \nu_2 > 0$. The proof is based on the resolvent expansion of the unperturbed problem near the spectral threshold Λ and on an analysis of a suitable Dirichlet-to-Neumann operator. This is a joint work with T. Weidl.

[1] Hänel, A. and Schulz, C. and Wirth, J., *Embedded eigenvalues for an elastic strip with cracks*, Quart. J. Mech. Appl. Math. **65**, (2012), 535–554.

Asymptotics of solutions to nonstationary Maxwell system in domain with small holes

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Joint work with B. Plamenevskii.

Let $\Omega(\varepsilon)$ be a bounded domain in $\mathbb{R}^3$ with finite number of small holes with diameters proportional to the small paramater ε. In $\Omega(\varepsilon)$ we consider a non-starionary Maxwell system with perfect conductivity or impendance boundary conditions on $\partial\Omega(\varepsilon)$ with time t runs $-\infty$ to $+\infty$. We derive complete asymptotic expansions of solutions as $\varepsilon \to 0$. This mathematical model describes behavior of the electromagnetic field inside the cavity resonator filled with plasma contaminated with small metal particles.

To this end, we use the method of composite asymptotic expansions (for elliptic problems in singularly perturbed domains, the method of compound expansions was presented in [1]). In this method, the asymptotics of solutions is constructed from solutions of “limit problems” not depending on ε. The specific character of the situation considered in present work is that one of limit problems is dynamic. Therefore, when describing the asymptotics of a solution to this problem, we used methods and results from the theory of nonstationary (hyperbolic) boundary value problems in domains with piecewise smooth boundary, presented in [2],[3].

[1] V.G. Maz'ya, S.A. Nazarov, B.A. Plamenevskii, *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains*, v. 1, Birkhäuser, Basel–Boston–Berlin (2000).

[2] B.A. Plamenevskii, *On the Dirichlet problem for the wave equation in a cylinder with edges*, Algebra i Analiz 10 (1998), no. 2, 197 – 228 // St. Petersburg Math. J. 10 (1999), no. 2, 373 – 397.

[3] S.I. Matyukevich, *On the nonstationary Maxwell system in domains with edges*, Algebra i Analiz 15 (2003), no. 6, 86 – 140 // St. Petersburg Math. J.15:6 (2004), 875 – 913.

Spectral asymptotics of operators of the tensor product type with almost regular marginal asymptotics

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An example of an operator with almost power spectral asymptotics arises naturally when considering an ordinary differential operator with singular self-similar weight. The known formula for the asymptotics of eigenvalue counting function in that case is

$$N(\lambda) = \lambda^D \cdot (s(\ln \lambda) + o(1)), \qquad \lambda \to +\infty,$$

where s is a positive continuous periodic function, dependent on the choice of the weight. Some properties of function s are described in [1,2](in particular, the fine structure of s is established for certain classes of weights). It is convenient for the sake of generality to also consider almost regular asymptotics of the form

$$N(\lambda) = \lambda^D \cdot \varphi(\lambda) \cdot (s(\ln \lambda) + o(1)), \qquad \lambda \to +\infty,$$

where φ is a *slowly varying function* (SVF).

As a part of the currently intensive development of the theory of small deviations of Gaussian random functions, tensor products of compact operators with almost regular marginal asymptotics containing periodic functions are considered for asymptotic analysis. We amend the abstract theorems developed in [3] to fit this case.

We infer that the same asymptotic behavior persists for the tensor product. It is also almost regular containing a periodic function. If the asymptotics of the operators are of different powers, it will resemble the stronger of two, only with the periodic term potentially changed. If the powers coincide, the slowly varying function will be the convolution of the original ones (in the case, when slowly varying functions are constant, it means the emergence of a logarithmic term). We establish the cases, where new periodic function could be shown to be non-degenerate under certain circumstances. We also establish cases, where it is guaranteed to degenerate into constant. Author was supported by RFBR grant 16-01-00258a, by "Native towns", a social investment program of PJSC "Gazprom Neft" and by the German-Russian Interdisciplinary Science Center (G-RISC) funded by the German Foreign Office via the German Academic Exchange Service (DAAD).

[1] Vladimirov A. A., Sheipak I. A. *On the Neumann Problem for the Sturm-Liouville Equation with Cantor-Type Self-Similar Weight*// Funct. Anal. Appl. 47 (2013), no. 4, 261 – 270.

[2] Rastegaev N. V. *On spectral asymptotics of the Neumann problem for the Sturm-Liouville equation with self-similar generalized Cantor type weight*// Journal of Mathematical Sciences (New York) 210 (2015), no. 6, 814 – 821.

[3] Karol A., Nazarov A., Nikitin Y., *Small ball probabilities for Gaussian random fields and tensor products of compact operators*// Trans. Amer. Math. Soc. 360 (2008), no. 3., 1443 – 1474.