The abstracts of the talks

11th St. Petersburg Conference in Spectral Theory

21 – 24 June 2019

FRIDAY 21 June

Resonances for one dimensional Stark Hamiltonians with weak electric field.

Richard Froese (University of British Columbia, Canada).

We study the asymptotic location of resonances for Stark Hamiltonians in one dimension for small electric field. This is joint work with Ira Herbst.

Sharp spectral transition for eigenvalues embedded into the spectral bands of perturbed periodic operators.

Darren Ong (Xiamen University, Malaysia).

We consider the Schrödinger equation, $Hu = -u'' + (V(x) + V_0(x))u = Eu$, where $V_0(x)$ is 1-periodic and $V(x)$ is a decaying perturbation. By Floquet theory, the spectrum of $H_0 = -\nabla^2 + V_0$ is purely absolutely continuous and consists of a union of closed intervals (often referred to as spectral bands). Given any finite set of points $\{E_j\}_{j=1}^N$ in any spectral band of H_0 obeying a mild non-resonance condition,we construct smooth functions $V(x) = \frac{O(1)}{1+|x|}$ such that $H = H_0 + V$ has eigenvalues $\{E_j\}_{j=1}^N$. Given any countable set of points $\{E_j\}$ in any spectral band of H_0 obeying the same non-resonance condition,and any function $h(x) > 0$ going to infinity arbitrarily slowly,we construct smooth functions $|V(x)| \leq \frac{h(x)}{1+|x|}$ such that $H = H_0 + V$ has eigenvalues $\{E_j\}$. On the other hand, we show that there is no eigenvalue of $H = H_0 + V$ embedded in the spectral bands if $V(x) = \frac{o(1)}{1+|x|}$ as x goes to infinity.
This is joint work with Wencai Liu.

Operator error estimates for homogenization of the nonstationary Schrödinger-type equations: dependence on time.

Mark Dorodny (St. Petesburg State University, Russia).

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider a selfadjoint matrix strongly elliptic second order differential operator $\mathcal{A}_\varepsilon$, with periodic coefficients depending on $\mathbf{x}/\varepsilon$. We find approximations of the exponential $e^{-i\tau\mathcal{A}_\varepsilon}$, $\tau \in \mathbb{R}$, for small ε in the $(H^s \to L_2)$-operator norm with suitable s. The sharpness of the error estimates with respect to τ is discussed. The results are applied to study the behavior of the solution $\mathbf{u}_\varepsilon$ of the Cauchy problem for the Schrödinger-type equation $i\partial_\tau \mathbf{u}_\varepsilon = \mathcal{A}_\varepsilon \mathbf{u}_\varepsilon + \mathbf{F}$.

Localization and delocalization for interacting 1D quasiperiodic particles.

Ilya Kachkovskiy (Michigan State University, USA).

We consider a system of two interacting one-dimensional quasiperiodic particles as an operator on $\ell^2(\mathbb{Z}^2)$. The fact that particle frequencies are identical, implies a new effect compared to generic 2D potentials: the presence of large coupling localization depends on symmetries of the single-particle potential. If the potential has no

cosine-type symmetries, then we are able to show localization at large disorder and all energies, even if the interaction is not small (with some assumptions on its complexity, the interaction can be stronger than the disorder). If symmetries are present, we can show localization away from finitely many energies, thus removing a fraction of spectrum from consideration. We also demonstrate that, in the symmetric case, delocalization can indeed happen if the interaction is strong, at the energies away from the bulk spectrum.
The talk is based on joint works with Jean Bourgain and Svetlana Jitomirskaya.

Asymptotics of spectral gaps for the almost Mathieu equation with a small coupling constant.

Alexander Fedotov (St. Petersburg State University, Russia).

To study the cantorian geometry of the spectrum of the almost Mathieu equation, a one-dimensional quasiperiodic difference Schrödinger equation, V.Buslaev and A.Fedotov suggested the monodromization method, a renormalization method based on ideas of the Bloch-Floquet theory used to study differential equations with periodic coefficients. In this talk we briefly describe basic ideas of the monodromization method and asymptotics of spectral gaps for the almost Mathieu equation with a small coupling constant obtained with its help.

SATURDAY 22 June

The Howland - Kato Commutator Problem.

Ira Herbst (University of Virginia, USA).

I will discuss the problem of determining all pairs of bounded measurable real functions f and g such that $i[f(P), g(Q)]$ is a non negative operator. This work is partly joint with Tom Kriete and with some help from Richard Froese.

Trace formula for functions of pairs of non commuting self-adjoint operators.

Vladimir Peller (Michigan State University, USA).

The talk will be devoted to attempts to generalize the Lifshts - Krein trace formula to functions of noncommuting self-adjoint operators. It follows from the results of Aleksandrov-Nazarov-Peller that as in the case of functions of one self-adjoint operator that for functions f on $\mathbb{R}^2$ of Besov class $B^1_{\infty,1}$ a Lipschitz time estimate in the trace norm holds. However, it turns out that the Lifshits - Krein trace formula cannot be generalized to the case of functions of two noncommuting self-adjoint operators.

The tan 2Θ-theorem in fluid dynamics.

Konstantin Makarov (University of Missouri, USA).

It is generally believed that a steady flow of an incompressible fluid is stable whenever the Reynolds number associated with the flow is sufficiently low, while it is experimentally proven that flows become turbulent for high Reynolds numbers (about several hundreds and beyond).

The first rigorous quantitative stability result for stationary solutions to the $2D$-Navier-Stokes equation (in bounded domains) is due to Ladyzhenskaya and her analysis shows that given a stationary solution v_{st}, any other solution v (with smooth initial data and the same forcing) approaches v_{st} exponentially fast

$$v - v_{\mathrm{st}} = O(e^{-\alpha t}), \quad t \to \infty,$$

whenever the generalized Reynolds number

$$\mathrm{Re}_{\,\mathrm{Lad}} = \frac{2v_*}{\nu\sqrt{\lambda_1(\Omega)}}$$

is less than one. Here ν is the viscosity of the incompressible fluid, $\lambda_1(\Omega)$ is the principal eigenvalue of the Dirichlet Laplacian in the bounded domain Ω, and v_* stands for the characteristic velocity of the stationary flow v_{st} and

$$\alpha = \nu\lambda_1(\Omega)(1 - \text{Re}_{\text{ Lad}}).$$

To better understand the functional-analytic as well as (Hilbert space) geometric aspects of the Navier-Stokes stability in any dimension, we introduce and study the (model) Stokes block operator, which is the Friedrichs extension of the block operator matrix

$$S = \begin{pmatrix} -\nu\mathbf{\Delta} & v_*\text{grad} \\ -v_*\text{div} & 0 \end{pmatrix}$$

initially defined on the set $C_0^\infty(\Omega)^n \oplus C^\infty(\Omega)$ of infinitely differentiable vector-valued functions in the Hilbert space $L^2(\Omega)^n \oplus L^2(\Omega)$, $n \geq 2$.

One of our principal results links the Ladyzhenskaya-Reynolds number $\text{Re}_{\text{ Lad}}$ to the norm of the operator angle Θ between the positive subspace of the Stokes operator and the positive subspace of its diagonal part.

That is, the following

Tan 2Θ-Theorem in Fluid Dynamics

$$\tan 2\|\Theta\| \leq \text{Re}_{\text{ Lad}},$$

holds.

The essence of this estimate is the remarkable fact that *the magnitude of the Reynolds number limits the rotation of the spectral subspaces of the block Stokes operator.*

We also show that the lowest positive eigenvalue $\lambda_1(S)$ of the Stokes operator S and the bottom of its negative (essential) spectrum satisfy the inequality

$$|\inf \text{spec}(S)| \leq \frac{1}{4}\left[\text{Re}_{\text{ Lad}}\right]^2 \lambda_1(S),$$

which is asymptotically sharp as $\nu \to \infty$ or $v_* \to 0$.

In particular, the Ladyzhenskaya (2D-) stability hypothesis $\text{Re}_{\text{ Lad}} < 1$ yields the following

Stability Laws

- the relative spectral shift δ defined as ratio of the shift of the spectrum from the origin to the left to the length of the spectral gap of the Stokes operator is bounded by

$$\delta = \frac{|\inf \text{spec}(S)|}{\lambda_1(S)} < \frac{1}{4}$$

- the maximal rotation angle $\|\Theta\|$ between the positive subspaces of the perturbed and unperturbed Stokes operators is bounded by

$$\|\Theta\| < \frac{\pi}{8}.$$

This is a joint work with L. Grubišić, V. Kostrykin, S. Schmitz, and K. Veselić.

Direct and inverse spectral problems for sloshing of a two-layer fluid in an open container.

Nikolay Kuznetsov (Institute for Problems in Mechanical Engineering, Russia).

Direct and inverse eigenvalue problems will be considered for a pair of harmonic functions with a spectral parameter in boundary and coupling conditions. The direct problem is relevant to sloshing frequencies of free oscillations of a two-layer fluid in a container. The upper fluid occupies a layer bounded above by a free surface and below by a layer of fluid of greater density. Both fluids are assumed to be inviscid, incompressible, and heavy, whereas the free surface and the interface between fluids are supposed to be bounded.

Orthogonal polynomials with unbounded recurrence coefficients.

Grzegorz Świderski (University of Wroclaw, Poland).

Let μ be a probability measure on the real line with all moments finite. Let $(p_n : n \geq 0)$ be the corresponding sequence of orthonormal polynomials. It satisfies

$$\begin{gathered} p_0(x) = 1, \quad p_1(x) = \frac{x - b_0}{a_0}, \\ a_{n-1}p_{n-1}(x) + b_n p_n(x) + a_n p_{n+1}(x) = x p_n(x) \quad (n \geq 1), \end{gathered} \tag{1}$$

for some sequences $a_n > 0$ and $b_n \in \mathbb{R}$. Conversely, Favard's theorem states that every sequence of polynomials satisfying (1) is orthonormal with respect to some measure μ. The measure μ is unique if the Carleman condition is satisfied, i.e. when

$$\sum_{k=0}^{\infty} \frac{1}{a_k} = \infty.$$

Because of the connections to Jacobi matrices, the approach starting with the sequences (a_n) and (b_n), sometimes is called *the spectral theory of orthogonal polynomials*, and the sequences are called *Jacobi parameters*. The aim of the spectral theory is to describe how the properties of Jacobi parameters are reflected in the measure μ.

The theory of bounded Jacobi parameters is well-developed. It covers, e.g., the cases of constant, periodic or almost periodic Jacobi parameters and compact perturbations theoreof. In the talk we are interested in the class of periodically modulated Jacobi parameters introduced by Janas-Naboko. They are defined by

$$a_n = \alpha_n \tilde{a}_n, \quad b_n = \beta_n \tilde{a}_n,$$

where $\tilde{a}_n$ is some regular sequence tending to infinity, α and β are N periodic sequences.

Under some regularity assumptions, it can be shown that μ is absolutely continuous with continuous density μ'. The crucial role lies in the analysis of Turán determinants, defined by

$$D_n = \det \begin{pmatrix} p_{n-1} & p_{n+N-1} \\ p_n & p_{n+N} \end{pmatrix} = p_n p_{n+N-1} - p_{n-1} p_{n+N}.$$

In fact,

$$\lim_{n \to \infty} a_{n+N-1} |D_n(x)| = \frac{c}{\mu'(x)}, \quad (x \in \mathbb{R})$$

for some explicit constant $c > 0$ depending only on α and β.

We will also present uniform asymptotics of the polynomials. The proof is based on analysis of transfer matrices combined with Turán determinants and an approximation procedure, which allow us to use the results known in the bounded case.

The next important object is the Christoffel-Darboux kernel

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y),$$

which is the kernel of the projection operator on the space of polynomials with degree less than n. We shall present the applications of the asymptotics of p_n to the limits of

$$\frac{1}{\rho_n} K_n\Big(x + \frac{u}{\rho_n}, x + \frac{v}{\rho_n}\Big),$$

where ρ_n is some explicit sequence tending to infinity. These limits are studied in the Random Matrix Theory and are connected to the asymptotic behaviour of eigenvalues around x of random matrices with dimension tending to infinity.

Some of the talk is based on a work in progress with Bartosz Trojan (Polish Academy of Sciences).

Uniform asymptotics of Hermite polynomials and associated Lagrangian manifolds.

Anna Tsvetkova (Moscow Institute of Science and Technology, Russia).

We discuss two approaches allowing one to construct the uniform asymptotics of Hermite polynomials $H_n(y)$ as $n \to \infty$ via the Airy function Ai. Both of the approaches are related to the Maslov canonical operator. Obtained formula is not well-known, but gives quite good approximation even for small n.

The first approach is based on the representation of Hermite functions $\psi_n(y) = (2^n n! \sqrt{\pi})^{\frac{1}{2}} e^{-\frac{y^2}{2}} H_n(y)$ as solutions of the Shrödinger equation for a harmonic oscillator

$$-\frac{1}{2}\frac{d^2\psi_n}{dy^2} + \frac{y^2}{2}\psi_n = \left(\frac{1}{2} + n\right)\psi_n, ||\psi_n||_{L_2(\mathbb{R}_y)} = 1.$$

The second approach is based on a reduction of the finite-difference equation for the Hermite polynomials

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y), H_0(y) = 1, H_1(y) = 2y$$

to a pseudodifferential equation.

To construct the asymptotics we use the Maslov canonical operator. This operator is connected with the Lagrangian manifolds which are defined by trajectories of the corresponding Hamiltonian systems. In the first case the Lagrangian manifold is a circle, in the second case the Lagrangian manifolds are a family of "distorted" horizontal parabolas.

It seems to us that both of the approaches are universal and can be applied to a wide class of orthogonal polynomials.

The presentation is based on the joint work with S. Yu. Dobrokhotov.

The research was supported by the Russian Science Foundation (project 16-11-10282).

SUNDAY 23 June

Random Matrices with Exchangeable Entries.

Werner Kirsch (University of Hagen, Germany).

Let X_N be a symmetric $N \times N$- matrix with entries $X_N(i,j)$. The empirical eigenvalue distribution is the measure $\mu_N = \frac{1}{N}\sum \delta_{\lambda_i}$ where the λ_i are the eigenvalues of the matrix $M_N = \frac{1}{\sqrt{N}} X_N$.

If the $X_N(i,j)$ are independent (except for the symmetry of the matrix) with a common probability distribution P, mean zero and variance σ^2 then the famous semi circle law states that μ_N converges weakly to the semi circle distribution, the density of which is $\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}$. This holds independent of the details of the distribution P.

In this talk we investigate how universal this law is if we weaken the assumption of stochastic independence of the $X_N(i,j)$. In particular, we consider random entries which are Curie-Weiss distributed. Such random variables constitute a simple model for magnetisms with a phase at a critical inverse temperature $\beta = 1$.

The Curie-Weiss model is a special case of exchangeable random variables. We also investigate random matrices with general exchangeable entries as well as band random matrices with such entries.

Ultrametricity and spectral statistics or random Schrodinger-like operators.

Sergei Nechaev (Interdisciplinary Scientific Center J.-V. Poncelet, Russia).

I discuss an explicit construction of the ultrametric landscape of potential barriers between optimal displacement of repulsive particles of a cylinder in terms of the Eisenstein series. The found landscape is ultimately connected to the Riemann "raindrop" function and to the spectral statistics of random Schrodinger-like operators with the off-diagonal randomness. The construction involves the regularization of the Riemann "raindrop" function in terms of the Dedekind eta-function and provides a "number-theoretic view" on the one-dimensional Anderson localization.

Partial Spectral Flow and the Aharonov–Bohm Effect in Graphene.

Vladimir Nazaikinskii (Ishlinsky Institute for Problems in Mechanica of the Russian Academy of Sciences, Russia).

Consider a graphene tube in the shape of a right circular open-ended cylinder whose height and radius are both much greater than the distance between neighboring carbon atoms. A magnetic field everywhere vanishing on the tube surface is adiabatically switched on. The tight-binding Hamiltonian H describing the electron π-states in graphene is changed as a result, $H = H_t$, $t \in [0, 1]$, and its eigenvalues move in the process. If the magnetic flux Φ through the tube in the final configuration is an integer multiple of the flux quantum Φ_0, $\Phi = n\Phi_0$, then the final Hamiltonian H_1 is unitarily equivalent to the initial one, and so the electron energy spectrum is the same as that of H_0. Further, since the lattice function space on which the tight-binding Hamiltonian acts is finite-dimensional, it follows that the spectral flow of the resulting family of Hamiltonians $\{H_t\}$ is zero. However, a more detailed analysis shows that the eigenfunctions corresponding to small eigenvalues are localized near the Dirac points K and K' of the reciprocal lattice in the momentum space, and if we count the spectral flow for such eigenfunctions separately, then we obtain two ≪partial spectral flows≫ $sf_K(H_t)$ and $-sf_{K'}(H_t)$, and one has

$$sf_K(H_t) = -sf_{K'}(H_t) = n(\text{the number of flux quanta}).$$

Physically, this means that electron and hole energy levels in graphene are created in pairs. Further, one has

$$sf_K(H_t) = sf_K(D_t),$$

where D_t is the Dirac operator approximating the tight-binding Hamiltonian H_t near the point K. We assign a precise mathematical meaning to the notion of partial spectral flow so that all the preceding assertions become rigorous. This research is joint work with M. I. Katsnelson (Radboud University, Nijmegen, The Netherlands) and J.Brüning (Humboldt-Universität zu Berlin, Berlin, Germany).

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Asymptotics of eigenfunctions of the two-dimensional operator associated with billiards with semi-hard walls.

Sergey Dobrokhotov (Moscow Institute of Physics and Technology, Russia).

We discuss asymptotic eigenfunctions of a two-dimensional operator with the degenerate coefficient $D(x)$ at the boundary of the region . Such operators arise, for example, in the problems of long waves on the water, trapped by the shores and islands. The constructed functions are related to analogs of Liouville tori of integrable geodesic flows with degenerate on metric and defined by the Hamilton system with the Hamiltonian $p^2D(x)$. The non-standard situation is that the momentum components of the trajectories on such tori tend to infinity on the curve $D(x) = 0$, although projections on the plane $\mathbb{R}^2$ form compact sets (usually diffeomorphic to rings on $\mathbb{R}^2$, similar to those which arise in the study of standard billiards (in this case, the Dirichlet or Neumann conditions are put on the boundary and we have the situation of hard wall)or in the study of the eigenfunctions of the Schroedinger operator in a potential well (then we have so-called the situation of "soft walls"). Thus geodetic flows appearing in considered problems can be called "billiards with semi-hard walls".

This work was done together with A.Anikin, V.Nazaikinskii and A.Tsvetkova and supprted by RSF grant 16-11-10282.

Spectral asymptotics for rough Riemannian manifold.

Julie Rowlett (Chalmers University of Technology, Sweden).

This talk is based on joint work with L. Bandara and M. Nursultanov. Our topological setting is a smooth compact manifold of dimension two or higher with smooth boundary. Although this underlying topological structure is smooth, the Riemannian metric tensor is only assumed to be bounded and measurable. This is known as a rough Riemannian manifold. These arise naturally in the context of harmonic analysis, as L. Bandara showed that they are geometric invariances of the Kato square root problem. The "roughness" of this geometric context can be seen for example by the fact that there is no canonical distance between points on a rough Riemannian manifold. I will discuss a certain class of weighted Laplace equations, with a range of boundary conditions including Dirichlet, Neumann, and mixed, for which we obtain spectral asymptotics. A major source of guiding inspiration for us has been the work of M. Sh. Birman and M. Z. Solomyak. Although their geometric context is Euclidean space, we are able to adapt some of their clever ideas and powerful techniques to our setting. This talk is dedicated to the continuing mathematical legacy of these great mathematicians.

MONDAY 24 June

Spectral Theory of the Fermi Polaron.

Marcel Griesemer (Stuttgart University, Germany).

The Fermi polaron refers to a system of free fermions interacting with an impurity particle by means of two-body contact forces. In this talk we present a general mathematical framework for defining many-body Hamiltonians with two-body contact interactions by means of a renormalization procedure. For the Fermi polaron in a two-dimensional box a novel variational principle, established within the general framework, links the low-lying eigenvalues of the system to the zero-modes of a Birman-Schwinger type operator. It allows us to show, e.g., that the polaron- and molecule energies, computed in the physical literature, are indeed upper bounds to the ground state energy of the system.
This is joint work with Ulrich Linden.

On the virtual level of Schrödinger operators with applications to N-body systems.

Andreas Bitter (Stuttgart University, Germany).

In this talk we study the behaviour of the resonance functions of the Schrödinger operator

$$H = -\Delta + V$$

in the case of a virtual level at the threshold of the essential spectrum. Based on an Agmon-type argument a new approach is presented to derive rates of decay of the resonance functions for $|x| \to \infty$. This technique is applied to multi-particle systems to analyse virtual levels of N-body Schrödinger operators. As a consequence, one can show that the Efimov-effect is absent in the case of $N \geqslant 4$ particles in dimensions $d \geqslant 3$ or for $N \geqslant 4$ fermions in dimension $d = 2$.

On the absence of the Efimov-effect for $N \geqslant 4$ particles.

Simon Barth (Stuttgart University, Germany).

According to the well-known Efimov-effect the resonance at the lower threshold of a two-body system turns into infinitely many bound states when adding a third particle. This comes from the behaviour of the resonance function for $|x| \to \infty$. Based on the talk of A. Bitter we prove the absence of the Efimov-effect for $N \geqslant 4$ particles in dimension $d = 3$ and for $N \geqslant 4$ fermions in dimension $d = 2$. Precisely, we show that the discrete spectrum of the corresponding N-body Schrödinger operator H is finite, provided every subsystem with $n \leqslant N - 2$ particles has no negative spectrum and no virtual level.

Mathematical scattering theory in quantum waveguides.

Alexander Poretsky (St. Petesburg State University, Russia).

Waveguide occupies a domain G having several cylindrical outlets to infinity and is described by a nonstationary equation of the form $i\partial_t \Psi = \mathcal{A}\Psi$, where $\mathcal{A}$ stands for an elliptic self-adjoint second order differential operator with variable coefficients (in particular, for $\mathcal{A} = -\Delta$, Δ being the Laplace operator, we have the Schrödinger equation). For a corresponding stationary boundary-value problem with spectral parameter, we introduce continuous spectrum eigenfunctions and the scattering matrix. By means of the limiting absorption principle, we establish a continuous spectrum eigenfunction expansion. Then we compute wave operators and prove their completeness. Finally, we describe the relation of the scattering operator to the scattering matrix.
The talk is based on a joint research with B.A. Plamenevskii and O.V. Sarafanov.

A review of simple localized solutions of the wave equation.

Alexey Kiselev (PDMI RAS, Russia).

A review is presented of simple explicit localized solutions of the wave equation $u_{xx} + u_{yy} + u_{zz} - \frac{1}{c^2}u_{tt} = 0$, $c = const > 0$. Two classes of solutions are addressed. First, we describe simple analytic solutions dependent on a certain parameter and becoming highly localized as it tends to infinity. These are the ones based on the complexified Bateman theory (see, e.g., [1, 2]) as well as those associated with the so-called "complex sources" [3,4,5,6]. Second, we discuss simple solutions of the homogeneous wave equation, having a singularity at a running point. Attention is paid to a detailed analytic investigation of the solution presented by Hörmander [7]. We establish that it isnothing but a specification of the classical Bateman solution [8]. Also, we are concerned with a certain specialized complexified Bateman solution, having similar properties [9]. We give a nontrivial definition of what should be called a non-complexified Bateman solution in the case of even number of spatial variables.
The results were obtained in cooperation with A.S. Blagovestchenskii, M.V.Perel, A.B.Plachenov and A.M. Tagirdzhanov.
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