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Symplectization, Complexification and Mathematical Trinities

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Abstract. This is the second of the series of three lectures given by Vladimir Arnold in June 1997 at the meeting in the Fields Institute dedicated to his 60th birthday.

> Augury is not algebra. Human mind is not a prophet but a guesser. It can see the general scheme of things and draw from it deep conjectures, which are often borne out by time.
>
> *A. S. Pushkin.*

The goal of this lecture is to explain some mathematical dreams.

There are several parts of mathematics and they are considered as independent parts or even as different mathematics, like say the theory of functions of real variables and the theory for complex variables which have been considered as completely different sciences. The origin of the Moscow mathematical school is based on a philosophical difference between real and complex mathematics discovered by the mathematician Bugaev. He was one of the founders of Moscow Mathematical Society a hundred and fifty years ago, a philosopher and the father of the well-known Russian symbolist poet Belyi. Bugaev observed that there were two main ideas in philosophy, the idea of the predestination and that of the free will (you can move your hand). He speculated that the mathematical version of the predestination idea was the theory of functions in complex variables where a germ at one point (or even a Taylor series at one point) contained all the information about the function by means of analytic continuation. But in XIXth century the idea of freedom was more important and Bugaev decided that one should develop in Russia the free will mathematical version — the theory of functions of real variables. So he sent his student Egorov and later Luzin to Paris where Lebesgue and Borel were working on

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the real variables mathematics. This was the origin of the Moscow mathematical school (which unified the real and the complex ideas only much later).

In this lecture I will use rather consciously some mysterious relations between different mathematics. I shall try to explain the way I have used them in the past. I hope in the future someone will make a theory of it. I am perhaps too old to formalize all this. The ideas are just facts, not theories. I have no axioms for what I shall describe, I have just examples, but I think these examples are natural and interesting. I shall explain these ideas starting with finite-dimensional linear algebra models while the most interesting part is infinite-dimensional where we work in differential geometry, the manifolds replacing the vector spaces, the diffeomorphisms replacing the linear operators and so on.

To start with the finite-dimensional situation is very natural here in Toronto since for the mathematicians Toronto is associated with the name of Coxeter. So I start with the list of Coxeter groups. These are finite groups generated by Euclidean reflections. They are described by Dynkin diagrams (we may call them in this way since they were invented by other mathematicians), see Fig. 1. In singularity theory there is an infinite series of E's but we will not discuss them here. So on the left there is the ADE list and on the right — diagrams with double lines including (unfortunately) the last Weyl group, having a triple line.

These groups preserve some lattices and hence are called crystallographic. We also have non-crystallographic Coxeter groups: the symmetry groups of the regular p-gon, the icosahedron symmetry group and the hypericosahedron symmetry group (Fig. 2). The hypericosahedron regular polyhedron H_4 lives in dimension 4 (the index is the dimension of the space). It has a nice description given by Coxeter, probably too nice to be put in Bourbaki's "Lie groups and algebras", who described H_3 but not H_4!

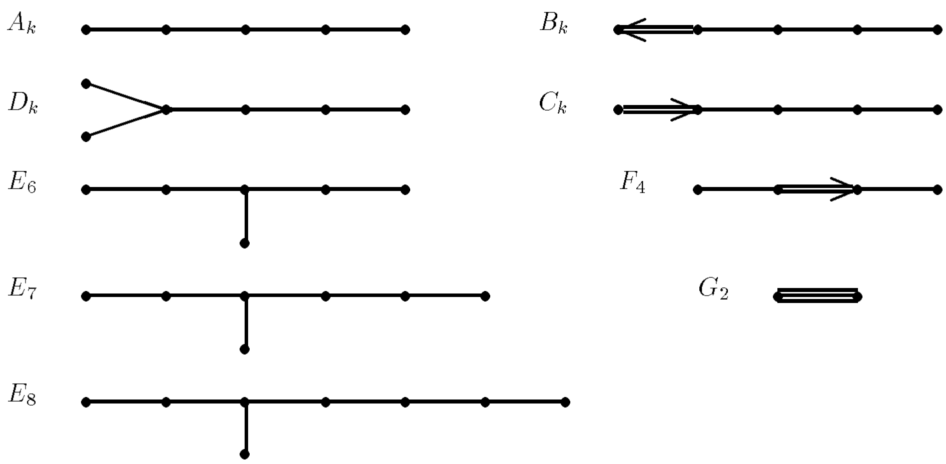

Fig. 1: Crystallographic Coxeter groups.

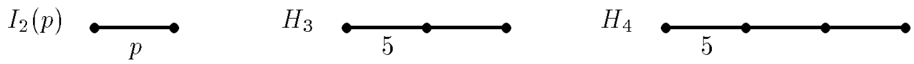

Fig. 2: Non-crystallographic Coxeter groups.

Any point in Fig. 1–2 represents a vector in the Euclidean space. Two points are connected by a single line if the angle between the vectors is $120°$, a double line represents the angle $135°$, the triple line — the angle $150°$, the line "p" from Fig. 2 — the angle $180°(1-\frac{1}{p})$. Since we need only directions I omit the description of rules for the vector lengths which is needed to restore the lattice generated by these vectors. One can construct the reflections in the mirrors orthogonal to the vectors. The resulting group is finite only in some special cases. Remind that a transformation group is called irreducible if the space of action admits no proper subspace invariant under all transformations represented by the group elements. Any Coxeter group is decomposed into the direct product of irreducible ones acting in mutually orthogonal subspaces. All the possible Dynkin diagrams for finite irreducible Coxeter groups are present in Fig. 1–2. So this list gives the classification of irreducible Coxeter groups which is one of the main classification theorem in mathematics[1].

The simplest nontrivial Coxeter group A_2 is characterized by two vectors of unit length with the angle $120°$ between them in the Euclidean plane (Fig. 3). The complete number of mirrors is 3 and the group is generated by any two of them. So A_2 is the symmetry group of the regular triangle. In the general case the generating reflections are described by the diagrams from Fig. 1-2. For instance for $p = 5$ on Fig. 2 we have the symmetry groups of the regular pentagon, of the icosahedron and of the hypericosahedron. The hypericosahedron is the convex body generated by 120 vertices in S^3, these 120 points forming a subgroup of $SU(2) \simeq S^3$. This subgroup is the preimage of the group of the 60 icosahedron preserving rotations under the natural two-fold covering mapping $S^3 \to SO(3)$.

Fig. 3: Generators and mirrors of the A_2 group.

The group A_k corresponds also to the special linear group $SL(k + 1)$ of $(k + 1) \times (k + 1)$ matrices with determinant 1 (or to the unitary group if you prefer the compact version). Similarly all the groups on the left of Fig. 1 have some lattices invariant under reflections, which are also classified by the diagrams, and they correspond to the simple Lie algebras as well. Thus B_k corresponds to $SO(2k+1)$, C_k corresponds to $Sp(2k)$ and D_k corresponds to $SO(2k)$. They have separate theories of eigenvalues, eigenspaces and Jordan blocks (for the unitary case just the eigenvalues theory). For example the ordinary spectral theory of matrices is just the theory of A, while those of B and D are the Euclidean geometries of odd- and even-dimensional spaces.

The theory of Coxeter groups is a description of linear algebra as of a special case of a more general theory. One can avoid to mention matrices, eigenvalues, eigenvectors and can replace everything with the root system geometry of the series

[1]Manin told me once that the reason why we always encounter this list in many different mathematical classifications is its presence in the hardware of our brain (which is thus unable to discover a more complicated scheme). I still hope there exists a better reason that once should be discovered.

A. Now if you take some result from linear algebra or some result on the matrices and reformulate it in terms of root systems then (with the meaning of notions appropriately changed) this result becomes a conjecture you might try to prove for all the root systems. Proving it you probably obtain some interesting and nontrivial results for the other cases from Fig. 1.

Of course you can consider all the resulting theories as a particular case of linear algebra (A_k), namely as the study of the linear algebra of a vector space with some additional structure. For example you might add an Euclidean structure which leads to the consideration of only the matrices which preserve the fixed scalar product (D_k and B_k). Or you might consider a symplectic 2-form making the vector space into the symplectic space and then put into consideration symplectic linear maps (C_k). So all the theories of Fig. 1 can be considered as subtheories of the linear algebra. But there is another way to think of them. We may consider them as not children of the linear algebra but as its sisters. This linear algebra is just the theory of one of the Coxeter groups series A. One can study the other Coxeter groups of the series B, C, D. So one may formulate theorems-sisters for different series and with more work it is possible to find their versions for the exceptional groups.

Now consider the infinite-dimensional case. The classification theorems extending the classification of Fig. 1 are due to E. Cartan. The first case corresponding to A_k is the group $\mathrm{Diff}(M)$ of diffeomorphisms of the manifold M. The corresponding Lie algebra consists of vector fields with the standard commutator and the geometry obtained in this way is called *differential geometry*. There are essentially 6 series of objects which are in the same relations to the group of diffeomorphisms as the first Coxeter group (simple Lie algebra) A_k to its sisters from Fig. 1. So there are many other geometries. The first is differential. Next we have hydrodynamics. It corresponds to the group $\mathrm{SDiff}(M, v)$ of volume preserving diffeomorphisms if the manifold M is equipped with a volume element v. Its Lie algebra consists of divergence-free vector fields. Next we have symplectic geometry with the group $\mathrm{Symp}(M, \omega)$ of symplectomorphisms. Its Lie algebra consists of locally Hamiltonian vector fields. There are some more geometries: complex, contact and so on. Of course all these geometric groups are subgroups of the first one but they are not normal subgroups because the identity component of $\mathrm{Diff}(M)$ is simple.

Now we extend the notion of sisters to our case and consider these infinite-dimensional groups as sisters. We start with some notions, ideas, theorems from differential geometry and topology and then translate them into the language similar to that of the root systems which is independent of the initial group. And then by the reverse process we try to understand what should be the symplectic version of what we've been considering or what should be the complex version or any other.

The result sometimes is very easy to guess. For example we have the Lie bracket of vector fields from $\mathrm{diff}(M)$ and the Poisson bracket of functions on a symplectic manifold (M, ω). So there is no doubt that the notion of Poisson bracket for functions is connected with the usual bracket of vector fields from $\mathrm{symp}(M, \omega) \subset \mathrm{diff}(M)$, we call this connection the *symplectization*. But in other cases complexifications, symplectizations or contactizations are highly nontrivial. And starting with one theory it might be difficult to find the analogs in another theory.

I had discovered some examples for which I had no doubts that the answers-conjectures obtained in this way were correct. But I was unable to prove them. The only thing I was able to do was to use them. And I really have used successfully these conjectures many times. The corollaries are nontrivial theorems but the method using which I arrived to them is rather illogical because there are no axioms to define exactly what the complexification is or what the symplectization means. There are only guesses at your disposal. You formulate conjectures and you can try to prove them. Then finally you or someone else prove the theorems and the obtained theory confirms the initial guesses.

One of the examples is what is known as Arnold conjecture in Lagrangian intersection theory[2]. This was my attempt to extend to the symplectic case the Poincaré theory of index of mappings and vector fields from the usual topology. There you have the notion of fixed points and the Euler-Poincaré theorem stating that the sum of indices of all fixed points on a manifold always equals one and the same number called the *Euler-Poincaré characteristic* of the manifold. I was trying to symplectize this and to guess what would be the answer. I shall not explain it in details. The main idea of symplectization is to consider models.

You start with a manifold M and you go to the symplectic manifold associated with it which is of course the cotangent bundle T^*M with the canonical symplectic structure. To any submanifold $N \subset M$ you associate a Lagrangian submanifold $L_N \subset T^*M$ corresponding to it. To symplectize the geometry of submanifolds of a smooth manifold you have to consider the geometry of Lagrangian submanifolds only.

This is the so called Weinstein principle. Alan Weinstein has formulated it in the following way: every interesting object in symplectic geometry is a Lagrangian submanifold. Since fixed points and their indices are certainly interesting, the symplectization of the usual intersection theory in M must give the Lagrangian intersection theory in T^*M.

The next idea is to forget the model manifold structure and to go to a general symplectic manifold. Thus in the sixties I was led to the conjecture that there should exist symplectic and contact topologies, and I started to develop them[3]

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[2]It seems that the first (correct?) proof of the "Arnold conjecture" for surfaces was published (after my rejecting about four preliminary versions of this publication in seventies) by Eliashberg in Syktyvkar in 1978 (there is still no English translation of the paper but Eliashberg has promised to publish it soon). Later after the celebrated paper by Conley and Zehnder (1983) based on the ideas of Rabinowitz there appeared many important contributions by Chaperon, Chekanov, Floer, Givental, Gromov, Hofer, Laudenbach, Sikorav, Viterbo, Weinstein and many others. Quantum and Floer cohomologies are the byproducts of this development. Since the first attempts there appeared different formulations extending my initial conjecture which minorates the number of the fixed points of an exact symplectomorphism of a compact symplectic manifold by the minimal number of critical points of a function on a manifold. Last year I was told that the original conjecture had been proved by Fukaya, Ono, Salamon, Ruan and others. However I was unable to check these technically very difficult proofs. Kontsevich was unable to report the details at my Paris seminar, whereas all these proofs are based on his lemmas on stable curves.

All these results are very important in symplectic topology. But it seems to me that my conjecture of 1965 on the number of fixed points of a symplectomorphism of a higher-dimensional annulus extending the Birkhoff-Poincaré theorem as well as the conjectures on the symplectic correspondences and on characteristic chords formulated in the paper "First steps of symplectic topology" (1986) are still open.

[3]I should stress that Gromov and Eliashberg, who were the first explorers of the new domains discovered in this road, were using a different notion of symplectic and contact topologies. They meant the study of homeomorphism invariants of objects from symplectic and contact geometries. And for me symplectic and contact topologies are the study of discrete invariants of the continuous objects from symplectic and contact geometries, be they homeomorphism invariant or not. In the same way I include into differential topology the discrete invariants of continuous (smooth) objects notwithstanding their homeomorphism invariance.

(Lagrange intersection theory, Maslov class, Lagrange and Legendre cobordisms). But I shall not discuss all this now; in the next lecture I shall give some examples of real conjectures and theorems which were obtained following the procedures of symplectization and contactization.

In the present lecture I shall discuss another project, namely the informal complexification. It seems simpler because everyone knows what are real and complex numbers. A complex vector space can be considered as a real vector space equipped with an additional structure, i.e. with the operator of multiplication by i. However the complexification of a theory is not just a restriction onto the spaces and introduction of a new operator. All the geometrical notions are changed. We should not consider complex subspaces and operators only as some subspaces and operators in the real space equipped with additional properties. Complex geometry is not just a subtheory of real geometry but is rather an independent sister-theory which is parallel to the real case. I shall show some examples of the informal complexification and the way how to use it.

Of course the complexification of the real line $\mathbb{R}$ is the complex line $\mathbb{C}$. Let us guess what is the complexification of Morse theory. In Morse theory you have nondegenerate functions. They must be considered smooth in the $\mathbb{R}$-case and holomorphic in the $\mathbb{C}$-case. Then you have critical points and critical values. If we consider smaller values and larger values around some critical value then to go from the first to the second one must add a handle. This is the main tool in the Morse theory which describes the modification of the regular level sets of a nondegenerate function. We use here the possibility to compare real numbers. But we have no natural inequalities for complex numbers. What shall be the complexification?

It is clear that the complexification of the function x^2+y^2 is z^2+w^2. And also the complexified notions of critical points and values remain the same. It agrees with the first example since the values lie in $\mathbb{R}$ and $\mathbb{C}$ correspondingly. The difference is that the complement to the critical values is disconnected in the $\mathbb{R}$-case and is connected in the $\mathbb{C}$-case. The preimages of different regular values are topologically the same in the complex case, you have no handles to be attached, you have no perestroikas. So what is happening? In the $\mathbb{R}$-case the zero-dimensional homotopy (one may wish to consider homology but I will deal with homotopy) group of the complement to the point is nontrivial:

$$\pi_0(\mathbb{R}\setminus\{0\}) = H_0(\mathbb{R}\setminus\{0\}) = \mathbb{Z}_2$$

contrary to the complex case. But for $\mathbb{C}$ we have to consider instead of the homotopy equivalence classes of mappings $S^0 \to \mathbb{R}\setminus\{0\}$ the homotopy equivalence classes of mappings $S^1 \to \mathbb{C}\setminus\{0\}$. So the complexification of π_0 is π_1 and we have:

$$\pi_1(\mathbb{C}\setminus\{0\}) = \mathbb{Z}.$$

Thus we obtain the first nontrivial fact that the complexification of the ring $\mathbb{Z}_2$ is the ring $\mathbb{Z}$.

Now we know what to do in the complex case with the critical values. We start from a regular value, make a turn around a critical value and return to the initial point. Thus we see that the perestroika complexifies to the monodromy. And the complexification of Morse theory is the Picard-Lefschetz theory[4]. In the example $f(z,w) = z^2+w^2$ the level variety is a cylinder and you have a vanishing cycle

[4] According to the recent paper by V. Vassiliev "Stratified Picard-Lefschetz Theory with Twisted Coefficients" [*Topics in Singularity Theory*, A. Khovansky, A. Varchenko, V. Vassiliev Ed., Advances in the Mathematical Sciences – 34 (AMS Translations 180), Providence RI (1997)

on it (which vanishes at the critical moment) and when you go around the critical value 0 you obtain the automorphism of the cylinder which maps the vanishing cycle to itself but makes the twist of the whole cylinder. So this Dehn twist, the Picard-Lefschetz transformation, is the complexification of the handle addition construction. From this simple example we see the complexification is not that trivial.

Let us continue. The complexification of the real projective space $\mathbb{R}P^n$ is the complex one $\mathbb{C}P^n$ and in particular the circle $S^1 \simeq \mathbb{R}P^1$ complexifies into the Riemann sphere $S^2 \simeq \mathbb{C}P^1$. Now turn to homologies. We have Stiefel-Whitney classes w_k in cohomologies with coefficients in $\mathbb{Z}_2$. And their complexifications are Chern classes c_k which belong to cohomologies with coefficients in $(\mathbb{Z}_2)_{\mathbb{C}} = \mathbb{Z}$. So all the examples agree.

Now I will show an example where these ideas of complexification work. Near 1970 Petrovsky asked me to help in evaluating a thesis of a mathematician Gudkov from Nizhni Novgorod (it was Gorky at that time). He was studying the Hilbert problem 16, the question on the plain algebraic curves of degree 6: what are the possible shapes of the set $f(x, y) = 0$, if $\deg f = 6$?

The classical answers for degree 2 were extended to degrees 3 and 4 by Newton and Descartes. But then the difficulties start. Hilbert was unable to solve the case of degree 6. And this problem was explicitly formulated in his list. One may also consider the affine version but it is more complicated and instead we may consider the projective one, dealing with the curves in $\mathbb{R}P^2$. Even to this, easier question, no answer was known at Hilbert's time.

The only known thing was the celebrated theorem of Harnack who had proved that the number of ovals (the zero-dimensional Betti number of the curve) was at most the genus plus one: $b_0(\Gamma) \le g + 1$, the genus being represented by Riemann's formula $g = \dfrac{(n-1)(n-2)}{2}$. For $n = 6$ one has $\max(b_0) = 11$. So the number of components of a curve of degree 6 is not greater than 11. A curve with the maximum possible number of ovals is called an M-curve. Hilbert formulated the question about the configuration of ovals for M-curves of degree 6.

The topologically possible configurations are counted by the rooted trees with 11 vertices. The number of such trees is enormous. However not all of them correspond to existing M-curves. Indeed, consider an example from Fig. 4. It is an impossible configuration by Bézout theorem: you have a line with 8 intersection points while the degree of the curve is 6.

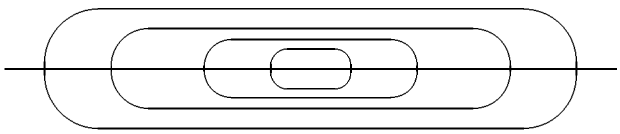

Fig. 4: Arrangement of ovals, impossible for a curve of degree 6.

So not all configurations are possible. Gudkov claimed to obtain the complete possible configurations list of ovals of degree 6 curves but Petrovsky was doubtful of his result. Let us describe it. The list contains three M-curves. Each of them

pp. 241–255], the complexification of the Goresky-MacPherson stratified Morse theory is the stratified Picard-Lefschetz theory extending that of F. Pham as well as the theory of the Petrovsky cycle crucial for the lacunae problems in hyperbolic PDEs.

possesses exactly one non-void oval, some number of other ovals lie inside and some lie outside. And for 3 items of the list the numbers of inside-outside are: $(1, 9)$, $(5, 5)$ and $(9, 1)$. And all other configurations are impossible. This is a wonderful theorem with a very involved and complicated proof, difficult to understand.

The thesis was the second version of the Gudkov's result. In the first one he had examined the case $(5, 5)$ and had been proving its nonexistence. And in the second he constructed the case $(5, 5)$ explicitly. He constructed also many curves of higher degrees and the whole picture was not clear, but he made some interesting observations about M-curves which he was able to construct.

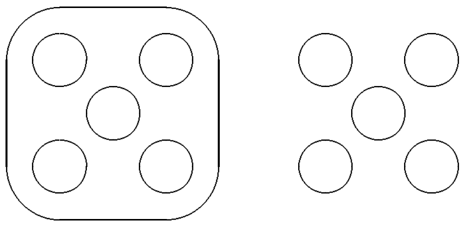

Fig. 5: Gudkov's $(5, 5)$ case for the topology of an M-curve of degree 6.

Let us consider the case $(5, 5)$, Fig. 5. If you put 4 ovals from inside to outside (or otherwise) then you obtain the case $(1, 9)$ (or $(9, 1)$) and the Euler characteristic of the set where the function is positive is changed by 8; you can go also in the opposite direction. So the Euler characteristic is well defined modulo 8. And this relation $\chi \equiv k^2 \pmod 8$ was observable in all examples of M-curves of degree $2k$ which Gudkov was able to construct for higher degrees. But there were no explanations for this behavior.

I was aware that congruences modulo 8 were standard in 4-dimensional topology. So my idea was that there existed somewhere a four-dimensional manifold which governed the topology of the real plane curve. But how to construct it? This was the place where the complexification came into the game and became very helpful.

To obtain something four-dimensional from a two-dimensional object you obviously need complexification. But the set under consideration, i.e. the positivity support of the function, is a manifold with boundary. Thus the problem was to complexify a manifold with boundary. Of course if you have a hypersurface, e.g. $\{x^2 + y^2 = 1\}$, then to complexify it is very easy; in our example: $\{z^2 + w^2 = 1\}$. But if you have an inequality $f(\mathbf{x}) \geq 0$ defining a manifold with boundary the description of the complexification is not so evident. I got it over in the following way.

Let us first algebrize, i.e. write the inequality $f(\mathbf{x}) \geq 0$ in algebraic terms. This is $f(\mathbf{x}) = y^2$. Of course this equation is equivalent to the preceding inequality but you have now no complex meaningless symbols. In the complex domain our formula defines the double covering of the complex manifold which is ramified along the boundary, i.e. along a complex hypersurface. Now we may complexify the positivity domain of our function: $f(\mathbf{z}) = w^2$. This is the complex analog of the manifold with boundary. It is the needed four-dimensional manifold.

Now to the complexified manifold we apply all the machinery of four-dimensional topology, calculating intersection forms, characteristic classes and other topological invariants. After this we return to the real picture and interpret everything in terms of the ovals topology. This gives the Gudkov theorem and many other results for higher degrees and dimensions.

Soon after my first steps many people continued all these ideas and the real algebraic geometry field became a very flourishing domain. I should mention here, to quote but a few, the important works by Rokhlin, Kharlamov, Viro, Nikulin, Shustin. For instance, Rokhlin proved the Gudkov congruence modulo 8 (I had proved it only modulo 4), Kharlamov found all the possible topological types of the surfaces of degree 4 in $\mathbb{R}P^3$ (which was one of the Hilbert's questions) and Nikulin found the components of the space of such surfaces. It is still unknown of what form are the curves of degree 8. The total number of cases verifying all the known conditions is at most 90 and only for 9 of them existence question is unclear. So there is still a challenge[5] even in degree 8.

Another example of complexification is connected with the paper "Modes and Quasimodes" on the quadratic forms stratification and its monodromy. Consider as degenerated the quadratic forms in Euclidean space which have multiple eigenvalues. It seems this condition gives one equation $\lambda_1 = \lambda_2$ but it turns that it does more. For symmetric matrices $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ of order 2 the equation $\lambda_1 = \lambda_2$ takes the form $(a-c)^2 + 4b^2 = 0$. Hence one equation $\lambda_1 = \lambda_2$ is equivalent to the set of two equations $a = c$ and $b = 0$. Similarly in any Euclidean space the codimension of the subvariety of degenerated quadrics among all quadrics is 2, not 1.

Suppose for instance you have a sputnik and you wish to make two axes equal in the adjusted ellipsoid of inertia, i.e. to make it an ellipsoid of revolution, which is better for the stability. And suppose your instrument is a weight that can move along one line. Then it is not sufficient, you really need two directions.

As the codimension of the variety of the degenerated ellipsoids is equal to 2 you may go around this set of ellipsoids with equal axes. You go as if around a line in a three-dimensional space. It is possible that during this turning a turnover occurs: two axes change their directions. The length of an axis remains the same, if you start from the largest axis of an ellipsoid then you return to the largest one but it is possible that the direction becomes opposite. This happens generically. Thus you have a flat connection, a monodromy in the space of nondegenerate ellipsoids.

Now consider the complexification. This means to consider Hermitian matrices instead of symmetric. In the Hermitian case the codimension of the variety of matrices having a multiple eigenvalue is 3, not 2. The fundamental group of the complement is trivial. But now on the transversal space (of dimension 3) there is a two-dimensional sphere linked with the multiple eigenvalues subvariety. And the complexification of the real theory of modes and quasimodes gives the theory of Berry phase from quantum mechanics and the theory of integer quantum Hall effect.

The complexification is a promising thing and there is a lot of questions. For example we considered the complexification of the boundary notion. But this notion is basic for the homology. What is the complexification of the homology? This is quite a nontrivial question followed by the questions about orientations, spin$^{\mathbb{C}}$-structures and so on. Here is plenty of field to speculate, formulate conjectures, construct theories but then there is the problem of proving them.

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[5]The Russian way to formulate a problem or a conjecture is to mention the simplest unknown case, making further simplification impossible. It is opposite to the French way of formulation where the problem appears in such a general form that no one can make a generalization.

And one must be careful because the complexification is by no means unique. There may be several complexifications of the same object. For example we have seen that the complexification of S^1 is S^2. But let us consider S^1 as the group $SO(2)$. The complexifications of $O(n)$ and $SO(n)$ are obviously $U(n)$ and $SU(n)$. So the complexification of $S^1 = SO(2)$ is $S^3 = SU(2)$. Our manifold is complexified differently: in one case as a projective variety, in the other as a Lie group. So we see that the complexification depends on the structure.

I mention also the informal complexification of the Coxeter reflection groups A, B, D which appeared in my recent paper on the Maxwell topological theorem on spherical functions, which extends the classical theorem $\mathbb{C}P^2/\mathrm{conj} = S^4$ to higher dimensions. These complexifications are Lie groups which might suggest a strange complexified version of the Chevalley theorem.

The recent complexification of the linking number by Frenkel and Khesin measures the "complex linking" of the topologically unlinked complex curves in complex 3-manifolds[6]. This complex linking is related to the string theory and hence to the quantum field theory. The complexification of the helicity invariant from hydrodynamics is certainly related to the Chern-Simons functional. But this relation should be formalized to become useful.

The next dream I want to present is an even more fantastic set of theorems and conjectures. Here I also have no theory and actually the ideas form a kind of religion rather than mathematics. The key observation is that in mathematics one encounters many trinities. I shall present below a list of examples. The main dream (or conjecture) is that all these trinities are united by some rectangular "commutative diagrams". I mean the existence of some "functorial" constructions connecting different trinities. The knowledge of the existence of these diagrams provides some new conjectures which might turn to be true theorems.

The first trinity that everyone knows is

$$(\mathbb{R},\ \mathbb{C},\ \mathbb{H}). \tag{1}$$

The next trinity is

$$(E_6,\ E_7,\ E_8). \tag{2}$$

The parallelism of these two trinities seems to be a nontrivial theorem in Galois theory for which I have no explanation, no proof and no formulation. Something similar was formulated and proved by Kazhdan at the Gelfand jubilee a few years ago but I have not seen its published version. I think in one of Kazhdan's papers the proof as well as the formulation may be found. I shall show beyond a simplified version.

A well known trinity comes from the Platonic theory. It is the trinity

$$(\langle\text{Tetrahedron}\rangle,\ \langle\text{Octahedron}\rangle,\ \langle\text{Icosahedron}\rangle). \tag{3}$$

The symmetry groups of these polyhedra are the Coxeter groups

$$(A_3,\ B_3,\ H_3). \tag{4}$$

[6]See B. Khesin's article "Informal Complexification and Poisson Structures on Moduli Spaces" [*Topics in Singularity Theory*, A. Khovansky, A. Varchenko, V. Vassiliev Ed., Advances in the Mathematical Sciences – 34 (AMS Translations 180), Providence RI (1997) pp. 147–155] and his papers with I. Frenkel, A. Todorov, A. Rosly and V. Fock quoted in this article]

Few years ago I had discovered an operation transforming the last trinity into another trinity of Coxeter groups:

$$(D_4,\ F_4,\ H_4). \tag{5}$$

I shall describe this rather unexpected operation later.

Let us continue. The first item of the next triple is the Möbius bundle, i.e. the two-fold covering mapping sending the Möbius strip boundary to the central line. The second is its complexification and the third is the quaternionic analog.

$$(S^1 \xrightarrow{S^0} S^1, \quad S^3 \xrightarrow{S^1} S^2, \quad S^7 \xrightarrow{S^3} S^4). \tag{6}$$

The complexification of the Möbius bundle is indeed the Hopf bundle. From the construction it is clear that already in the Möbius case one should consider the base as a projective space and the total space as a group. For the Hopf bundle it is well known. This agrees with the previous assertion that $(S^1)_{\mathbb{C}} = S^2$ when S^1 is a projective line and $(S^1)_{\mathbb{C}} = S^3$ when S^1 is considered with the group structure.

Next come the complex polynomials in one variable, the Laurent polynomials and the modular polynomials, i.e. rational functions with 3 poles at 0, 1 and ∞ (if one enlarge the number of poles then moduli appear but 3 points can always be normalized):

$$(\mathbb{C}[z], \quad \mathbb{C}[z, z^{-1}], \quad \mathbb{C}[z, z^{-1}, (z-1)^{-1}]). \tag{7}$$

A recent paper of Turaev and Frenkel published in the "Arnold and Gelfand seminars" contains the trinity

$$(\langle \text{Numbers} \rangle, \langle \text{Trigonometric Numbers} \rangle, \langle \text{Elliptic Numbers} \rangle). \tag{8}$$

The first is the set of ordinary complex numbers. The second is a quantum version, it consists of the deformations with one parameter. The numbers of the third type are two-parameter deformations.

The theory of modes and quasimodes leads to the geometrical triples

$$(\langle \text{Quadratic forms} \rangle, \langle \text{Hermitian forms} \rangle, \langle \text{HyperHermitian forms} \rangle). \tag{9}$$

and

$$(\langle \text{Flat Connection Monodromy} \rangle, \langle \text{Vector Bundle Curvature} \rangle, \langle \boxed{?} \rangle). \tag{10}$$

The first two items in both trinities are well-known. A hyperHermitian form in a quaternionic vector space is a real quadratic form invariant under the action of the group $S^3 = SU(2)$ of unitary quaternions. The third term in (10) is probably known to the experts in hyperKählerian geometry. The complexification of the curvature should be some 4-form, maybe called hypercurvature, and it should measure the degree to which the Bianchi identity fails.

I think that the relation between the hydrodynamical helicity (the asymptotic Hopf invariant) and the Chern-Simons functional mentioned above might be completed to form a trinity. But it is not clear whether the third item should be of smaller or of higher dimension.

The next trinity is a well-known homological triple consisting of Whitney, Chern and Pontryagin classes[7]:

$$(w_i,\ c_i,\ p_i). \tag{11}$$

[7] I was told by Gabrielov after the Toronto talk that in the polylogarithmic expressions for the Chern and Pontryagin forms he had studied with Gelfand, Losik and McPherson one meets the rational functions with 2 and 3 poles respectively. This confirms the parallelism between the

Givental proposed the trinity consisting of the homology, of the complexification, i.e. K-theory, and of the elliptic homology:

$$(H^*,\, K,\, \langle \varepsilon\ell\ell \rangle). \tag{12}$$

Now I shall explain some relations between different trinities. The relations in many cases are nontrivial and I shall show only the simplest ones. Consider first the polyhedra (3) and count the numbers of edges. They are 6 for the tetrahedron, 12 for the octahedron and 30 for the icosahedron. Each of these numbers is a product of two consecutive integers: $6 = 2 \cdot 3$, $12 = 3 \cdot 4$ and $30 = 5 \cdot 6$. Take the first factor and subtract one. We obtain the relations with the very first trinity (1):

$$\begin{matrix} \mathbb{R} & \mathbb{C} & \mathbb{H} \\ 1 & 2 & 4 \end{matrix}$$

In most cases the numerology is more delicate but is as astonishing as in this simple example. To find the relations one needs sometimes to work hard. For instance consider the line (10) where we do not know exactly what to put on the right. It must be some 4-form related to Pontryagin numbers or maybe some quaternization of the real locally trivial bundles or a complexification of the complex ones. But it seems nontrivial to find all these relations.

Maybe there is some complexified version of the quantum Hall effect, the three-dimensional transversal being replaced by a five-dimensional one. It would have been easy to predict the quantum Hall effect and the Berry phase theory simply by complexifying the theory of monodromy of quadratic forms from the "Modes and Quasimodes". This opportunity was lost. We may also miss more opportunities not studying the quaternionic version of the modes and quasimodes theory.

The relations between the lines (3) and (4) are obvious: the last consists of the symmetry groups of the first. But what is the relation with the line (5)? Consider first the group A_3. It is generated by reflections in 3 mirrors. But the total number of mirrors is higher. It is a good exercise for schoolchildren to find the whole configuration and the total number of mirrors for the tetrahedron symmetry group. The answer is 6 since there exists exactly one mirror through each edge. But it is not so easy to imagine the 3-dimensional space decomposition into parts given by these 6 planes in $\mathbb{R}^3$. Even to count the number of parts is some exercise. The answer is 24 and it is the order of the Weyl (or Coxeter) group, the parts being called chambers.

It is better to represent these chambers in the projective plane rather than in the 3-space since every mirror contains the origin of $\mathbb{R}^3$. So we have 6 lines in the projective plane and now it is not difficult to draw the picture (an arrangement in the contemporary language). It is represented in Fig. 6, left.

trinities (7) and (11). But the relation of the Whitney class to the ordinary polynomials, i.e. to the one pole case, remains mysterious.

Fig. 6: Springer cones for the A_3 group and their decomposition into Weyl chambers.

Each pair of opposite chambers in $\mathbb{R}^3$ is represented by a triangle (on the right side of Fig. 6 the triangle corresponding to two Weyl chambers is hatched). The walls of the chamber form just 3 lines of 6. The other 3 are added on the left side of the picture. Together they decompose the projective plane into 12 triangles, so the total number of pieces in $\mathbb{R}^3$ is $2 \cdot 12 = 24$.

Let us describe this decomposition in more details. The chamber walls form 3 planes in the 3-dimensional space and decompose it into 8 parts. I call them Springer cones (since Tony Springer has never considered them). For any Weyl chamber you construct a collection of walls — planes, which decompose the n-space into 2^n Springer cones. Each Springer cone contains several Weyl chambers, as in Fig. 6. On the projective plane the decomposition of the space into 8 parts is represented as the decomposition into 4 parts, each representing two opposite cones. Thus we get the decomposition of the projective plane into 4 (Springer) triangles by continuations of the walls of one chamber. Now we count the numbers of Weyl chambers in the different parts and obtain the decomposition of the total Weyl number:

$$24 = 2\,(1 + 3 + 3 + 5).$$

If we do the same thing for the case of octahedron we obtain the following decomposition of the corresponding Weyl number:

$$48 = 2\,(1 + 5 + 7 + 11).$$

For the icosahedron case we get:

$$120 = 2\,(1 + 11 + 19 + 29).$$

If you are experienced with Coxeter groups you know the quasihomogeneous weights for D_4. They are $(2, 4, 4, 6)$. If you compare these numbers with the numbers from the decomposition of 24 above it will not be strange that the quasihomogeneous weights for F_4 are $(2, 6, 8, 12)$. Now if you even know nothing about H_4 you may guess the quasihomogeneous weights: $(2, 12, 20, 30)$. And so we found one relation between lines (4) and (5).

Note, that the weights of D_4, F_4 and H_4 provide the numbers of vertices, faces and edges of the tetrahedron, octahedron and icosahedron. And that the number of the vertices of the octahedron equals the number of the edges of the tetrahedron, while the number of the vertices of the icosahedron is equal to the number of the edges of the octahedron. It is a mistery for me — the complexification seems to transform the edges into the vertices!

In other cases the parallelism might be even more strange. I add the triple of the triangles which are sold in the stationery shops:

$$\Big((60, 60, 60),\, (45, 45, 90)\, (30, 60, 90)\Big). \tag{13}$$

The theory described above suggests that the second is the informal complexification of the first and the third is the quaternionic version, but this conjecture is as mysterious as the statement that the octahedron is the complex and the icosahedron is the quaternionic version of the tetrahedron.

I have heard from John MacKay that the 27 straight lines on a cubical surface, the 28 tangents of a quartic plane curve and the 120 tritangent planes of a canonic sextic curve of genus 4, form a trinity parallel to E_6, E_7 and E_8.

One might even speculate, whether the sporadic simple groups are the quaternionic versions of the classical ones.

Question. What is the complexification of the Kazarian cylinder? That is of the functions on the cylinder.

Answer. The complexification of the real trigonometric polynomials stratification theory is the theory of complex trigonometric polynomials stratification; it may be found in "Functional Analysis" 1996. There also must be a quaternization.

The theory of Springer cones is the real version of the complex L2 (Lyashko-Looijenga) mapping, sending the complex polynomial in one variable to the unordered set of its critical values. The complex trigonometric version of the L2 mapping is the L3 (Lyashko-Looijenga-Laurent) mapping. One can try to consider the real trigonometric polynomial theory as an affine Coxeter group $\tilde{A}$ version of the Springer cones theory, in the spirit of Dubrovin's work on the affine Coxeter groups associated to the Frobenius varieties of conformal string theory. Trying to find this extension of the Springer cones theory I have constructed a polyhedral model of the real trigonometric M-polynomials stratified variety. But the relation of these polyhedra to the Dubrovin affine Coxeter group remains conjectural (perhaps to get the trigonometric case of the Springer cones theory one should apply Dubrovin's construction to the real forms, different from the considered real form, which correspond to the same complex object as it was in Dubrovin's study).

The real version of the complex theory of the L3 mapping is related to the Kazarian theorem. One might also consider another complexification of the cylinder $S^1 \times \mathbb{R}$ as was suggested to me by A. Tyurin. It is $S^3 \times \mathbb{R}$. Indeed, the neighborhood of S^1 in $\mathbb{C}$ is $S^1 \times \mathbb{R}$ while that of $S^3 = (S^1)_{\mathbb{C}}$ in $\mathbb{H} = (\mathbb{C})_{\mathbb{C}}$ is $S^3 \times \mathbb{R}$. There also should exist some modular version of L3 mapping theory with 3 singular points. But it has to be constructed yet.

Question. Is the case of 3 singular points related to quaternions?

Answer. Certainly it should be related to quaternions but I do not know what the relation is.

Question. Did Gudkov get the recommendation for his thesis?

Answer. The thesis was of course defended even though I was never able to read it. But as a result I invented all the matter I have explained to you. I was working hard for a month and after this I proved his conjecture modulo 4. The most difficult thing was some lemma which I was able to guess but was unable to prove. I always had very good undergraduate students and at that time I asked Varchenko to help me. He returned in a few days saying that the lemma might be proved by some ingenious arguments that he explained to me. This allowed me to continue my work. But when I came to the end I found out that the arguments were not proving the lemma and it was a catastrophe because everything depended on this lemma. I had to work hard once more to prove the lemma and I finally succeeded. But without Varchenko's statement on the correctness of that lemma I would have never performed the rest of the work, being stopped by the lemma. Unfortunately Varchenko had declined to sign the final paper as a coauthor.

D. A. Gudkov became the leader of a strong team in real algebraic geometry at Nizhni Novgorod (Utkin, Polotovskii, Shustin, ...). Some of the results of Gudkov and his students were recently rediscovered by C. T. C. Wall. Not long ago the American Mathematical Society has published a volume, dedicated to Gudkov's memory. You can find there the biography of this extraordinary person and the description of his plenty contributions to real algebraic geometry.