THE COHOMOLOGY RING OF THE COLOR
ED BRAID GROUP

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The cohomology ring is obtained for the space of ordered sets of \( n \) different points of a plane.

Artin's colored braid group of the space \( M_n \) of ordered sets of \( n \) pairwise different points of a plane.† It is not difficult to show that \( M_n \) is the space \( K(\pi, 1) \) for the group \( \text{I}(n) \):

\[
\pi_1(M_n) = I(n), \quad \pi_i(M_n) = 0 \quad \text{for} \quad i > 1.
\]

From this it follows that the cohomologies of \( \text{I}(n) \) coincide with those of \( M_n \) (what we have in mind is the trivial action of \( Z \)):

\[
H^*(\text{I}(n)) \cong H^*(M_n, Z).
\]

In the present note a description is given of this cohomology ring. We use a realization of \( M_n \) in the form of a complex affine space \( C^n = \{z = (z_1, \ldots, z_n)\} \) with "eliminated diagonals:"

\[
M_n = (Z \subset C^n; \quad z_k \neq z_l, \quad \forall k \neq l).
\]

We shall denote by \( A(n) \) the external graduated ring \( C_n^k \) generated by one-dimensional elements \( \omega_{k,l} = \omega_{l,k}, \)

\[
1 \leq k \neq l \leq n, \quad C_n^k \text{ satisfying the relationships}
\]

\[
\omega_{k,l} \omega_{m,k} + \omega_{k,l} \omega_{m,l} + \omega_{m,k} \omega_{m,l} = 0.
\]

THEOREM. The homology ring of the colored braid group is isomorphic to \( A(n) \). The isomorphism \( H^*(M_n Z) \cong A(n) \) is set up by the formulas

\[
\omega_{k,l} = \frac{1}{2\pi i} \frac{dz_k - dz_l}{z_k - z_l}.
\]

In other words, the one-dimensional generators \( \omega_{k,l} \) correspond to circuits around the diagonals \( z_k = z_l \).

COROLLARY 1. The cohomology groups of the colored braid group are torsion-free.

COROLLARY 2. The Poincaré polynomial of the manifold \( M_n \) is

\[
p(t) = (1 + t)(1 + 2t) \ldots (1 + (n - 1)t).
\]

In other words, the cohomology groups of the manifold \( M_n \) (or of the group \( \text{I}(n) \)) are the same as for the direct product of a circle, a bouquet of two circles, \ldots, a bouquet of \( (n-1) \) circles.

COROLLARY 3. The additive basis of the ring \( A(n) \) consists of all products of the form

\[
\omega_{k_1,l_1} \omega_{k_2,l_2} \ldots \omega_{k_p,l_p}, \text{where} \quad k_1 < l_1, \quad l_1 < l_2 < \ldots < l_p.
\]

COROLLARY 4. The subring of the ring of external differential forms \( C_n^k \) generated by the forms (2) is isomorphic to \( A(n) \).

COROLLARY 5. An external polynomial in the differential forms (3) is cohomologous to zero in \( M_n \) if and only if it is equal to zero.

† The name is explained by the other definition: \( \text{I}(n) \) is the kernel of the natural homomorphism \( B(n) \rightarrow S(n) \)

of the group of braids consisting of \( n \) strands onto the symmetric group of permutations of the ends of the braid. In other words, \( \text{I}(n) \) consists of braids each strand of which is individualized (tinted in its own color) and ends where it begins.


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COROLLARY 6. The symmetrization of an arbitrary external polynomial of degree greater than 1 in the differential forms (2) is equal to zero.

Example. The non-obvious identity

$$\sum_{\text{all}} \omega_1, \omega_2, \omega_3, \omega_4, \omega_5 = 0,$$

holds, where the summation is carried out over all 120 permutations of the digits 1, ..., 5.

It is easy to prove

LEMMA 1. There exists a stratification $M_n \rightarrow M_{n-1}$: its stratum is a plane lacking $n-1$ points. The action of the fundamental group of the base $M_{n-1}$ in a cohomology of the stratum is trivial. The stratification $p$ has a secant.

In fact, let us assume $p(z_1, ..., z_n) = z_1, ..., z_{n-1}$. Then the stratum $F_{n-1} = \{z \in C: z = z_1, ..., z_{n-1}\}$. The stratum $F_{n-1}$ is homotopically equivalent to a bouquet of $n-1$ circles. The group of one-dimensional (co)homologies for the stratum is isomorphic to $Z + \cdots + Z (n-1$ times). The fundamental group of the base is the colored braid group resulting from $n-1$ strands, $I(n-1)$. Its action in the stratum is the ordinary action of a braid group in a plane with eliminated points. But the braids in $I(n-1)$ are colored, and they do not permute the eliminated points. Consequently, $I(n-1)$ acts trivially in a (co)homology of the stratum. The secant may be given by the formula

$$z_n = \frac{z_1 + \cdots + z_{n-1}}{n-1} + 2 \max_{1 \leq i < j \leq n-1} |z_i - z_j| + 1.$$

The simple proof of Theorem 1 given above is due to D. B. Fuks.

We shall consider a cohomological spectral sequence of the stratification $M_n \rightarrow M_{n-1}$. Since $\pi_1(M_{n-1})$ acts trivially in a cohomology of the stratum $F_{n-1}$, the term $E_2^* = H^*(M_{n-1}, H^*(F_{n-1}))$ is the same as in the direct product. The only possible differential $d_2$ is in fact zero (this easily follows from the existence of the secant of the surface). Thus, $E_2 = E_{\infty}$. So the (co)homology groups of $M_n$ are the same as in the direct product of $M_{n-1}$ and $F_{n-1}$. Putting in succession $n = 2, 3, \ldots$ (with $M_1 = C$), we find that the (co)homologies of $M_n$ are the same as in the direct product of a circle, a lemniscate, ..., a bouquet of $n-1$ circles. Corollaries 1 and 2 are proved.

We shall construct an additive basis for $H^*(M_n, Z)$. It follows from our spectral sequence that it can be obtained from the image of the additive basis of $H^*(M_{n-1}, Z)$ under the map $p^* \pi^*$ by adding the products of its elements by $n-1$ one-dimensional classes of cohomologies which transform into the generators $H^*(M_{n-1}, Z)$ under the map $l^*$ (where $l: F_{n-1} \rightarrow \pi$). We note that we may take as these one-dimensional classes cohomology classes of the differential forms $\omega_1, \omega_2, \omega_3, \ldots, \omega_n, \omega_n$ of (2). Putting in succession $n = 2, 3, \ldots$, we see that the products of the type (3) of the differential forms (2) form the additive bases of $H^*(M_n, Z)$.

The differential forms (2) satisfy the relationships (1). This can be verified by direct substitution. The cohomology classes of the differential forms (2) in the ring $H^*(M_n, Z)$ a fortiori satisfy the relationships (1). We can therefore construct the ring homomorphism $\varphi: A(n) \rightarrow H^*(M_n, Z)$ by associating with the generators $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$ of $A(n)$ the differential forms of $H^*(M_n, Z)$ in accordance with formula (2). We have shown above that $\varphi$ has no kernel. It is easy to prove

LEMMA 2. The ring $A(n)$ is generated additively by the products (3).

For it follows from the anticommutative property that $A(n)$ is generated by the products $\omega_{k_1} \omega_{k_2} \cdots \omega_{k_p}$, where $k_1 < k_2 < \cdots < k_p$ and $i \leq i$. The relationship (1) enables us to get rid of equal $l$. For example,

$$\omega_{k_1} \omega_{k_2} \cdots \omega_{k_p} = \omega_{k_1} \omega_{k_2} \cdots \omega_{k_p}.$$

In both the summands the greater index of the first factor is strictly less than $l$. Thus all the products $\omega_{k_1} \omega_{k_2} \cdots \omega_{k_p}$ can be expressed additively in terms of products in which $k_1 < k_2 < \cdots < k_p$. The lemma is proved.

It follows from this that the ring homomorphism $\varphi: A(n) \rightarrow H^*$ has no kernel. For the products (3) which generate $A(n)$ additively transform into independent elements of $H^*$ (we have established above that they form in $H^*$ an additive basis). Consequently $\varphi$ has no kernel; so $\varphi$ is a ring isomorphism. Theorem 1 is proved.
We have at the same time proved Corollary 3, since we already know that in the ring \( H^* \) the products (3) form an additive basis. Corollaries 4 and 5 follow from the fact that, on the one hand, the cohomology classes of the forms generated by the forms (2) form the ring \( H^*(M_N, Z) \), isomorphic to \( A(n) \); but on the other hand, the differential forms (2) themselves satisfy the relationships (1).

Corollary 6 follows from Corollary 5 and the finiteness of the cohomology groups \( H^i(B(n)) \), \( i > 1 \) (\( B(n) \) is the braid group formed from \( n \) strands [1]).

**Note.** Let \( M \) be the manifold obtained from \( C^n \) by discarding an arbitrary number of hyperplanes

\[
M = \{ z \in C^n : a_k(z) \neq 0, k = 1, \ldots, N \}.
\]

Probably, the ring \( H^*(M, z) \) is torsion-free and is generated by the one-dimensional classes \( \omega_k = (1/2\pi i)(da_k/\omega_k) \), an external polynomial in \( \omega_k \) being cohomologous to 0 in \( H^* \) only when it is zero.

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