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RECOGNIZING TEXTILE STRUCTURES BY FINITE TYPE KNOT INVARIANTS

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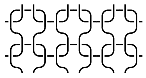
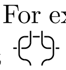
ABSTRACT

Typical examples of textile structures are separated by finite type invariants of knots in non-trivial (in particular, non-orientable) manifolds. A new series of such invariants is described.

Keywords: Periodic knot, finite type invariant, textile structure, knots in non-orientable manifolds

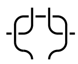
1. Introduction

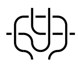
The topological study of textile structures was started in [8]. By some idealization, textile structures can be considered as specific knots or links. In this way their study can be reduced to the study of double periodic links in $\mathbf{R}^3$, i.e. one-dimensional submanifolds in $\mathbf{R}^3$ invariant under the action of some sublattice $\mathbf{Z}^2 \subset \mathbf{R}^3$. Up to some specialization (the choice of a linear identification of the quotient space $\mathbf{R}^2/\mathbf{Z}^2$ with a standard torus $\mathbf{T}^2$) it coincides with the study of links in $\mathbf{T}^2 \times \mathbf{R}^1$. For example, the single jersey structure  can be represented by the knot  in a fundamental domain of this action. For a review of fabric types, see e.g. [5] and [16]; these types provide a wealth of natural examples of knots in this manifold. In the present article, we demonstrate how the finite type invariants (of degree up to 2) separate these structures, and elaborate some methods of reducing the related calculations. We believe that this gives us the first examples of practical calculations of finite type invariants of sufficiently complicated knots in non-trivial (in particular, non-orientable) manifolds. Here we restrict ourselves only on the case of doubly periodic knots, i.e. one-component links; they occur mainly as knitted (i.e. jersey-like) fabrics.

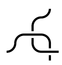
Different invariants of knots in non-trivial manifolds have been studied in many works. In particular, some Kauffman-type polynomial invariants of knots in manifolds of type $M^2 \times \mathbf{R}^1$ (M^2 an orientable surface) are known, see [12] and references in therein, and also [8]. Many of these invariants yield also finite type invariants by using the Birman–Lin construction [4]. In particular, the invariant of [8] separates many textile structures, however, there are many structures, well-known in the textile practice, that have one and the same value of this invariant. Theory of knots in manifolds $M^2 \times \mathbf{R}^1$ is closely related to the virtual knot theory of [10]. In the framework of this theory, the Khovanov homology theory was generalized to the knots in manifolds of type $M^2 \times \mathbf{R}^1$, see [13], [14]. This invariant is more powerful than the analog of the Kauffman polynomial, but its explicit calculation is very complicated.

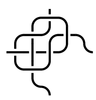
The general theory of finite type invariants for links in arbitrary 3-dimensional manifolds was developed in [9] and [19]; the special case of 3-manifolds of the form $M^2 \times \mathbf{R}^1$ (M^2 orientable) was studied in [6], [7], [1], [2] and many other works. However, only few explicit examples of such invariants are known. In §2 we recall some such examples and introduce several new invariants. In particular, in subsection 2.2 we describe an infinite series of invariants of all degrees, whose first representatives are the degree 1 Fiedler's invariants [6], [7]. These invariants are characterized by the condition that their principal parts take zero value on all chord diagrams with crossing chords. In §3 we apply these and other invariants to distinguish classical knitted and knotted structures. Namely, we compare the following well-known structures (represented by their diagrams in the standard rectangular chart of $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, see [8]).

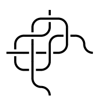
The unknit (1.1)

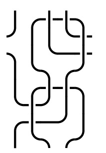
Plain knit (single jersey)  (1.2)

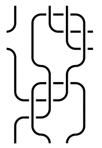
Plain knit with closed loops  (1.3)

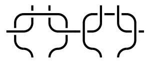
Wire netting  (1.4)

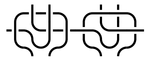
Weaver's knot  (1.5)

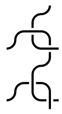
Fake weaver's knot  (1.6)

Tricot with open loops  (1.7)

Tricot with closed loops  (1.8)

1+1 rib with open loops  (1.9)

1+1 rib with closed loops  (1.10)

1+1 wire netting  (1.11)

The mirror images of all these structures obtained by replacing all overcrossings with undercrossings and vice versa will be considered as well. We shall denote the mirror image of the structure (x) by $\overline{(x)}$. It is quite obvious that the structure $\overline{(1.9)}$ is equivalent to (1.9), $\overline{(1.10)}$ to (1.10), and $\overline{(1.11)}$ to (1.11).

We shall assume that three orientations are always fixed: 1) the canonical (counterclockwise) orientation of the standard torus, 2) the additional orientation "towards the observer" of the line $\mathbf{R}^1$ orthogonal to the torus (and hence of the entire space $\mathbf{T}^2 \times \mathbf{R}^1$), and 3) an orientation of the knot. Two knot diagrams are considered equivalent if they can be reduced to one another by an orientation-preserving diffeomorphism of $\mathbf{T}^2$.

Theorem 1. *The structures (1.1)—(1.11) and their mirror images are divided by the first degree invariants into five groups so that elements of any group are not separated from one another, but are separated from all elements of other groups*:

- (1.1), (1.6), $\overline{(1.6)}$, (1.7), $\overline{(1.7)}$, (1.8), $\overline{(1.8)}$, (1.9), (1.10), (1.11);
- (1.2), $\overline{(1.3)}$;
- $\overline{(1.2)}$, (1.3);
- (1.4), (1.5);
- $\overline{(1.4)}$, $\overline{(1.5)}$.

Namely, let $\overset{1}{\sim}$ be the 1-equivalence relation that identifies structures not separated by degree 1 invariants. The following relations are proved: (1.1) $\overset{1}{\not\sim}$ (1.2) $\overset{1}{\not\sim}$ $\overline{(1.2)}$ $\overset{1}{\not\sim}$ (1.1) in §3.1; (1.2) $\overset{1}{\sim}$ $\overline{(1.3)}$ and $\overline{(1.2)}$ $\overset{1}{\sim}$ (1.3) in §3.2; non-equivalence of (1.4) and $\overline{(1.4)}$ to each other and to either of (1.1), (1.2), $\overline{(1.2)}$, (1.3) and $\overline{(1.3)}$ in §3.3; (1.5) $\overset{1}{\sim}$ (1.4) and $\overline{(1.5)}$ $\overset{1}{\sim}$ $\overline{(1.4)}$ in §3.4; the equivalence (1.6) $\overset{1}{\sim}$ (1.1) $\overset{1}{\sim}$ $\overline{(1.6)}$ in §3.5; (1.7) $\overset{1}{\sim}$ (1.1) $\overset{1}{\sim}$ $\overline{(1.7)}$ in §3.6; (1.8) $\overset{1}{\sim}$ (1.7) and $\overline{(1.8)}$ $\overset{1}{\sim}$ $\overline{(1.7)}$ in §3.7; (1.9) $\overset{1}{\sim}$ (1.1) in §3.8; (1.10) $\overset{1}{\sim}$ (1.1) in §3.9; and (1.11) $\overset{1}{\sim}$ (1.1) in §3.10.

Remark 1. We do not claim here that (1.6) and $\overline{(1.6)}$ (respectively, (1.7) and $\overline{(1.7)}$, respectively, (1.8) and $\overline{(1.8)}$) are non-equivalent to one another.

Theorem 2. *The second degree invariants separate all the structures* (1.1)*,* (1.6)*,* (1.7)*,* (1.8)*,* (1.9)*,* (1.10)*,* (1.11) *from each other, with the unique exception that they cannot separate* (1.7) *from* (1.8)*. Also, they separate the structure* (1.2) *from* $\overline{(1.3)}$, $\overline{(1.2)}$ *from* (1.3)*,* (1.4) *from* (1.5)*, and* $\overline{(1.4)}$ *from* $\overline{(1.5)}$*.*

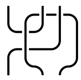
Namely, in §§3.5, 3.6, 3.7, 3.8, 3.9 and 3.10 we calculate the values that basic second degree invariants take on structures (1.6), (1.7), (1.8), (1.9), (1.10) and (1.11) respectively. The comparison of these values proves that all these structures are not 2-equivalent to each other except for the pair (1.7) and (1.8).

Also, we prove in §3.2 that (1.3) is not 2-equivalent to $\overline{(1.2)}$, and $\overline{(1.3)}$ is not 2-equivalent to (1.2); in §3.4 we prove that (1.4) is not 2-equivalent to (1.5) and $\overline{(1.4)}$ is not 2-equivalent to $\overline{(1.5)}$; in §3.7 we prove that (1.8) is 2-equivalent to (1.7).

Second degree invariants do not separate a knot and its mirror image that are not separated by first degree invariants; therefore we obtain also the following fact.

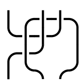
Corollary 1. *Any of structures* (1.1)*,* (1.6)*,* $\overline{(1.6)}$*,* (1.7)*,* $\overline{(1.7)}$*,* (1.8)*,* $\overline{(1.8)}$*,* (1.9)*,* (1.10)*,* (1.11) *is 2-separated from all other structures of this list, with only the following exceptions: four structures* (1.7), (1.8), $\overline{(1.7)}$ *and* $\overline{(1.8)}$ *are not 2-separated from each other, and* (1.6) *is not 2-separated from* $\overline{(1.6)}$*.*

Nevertheless, the structures (1.7) and (1.8), $\overline{(1.7)}$ and $\overline{(1.8)}$ can be separated even by degree one invariants, if we consider them up to a more refined classification. Indeed, the tricot fabric, whose elementary cell is represented by (1.7), has an additional symmetry to the group of parallel shifts of the lattice $\mathbf{Z}^2$. In the picture (1.7) this additional symmetry is represented by the vertical shift by half the height of the cell, and simultaneous reflection with respect to a vertical line. The quotient of $\mathbf{R}^3$ by the entire group of symmetries generated by $\mathbf{Z}^2$ and this additional symmetry is equal to $\mathbf{K}^2 \times \mathbf{R}^1$, where $\mathbf{K}^2$ is the Klein bottle. This additional factorization reduces the cell (1.7) to the picture



(1.12)

Note that the boundary of the obtained rectangular cell is subject to the following identification: the vertical margins are joined together preserving the height in the page, while top and bottom boundaries are identified in such a way that the first, the second and the third from the left endpoints of our knot in the top boundary are identified respectively with the first, the third and the second endpoints in the bottom boundary. The similar picture for (1.8) is



(1.13)

The isotopy problem of these two knots in $\mathbf{K}^2 \times \mathbf{R}^1$ is exactly the problem on the equivalence of the knots (1.7), (1.8) up to isotopies in $\mathbf{T}^2 \times \mathbf{R}^1$ preserving the additional symmetry.

Theory of finite type invariants in non-oriented manifolds was developed in [19]. Using it, we obtain the following result.

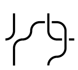
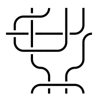
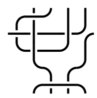
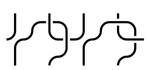
Theorem 3. *All knots* (1.12), (1.13), $\overline{(1.12)}$, *and* $\overline{(1.13)}$ *in* $\mathbf{K}^2 \times \mathbf{R}^1$ *are separated from each other and from the unknit (i.e. from a knot with non-crossed projection to* $\mathbf{K}^2$*) by invariants of degree 1, with only two exceptions that* (1.12) *is not separated by these invariants from* $\overline{(1.13)}$, *and* (1.13) *from* $\overline{(1.12)}$.

We plan to prove in a future work that the last two pairs of knots can be separated by a degree 2 invariant of knots in $\mathbf{K}^2 \times \mathbf{R}^1$.

2. Invariants of knots in $M^2 \times \mathbf{R}^1$

In this section, we recollect briefly the main facts on finite degree invariants of knots in arbitrary 3-dimensional manifolds (and especially in the manifolds of type $M^2 \times \mathbf{R}^1$, M^2 an orientable surface), and describe several such invariants.

The complete invariant of degree zero of knots in M^3 is the homotopy class of the knot in the loop space of all continuous mappings $S^1 \to M^3$, i.e., the corresponding conjugacy class in the group $\pi_1(M^3)$. In the case $M^3 = \mathbf{T}^2 \times \mathbf{R}^1$ these conjugacy classes coincide with elements of $H_1(\mathbf{T}^2)$ and are described by two integer numbers: the rotation indices along the horizontal and the vertical (in our pictures) generators of the torus. These numbers are equal to $(1, 0)$ for structures (1.1), (1.2), (1.3), (1.9), and (1.10); $(1, -1)$ for (1.4), (1.5) and (1.6); $(0, 1)$ for (1.7) and (1.8); and $(2, -1)$ for (1.11). All these classes can be reduced to one another by appropriate linear (i.e. induced by $SL(2, \mathbf{Z})$-transformations $\mathbf{R}^2 \to \mathbf{R}^2$) diffeomorphisms of $\mathbf{T}^2$. In particular, the transformations reducing these elements to the vector $(1, 0)$ move the pictures (1.4), (1.5), (1.6) and (1.11) to the forms

 ;  ;  and  (2.1)

respectively. The similar forms for (1.7) and (1.8) can be obtained by the clockwise rotation of the initial pictures by $\pi/2$.

In general, two double periodic structures in $\mathbf{R}^3$ are equivalent if the corresponding knots in $\mathbf{T}^2 \times \mathbf{R}^1$ can be reduced to one another by compositions of smooth isotopies and linear diffeomorphisms of $\mathbf{T}^2 \times \mathbf{R}^1$ induced by appropriate $SL(2, \mathbf{Z})$-transformations of $\mathbf{T}^2$.

2.1. *Finite type invariants in orientable 3-manifolds*

In this subsection we recollect necessary facts about finite type invariants in arbitrary orientable 3-dimensional manifolds, see [11], [9], [19]. These invariants can be defined in exactly the same way as for knots in $\mathbf{R}^3$, see [17], [4].

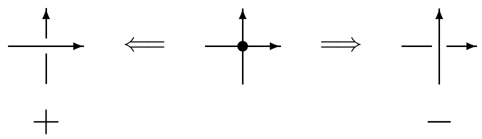
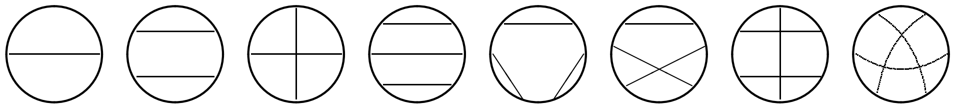


Fig. 1. Resolutions of a transverse self-intersection



1_1 2_1 2_2 3_1 3_2 3_3 3_4 3_5

Fig. 2. Examples of chord diagrams

A self-intersection point $f(x) = f(y)$, $x \neq y$, of a smooth map $f : S^1 \to M^3$ is called *transverse* if the derivatives of f at x and y are not collinear in $T_{f(x)}M^3$. Any transverse self-intersection of a map $f : S^1 \to M^3$ can be resolved in two essentially different ways by small local moves of f, see Fig. 1. These two local resolutions cannot be connected by a short local path in the space of embeddings $S^1 \to M^3$: they are separated in a neighborhood of f by a piece of the *discriminant subvariety* in $C^\infty(S^1, M^3)$ consisting of maps with self-intersections. This subvariety is a singular hypersurface; its smooth points are exactly the maps with unique transverse self-intersection. If M^3 is oriented, then there is an invariant way to call one of these two resolutions as *positive*, and the other as *negative*; for the canonical orientation in $\mathbf{R}^3$ this discrimination is indicated by indices $+$ and $-$ in Fig. 1. Indeed, if we fix an affine chart in M^3 close to the self-intersection point $f(x) = f(y)$, and a parameterization of S^1, then the determinant of the triplet of vectors $f'(x)$, $f'(y)$, and $f(y) - f(x)$ is a well-defined function in a neighborhood of the point f in the space $C^\infty(S^1, M^3)$. The derivative of this function defines an invariant transversal orientation of the discriminant variety at the point f, and hence the desired difference between two possible resolutions of f.

Given a numerical invariant I of knots in M^3 (i.e. of smooth embeddings $S^1 \to M^3$) and an arbitrary map $f : S^1 \to M^3$ with k transverse self-intersection points $f(x_i) = f(y_i)$, $f'(x_i) \nparallel f'(y_i)$, $i = 1, \dots, k$, which does not have any other self-intersections or singular points, we can eliminate all these singularities in 2^k different ways, replacing any self-intersection point as it is shown in the left- or right-hand part of Fig. 1. The *residue* of the invariant I on the singular knot f is defined as the alternated sum of values of I on all these 2^k non-singular knots obtained from f; the value of I on such a desingularization should be taken with the coefficient 1 or -1 depending on the parity of the number of negative local resolutions defining this desingularization.

By definition, a knot invariant is *of degree* $\leq k$ if its residue at any singular knot with more than k transverse self-intersections is equal to 0. An invariant is of degree k if it is of degree $\leq k$ but not of degree $\leq k-1$.

Definition 1. A *chord diagram of degree* k (or simply a k*-chord diagram*) is an arbitrary collection of $2k$ distinct points in S^1 matched in pairs. (For examples of such diagrams, see Fig. 2, where the matched points are connected by thin chords). A smooth map $f : S^1 \to M^3$ *respects* some chord diagram if it joins together the points of any of its pair. Two k-chord diagrams are *equivalent* if they can be transformed into one another by orientation-preserving diffeomorphisms of S^1. Given an equivalence class A of k-chord diagrams, we say that two maps $f_1, f_2 : S^1 \to M^3$ belong to one and the same A*-route* of degree k, if they both respect some k-chord diagrams $\bar{A}_1, \bar{A}_2$ of class A, and can be reduced to one another by the composition of 1) a homotopy in the class of maps $S^1 \to M^3$ respecting $\bar{A}_1$, and 2) an orientation-preserving reparameterization of S^1 moving $\bar{A}_1$ to $\bar{A}_2$. Thus, the A*-routes* in M^3 are the equivalence classes of singular maps $S^1 \to M^3$ under this equivalence relation.

It is easy to see that any knot invariant of degree k defines equal residues at all singular knots with k transverse self-intersections, belonging to one and the same A-route of degree k.

The function, defined thus by a knot invariant of degree k on the set of all possible A-routes of degree k, is called the *principal part* of this invariant. Principal parts of all degree k invariants satisfy two standard conditions. The simplest of them, called 1T-*relation*, claims that any such principal part takes zero value on any A-route of degree k such that

1) any chord diagram of class A contains a chord, whose endpoints x_i, y_i are not separated in S^1 by the endpoints of other chords of this diagram (i.e. one of segments $[x_i, y_i]$ or $[y_i, x_i] \subset S^1$ does not contain points x_j or y_j, $j \neq i$, as in pictures 1_1, 2_1, 3_1, 3_2, 3_3 of Fig. 2), and

2) the loop $f : [x_i, y_i] \to M^3$ or $f : [y_i, x_i] \to M^3$, defined by the image of this segment under a map f from our A-route, is contractible in M^3.

The second series of restrictions (2.2), called 4T-relations, is more complicated; it can be derived from the consideration of singular maps with $k-2$ self-intersections and one triple point. Let us consider any such generic map, i.e. a map $f : S^1 \to M^3$ with $k-2$ transverse double self-intersections, one triple self-intersection such that three derivatives of f at this point are linearly independent in T_*M^3, and having no other self-intersection or singular points. The triple point of this map can be resolved in six different ways, splitting it into two double self-intersection points, see Fig. 3, so that f splits in six different ways into singular knots with k self-intersections. Let I be a degree k invariant, and $I(m)$, $m = 1, \ldots, 6$ be the value of its principal part on the singular knots obtained from f as indicated in Fig. 3 in the sector labelled by m. Then

$$I(1) - I(4) = I(2) - I(5) = I(3) - I(6). \tag{2.2}$$

In general, these necessary conditions are not sufficient: it can happen that 1T-

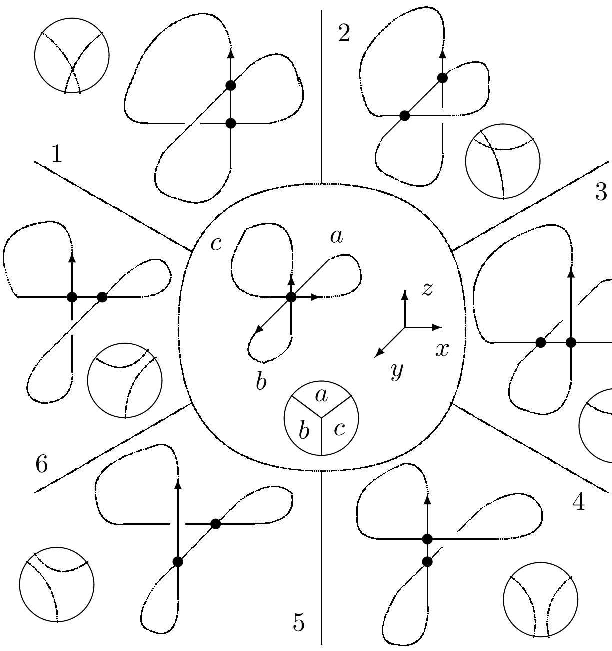


Fig. 3. Resolutions of a triple point

and 4T-relations are satisfied for a function on the set of all A-routes of degree k in M^3, but there is no degree k knot invariant with the principal part equal to this function; see [19]. In the case of $M^3 = M^2 \times \mathbf{R}^1$ the situation is much better.

Proposition 1 (see [2], [1]). *Suppose that $M^3 = M^2 \times \mathbf{R}^1$, M^2 an orientable surface (maybe with boundary), and I_k is a $\mathbf{R}$-valued function on the set of all A-routes of degree k in M^3. If I_k satisfies 1T- and 4T-relations, then there exists a $\mathbf{R}$-valued degree k invariant of knots in M^3, whose principal part coincides with this function I_k.* □

2.1.1. *Example: invariants of degree 0 and 1*

Now we recall the construction of all first degree invariants of knots in a closed orientable 3-manifold M^3, see [19]. In the special case of $M^3 = M^2 \times \mathbf{R}^1$, they constitute a minor extension of the Fiedler's invariants defined in [6], and coincide with them in the most interesting for us case of $M^2 = \mathbf{T}^2$.

By definition, an invariant of degree 0 should take equal values on all knots in M^3 that are homotopic to one another as maps $S^1 \to M^3$. Therefore such invariants can be identified with functions on the group $\pi_1(M^3)$ taking equal values on conjugate elements: $I(a) = I(b)$ if $a = s^{-1}bs$ for some $s \in \pi_1(M^3)$. Below we compare by invariants of positive degrees only the knots that are not separated by degree 0 invariants, i.e. represent one and the same such homotopy class.

Let a be an arbitrary element of the group $\pi_1(M^3)$ and $\bar{a}$ the corresponding homotopy class of maps $S^1 \to M^3$, i.e. the conjugacy class of a in $\pi_1(M^3)$. Let us denote the space of smooth maps $S^1 \to M^3$ representing this class $\bar{a}$ as $\Omega_{\bar{a}}(M^3)$. The discriminant variety $\Sigma \subset \Omega_{\bar{a}}(M^3)$ consists of all maps in this space having self-intersections or singular points. Irreducible components of Σ are in the one-to-one correspondence with decompositions of the element a into the product $a = b \cdot c$ considered up to simultaneous conjugacy: $(b, c) \sim (b', c')$ if there is $s \in \pi_1(M^3)$ such that $b' = s^{-1} \cdot b \cdot s$, $c' = s^{-1} \cdot c \cdot s$. In the special case when the group $\pi_1(M^3)$ is Abelian, these components are counted by the (unordered) decompositions of the class a into the *sums* of two elements $a = b + c$.

Every such irreducible component defines a subvariety of codimension 1 in $\Omega_{\bar{a}}(M^3)$. This subvariety is a cycle (i.e. has no boundary) if and only if none of b and c is the unit element of $\pi_1(M^3)$. The intersection indices of such cycles with 1-homology classes in $\Omega_{\bar{a}}(M^3)$ are well-defined: we represent any such class by a generic smooth closed curve (i.e. by a family of maps $S^1 \to M^3$ parameterized by the points of a circle) and count the intersection points of this curve with Σ, taken with their signs, depending on the orientation of the intersection, see Fig. 1.

If (and only if) this intersection index is equal to zero for all 1-homology classes in $\Omega_{\bar{a}}(M^3)$, then the linking numbers with our component of Σ define a knot invariant in M^3. Namely, we fix a sample knot K_0 in M^3, postulate that all our invariants take zero value on this knot, and define the value of the invariant on any other knot as the intersection number of our component of Σ with any path connecting the sample knot K_0 with the knot in question. It follows immediately from the definitions, that any invariant obtained in this way is of degree 1, and, conversely, all degree 1 invariants of knots in $\Omega_{\bar{a}}(M^3)$, taking zero value on the sample knot, are linear combinations of invariants defined in this way.

The above homological condition (zero intersection indices with all 1-homology classes in $C^\infty(S^1, M^3)$) is not satisfied if M^3 is sufficiently complicated, e. g. for $M^3 = S^2 \times S^1$, see [19]. However, if $M^3 = M^2 \times \mathbf{R}^1$, then all cycles of codimension 1 in $\Omega_{\bar{a}}(M^3)$, defined by irreducible components of Σ, are cohomologous to zero in this sense. This follows formally from Proposition 1, but can also be proved immediately, because the loop space of M^3 in this case is very simple, cf. the proof of Lemma 2 in §4. Therefore the following statement holds.

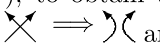
Proposition 2. *For any connected orientable surface M^2, the group of first degree $\mathbf{Z}$-valued invariants of knots in $M^2 \times \mathbf{R}^1$ is free Abelian with canonical generators labelled by pairs of non-unit elements b, c of $\pi_1(M^2)$ considered up to simultaneous conjugacies: $(b, c) \equiv (s^{-1}bs, s^{-1}cs)$ for any $s \in \pi_1(M^2)$.* □

Certainly, any such equivalence class of pairs (b, c) defines correctly an unordered pair of elements $(\bar{b}, \bar{c}) \equiv (\bar{c}, \bar{b})$ in $H_1(M^2)$, $\bar{b} \neq 0 \neq \bar{c}$. This pair of homology classes is called the *passport* of our component of the discriminant. The Fiedler's invariants of [6] are exactly the sums of all invariants from Proposition 2 corresponding to conjugacy classes $\{(b, c)\}$ defining one and the same pair of homology classes $(\bar{b}, \bar{c})$.

If $\pi_1(M^2)$ is Abelian, then any such sum consists of only one summand.

2.2. *Higher degree invariants with non-crossed chord diagrams*

Every unordered collection of $k+1$ non-zero elements of the group $H_1(M^2)$ defines well a degree k invariant of knots in $M^2 \times \mathbf{R}^1$, generalizing the Fiedler's degree 1 invariant from the previous subsection.

Given such a collection Γ of elements $\gamma_0, \dots, \gamma_k \in H_1(M^3) \setminus 0$ for an arbitrary orientable 3-manifold M^3, the corresponding function I_Γ on the space of all A-routes of degree k in M^3 is defined as follows. If the k-chord diagram A has at least one pair of crossing chords (i.e. chords whose four endpoints alternate in S^1, as e.g. in diagrams 2_2, 3_3, 3_4, 3_5 of Fig. 2), then the value of I_Γ on any A-route is equal to 0. If A has no such crossing chords, then for any immersion $f : S^1 \to M^3$, respecting this chord diagram and having no other self-intersections, the variety $f(S^1)$ defines naturally $k+1$ elements of $H_1(M^3)$; to obtain these elements, we smooth any self-intersection of $f(S^1)$ by the rule  and take the classes of $k+1$ separate circles, into which this smoothing splits our curve. The value of the desired function I_Γ on an A-route is equal to 1 (respectively, to 0) if the obtained unordered collection of elements of $H_1(M^3)$ coincides (respectively, does not coincide) with the given collection $(\gamma_0, \dots, \gamma_k)$.

Theorem 4. *For any collection Γ of non-zero elements $\gamma_0, \dots, \gamma_k$ of $H_1(M^3)$, this function I_Γ on the space of A-routes satisfies the 1T- and 4T-relations. In particular, if $M^3 = M^2 \times \mathbf{R}^1$, M^2 orientable, then by Proposition 1 this function I_Γ can be extended to a well-defined degree k invariant of knots in $M^2 \times \mathbf{R}^1$.*

Proof. Consider a generic singular knot $f : S^1 \to M^3$ with one triple point and $k-2$ double points, see Fig. 3. If one of its six decompositions into singular knots with k double points defines a chord diagram without crossing chords, then exactly two other decompositions also have diagrams with this property; in Fig. 3 they are decompositions 4, 5 and 6. The collections of $k+1$ homology classes, corresponding to these three decompositions, also coincide, therefore our function satisfies the 4T-relation. The 1T-relation follows now from the condition that none of elements γ_i is trivial. □

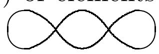
Remark 2. A majority of degree 2 invariants of this series coincides (modulo degree 1 invariants) with certain invariants $I_3^K(a,b)$ from Theorem 2.10 of [7]. Namely, for any fixed homology class $[K]$ of considered knots, $[K] \in H_1(M^2)$, any our weight system $I_{(\gamma_0,\gamma_1,\gamma_2)}$ with $\gamma_0 \neq \gamma_1 \neq \gamma_2 \neq \gamma_0$ and $\gamma_0 + \gamma_1 + \gamma_2 = [K]$ coincides (maybe up to a sign) with principal parts of six Fiedler's invariants: $I_3^K(\gamma_1+\gamma_0,\gamma_0)$, $I_3^K(\gamma_2+\gamma_0,\gamma_0)$, $I_3^K(\gamma_0+\gamma_1,\gamma_1)$, $I_3^K(\gamma_2+\gamma_1,\gamma_1)$, $I_3^K(\gamma_0+\gamma_2,\gamma_2)$, and $I_3^K(\gamma_1+\gamma_2,\gamma_2)$, which, therefore, can be reduced to one another by adding invariants of degree 1 (at least in the case of orientable M^2).

If however some two of our classes γ_i coincide (say, $\gamma_0 = \gamma_2 \neq \gamma_1$), then the corresponding weight system $I_{(\gamma_0,\gamma_1,\gamma_2)}$ coincides with the principal parts of only two

Fiedler's invariants $I_3^K(\gamma_1+\gamma_0,\gamma_0)$ and $I_3^K(\gamma_1+\gamma_2,\gamma_2)$. Thus the only our degree 2 weight systems $I_{(\gamma_0,\gamma_1,\gamma_2)}$ not covered by the Fiedler's invariants correspond to a quite useless case $\gamma_0=\gamma_1=\gamma_2\neq 0$ (implying that $[K]$ is divisible by 3).

It is worth noting also that the Fiedler's invariants $I_3^K(a,b)$ are applicable to knots in non-orientable manifolds of the form $M^2\times\mathbf{R}^1$, in contrast to the above-defined weight systems I_Γ.

Any A-route in M^3 with the non-crossed 2-chord diagram can be represented by the oriented singular knot  embedded somehow into M^3.

Definition 2. The *passport* of an embedded singular knot with two self-intersections and non-crossed chord diagram is the triplet of classes (α,β,γ) of elements of $H_1(M^3)$, β being represented by the image of the middle cycle of  and α and γ by images of two other cycles; the passports (α,β,γ) and (γ,β,α) are considered as identical.

2.3. *Degree 2 invariants defined by crossed diagrams*

Any manifold of the form $M^2\times\mathbf{R}^1$, M^2 orientable, can be embedded into S^3. Therefore any finite degree invariant of knots in S^3 induces an invariant (of the same degree) of knots in $M^2\times\mathbf{R}^1$. Generally, this invariant depends on the embedding. Therefore, in order to compare our knots in $M^2\times\mathbf{R}^1$, we need to fix such an embedding.

In the case $M^2=\mathbf{T}^2$ we identify $\mathbf{T}^2\times\mathbf{R}^1$ with the complement of the Hopf link (i.e. two unknotted linked circles) $C_1\sqcup C_2\subset S^3$ in such a way that

1) any line $x\times\mathbf{R}^1$, $x\in\mathbf{T}^2$, tends to C_1 (respectively, to C_2) when the parameter in $\mathbf{R}^1$ tends to $+\infty$ (i.e., "to the reader" in our pictures) (respectively, to $-\infty$);

2) the horizontal (respectively, vertical) generator of $H_1(\mathbf{T}^2)\sim\mathbf{Z}^2$ in our pictures generates the kernel of the induced homomorphism $H_1(\mathbf{T}^2)\to H_1(S^3\setminus C_2)$ (respectively, $H_1(\mathbf{T}^2)\to H_1(S^3\setminus C_1)$).

Also, we shall assume that the class in $H_1(\mathbf{T}^2\times\mathbf{R}^1)$ of all our knots is equal to the horizontal generator of this group, oriented from the left to the right. In particular, we consider the knots (1.4), (1.5), (1.6) and (1.11) in the form (2.1), and knots (1.7) and (1.8) in the form obtained from the original pictures by the clockwise rotation by $\pi/2$. Immediate calculations give us the following result.

Theorem 5. *The canonical embedding* $\mathbf{T}^2\times\mathbf{R}^1\to S^3$ *transforms the knots* (1.2)–(1.11) (*of which* (1.4), (1.5), (1.6) *and* (1.11) *are in the form* (*2.1*), *and* (1.7) *and* (1.8) *are rotated by* $\pi/2$) *and their mirror images into the knots of following types:*

(1.2), (1.3) *and* (1.4) *to the unknot;*

$\overline{(1.2)}$, (1.9), (1.6), (1.7), $\overline{(1.7)}$, (1.8) *and* $\overline{(1.8)}$ *to* $III_1\#\overline{III_1}$;

$\overline{(1.3)}$ *and* (1.10) *to* $\overline{VI_1}$;

$\overline{(1.4)}$ *and* (1.11) *to* III_1;

(1.5) *and* $\overline{(1.6)}$ *to* $\overline{VIII_{20}}$;

$\overline{(1.5)}$ *to* VI_3. □

Of course, the non-equivalence of types of knots in S^3 obtained from some knots in M^3 by one and the same embedding $M^3 \to S^3$ implies the non-equivalence of these knots in M^3. Unfortunately, for $M^3 = \mathbf{T}^2 \times \mathbf{R}^1$ it generally does not imply the difference of textile structures defined by these knots: indeed, one of these knots can be transformed to a knot equivalent to the other by a non-trivial diffeomorphism $\mathbf{T}^2 \to \mathbf{T}^2$. Therefore, separating textile structures, we usually need to prove something additionally: e.g. that an invariant separating the embedded knots in S^3 takes equal values on all knots obtained from one another by such diffeomorphisms of $\mathbf{T}^2 \times \mathbf{R}^1$. Let us give the first example.

Denote by $I_\oplus$ the basic degree 2 invariant of knots in S^3, defined by the chord diagram $\oplus$, and also the invariant of knots in $\mathbf{T}^2 \times \mathbf{R}^1$ induced from it by our inclusion.

Proposition 3. *Suppose that the knot $K : S^1 \to \mathbf{T}^2 \times \mathbf{R}^1$*

1) maps the fundamental class of S^1 to the homology class $(1, 0) \in H_1(\mathbf{T}^1, \mathbf{Z})$ (i.e. to the class of the horizontal line in our pictures, oriented to the right);

2) is not separated from the unknit (1.1) by any invariants of degree 0 or 1.

Then the invariant $I_\oplus$ takes one and the same value on K and on all knots obtained from K by all diffeomorphisms of $\mathbf{T}^2 \times \mathbf{R}^1$ preserving the direct product structure, orientations of $\mathbf{T}^2$ and $\mathbf{R}^1$, and the homology class of K (i.e. such that the corresponding operator in $H_1(\mathbf{T}^2)$ is equal to $\binom{10}{q1}$, $q \in \mathbf{Z}$).

Proof. Let us join K and the unknit (1.1) by a generic path in $C^\infty(S^1, \mathbf{T}^2 \times \mathbf{R}^1)$. Since K is 1-equivalent to the unknit, this path crosses any irreducible closed component of Σ (see §2.1.1) an even number of times, in such a way that positive crossings can be matched with negative ones. Let us connect any two matched crossings by a generic path inside this component of Σ. This path crosses several times the self-intersection locus of the discriminant. This self-intersection locus consists of maps $S^1 \to \mathbf{T}^2 \times \mathbf{R}^1$ with two self-intersections, and splits naturally into two pieces depending on the 2-chord diagrams defined by the configuration of preimages of these self-intersection points. The value $I_\oplus(K) - I_\oplus(\text{the unknit})$ is equal to the number of all such crossings (in all our paths) corresponding to the crossed chord diagram $\oplus$ and taken with appropriate signs. Let K_1 be a knot obtained from K by some diffeomorphism $A : \mathbf{T}^2 \times \mathbf{R}^1 \to \mathbf{T}^2 \times \mathbf{R}^1$ of the form $\binom{10}{q1} \times \mathrm{Id}$. Then K_1 also satisfies conditions 1) and 2) of our Proposition, and the diffeomorphism A moves the unknit to itself, all these paths into similar paths, and their intersection points with the self-intersection locus of the discriminant into intersection points of the same sign. Therefore $I_\oplus(K) = I_\oplus(K_1)$. $\square$

Corollary 2. *$I_\oplus$ is a well-defined invariant of textile structures defined by a single string in $\mathbf{T}^2 \times \mathbf{R}^1$, such that*

1) its homology class is not equal to a non-zero element of $H_1(\mathbf{T}^2 \times \mathbf{R}^1)$ multiplied by some $a \geq 2$, *2) it cannot be separated from the unknit in the same homology class by invariants of degree 1.* □

Remark 3. If $\pi_1(M^2) \neq 0$ then the invariant $I_{\oplus}$, similarly to other invariants of this origin, can be split into sums of many more specific invariants, in correspondence with the splitting of the set of $\bigoplus$-routes into some equivalence classes. The corresponding equivalence relation is spanned by the following elementary relations: two $\bigoplus$-routes are equivalent if one can approach one and the same singular knot with a generic triple point (see Fig. 3) along both these routes. For example, if the homology class of our knots in M^3 is equal to 0, and rank $H_1(M^3, \mathbf{Z}) \geq 2$, then the area of an oriented triangle in $H_1(M^3, \mathbf{Z})$ spanned by homology classes of three (cyclically ordered) loops of any singular knot with a triple point, obtained by such an approach, is an invariant of this equivalence and separates infinitely many equivalence classes.

2.4. *The case of non-orientable* M^2

The general theory of finite type knot invariants in non-orientable 3-manifolds was developed in [19]. Here we describe only the first degree invariants of knots in $M^2 \times \mathbf{R}^1$, where M^2 is a non-oriented surface.

In the same way as in the orientable case, irreducible components of the discriminant Σ in the space of maps $S^1 \to M^3$ are in one-to-one correspondence with pairs (b, c) of elements of the group $\pi_1(M^3)$, considered up to simultaneous conjugation: $(b, c) \sim (s^{-1}bs, s^{-1}cs)$ for any $s \in \pi_1(M^3)$. Such a component defines a mod 2 cycle (and has no boundary) if and only if both b and c are not equal to the unit element. To define an integral cycle, such a component should have a global transversal orientation at its regular points, i.e. at the points corresponding to maps $f : S^1 \to M^3$ with transverse self-intersections only. Any local branch of this component in a neighborhood of a self-intersection point of the discriminant, i.e. a point corresponding to a map f with two self-intersection points, should be transversally oriented.

Proposition 4 (see [19]). *An irreducible component of* Σ*, defined by the pair of elements* b, c *of* $\pi_1(M^3)$*,* does not *have a global transversal orientation in* $C^\infty(S^1, M^3)$ *if and only if there exists an element* $s \in \pi_1(M^3)$ *violating the orientation of* M^3 *and such that either* $b = s^{-1}bs, c = s^{-1}cs$*, or* $b = s^{-1}cs, c = s^{-1}bs$*.* □

If such an element s does not exist, then our component $\{(b, c)\}$, $b \neq 1 \neq c$, of the discriminant variety defines an integral 1-cohomology class in the space of smooth maps $S^1 \to M^3$. If (and only if) this cohomology class is trivial, then this component defines well a degree 1 invariant of knots in M^3.

3. Proof of Theorems 1 and 2

According to §2.1.1, any basic first degree (Fiedler) invariant of knots in $\mathbf{T}^2 \times \mathbf{R}^1$, corresponding to an irreducible component of the discriminant, can be described by its passport, i.e. an unordered pair of pairs of integers,

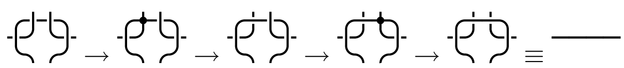
$$((m_1, n_1)(m_2, n_2)) \cong ((m_2, n_2)(m_1, n_1)), \tag{3.1}$$

$m_1^2 + n_1^2 \neq 0$, $m_2^2 + n_2^2 \neq 0$. Here (m_1, n_1) and (m_2, n_2) are elements of $\mathbf{Z}^2 \equiv H_1(\mathbf{T}^2)$ represented by two loops into which the intersection point splits any generic singular knot corresponding to a point at this component of the discriminant.

The invariant (3.1) takes non-trivial values only on the knots representing the homology class $(m_1 + m_2, n_1 + n_2)$. For any such homology class we choose a sample knot representing it, and postulate that all our invariants take zero value on this knot; everywhere below this is a knot with the non-crossed diagram. The value of our invariant (3.1) on any different knot is equal to the intersection number of our component of Σ with the path in $C^\infty(S^1, M^3)$ connecting the sample knot with the knot in question. For all knots (1.1)–(1.11) their homology classes are not multiples of other elements in $H_1(\mathbf{T}^2)$, therefore all sample knots can (and will) be chosen as knots whose diagrams have no crossing points in $\mathbf{T}^2$; all these sample knots are $SL(2, \mathbf{Z})$-equivalent to each other and to the knot (1.1).

3.1. *Non-triviality of the single jersey and its mirror image*

Single jersey (1.2) can be transformed into the trivial $(1, 0)$-structure by the sequence of two surgeries:


(3.2)

Passports of the first and the second surgeries in this path are equal to

$$((0, 1)(1, -1)) \quad \text{and} \quad ((0, -1)(1, 1)) \tag{3.3}$$

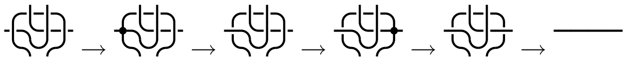
respectively, in particular they define different Fiedler invariants. The signs of these surgeries are equal to + and −, respectively.

On the other hand, structure $\overline{(1.2)}$ can be transformed into the trivial one by a similar chain of surgeries consisting of mirror images of corresponding elements of (3.2). The passports of these surgeries are respectively the same, but the signs are opposite. Therefore we get

Proposition 5. *Three structures* (1.1), (1.2) *and* $\overline{(1.2)}$ *can be separated from one another by either of two first degree invariants defined by the passports* (3.3). □

3.2. *Plain knit with closed loops is 1-separated from the unknit and 2-separated from the plain knit with open loops*

Let us transform the knot in question (1.3) into the unknit by the sequence of two surgeries


(3.4)

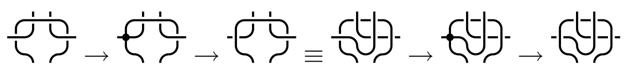
The signs of these surgeries are equal to $-$ and $+$ respectively; their passports are

$$((0,1)(1,-1)) \text{ and } ((0,-1)(1,1)). \tag{3.5}$$

Considering additionally the sequence consisting of knots which are mirror opposite to elements of (3.4), and comparing formulae (3.5) and (3.3), we obtain the following fact.

Proposition 6. *The plain knit with closed loops* (1.3) *is separated from its mirror opposite* $\overline{(1.3)}$ *and the unknit by either of two first degree invariants defined by passports* (3.5). *On the other hand, this structure* (1.3) *cannot be separated by first degree invariants from* $\overline{(1.2)}$, *and* $\overline{(1.3)}$ *from* (1.2). □

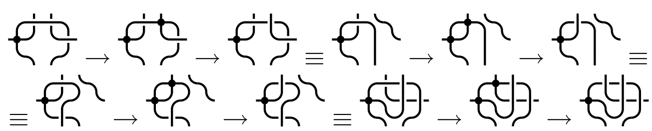
Now, let us try to separate the latter two pairs of structures by the second degree invariants. A homotopy connecting the structures $\overline{(1.2)}$ and (1.3) can be chosen in the following way:



(3.6)

It contains two surgeries with signs equal to $-$ and $+$ and passports equal to $((0,1)(1,-1))$; this proves once more that these two structures cannot be separated by first degree invariants.

To calculate the difference of values of a degree 2 invariant on the knots $\overline{(1.2)}$ and (1.3), we consider the same sequence (3.6). However, now these two surgeries should be taken not only with (the same) signs $-$ and $+$, but also with certain weights determined by our second degree invariant. Only the difference of these two weights is important for our calculation; this difference can be found as follows. Let us join these two surgeries by a generic path inside the discriminant as follows:



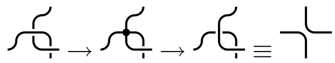
(3.7)

This path contains several surgeries of second order, representing some A-routes of degree 2. The desired difference of weights is equal to the sum of values of the principal part of our invariant on these A-routes, taken with signs $+$ or $-$ depending on the directions of intersection. In our case, the second and the fourth surgeries of (3.7) have crossed diagrams and signs equal to $-$. The first and the third surgery have non-crossed diagrams, signs equal to $+$, and passports equal to $((0,1)(1,0)(0,-1))$ and $((0,1)(0,0)(1,-1))$ respectively. Therefore we obtain the following fact.

Proposition 7. *The first and the last structures of the sequence* (3.6) *are separated by both the invariant* $I_\oplus$ *corresponding to the crossed 2-diagram (see subsection 2.3) and by the second degree invariant with the non-crossed chord diagram and passport* $((0,1)(1,0)(0,-1))$. □

3.3. *Wire netting and its mirror image are 1-separated from the unknit, single jersey, and single jersey with closed loops*

The sequence



(3.8)

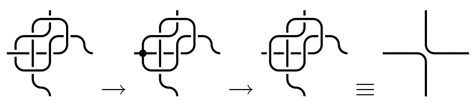
links the wire netting (1.4) with the unknit and experiences exactly one crossing of the discriminant, with passport equal to $((0,-1)(1,0))$ and the sign $+$. The mirror image $\overline{(1.4)}$ of the wire netting can be linked to the unknit by a similar surgery with the same passport and the different sign. Therefore all these three knots are separated from each other by the invariant of degree 1 corresponding to this passport.

Moreover, we see that the structures (1.4) and $\overline{(1.4)}$ are separated from the unknit by only one basic invariant of degree 1, while any of structures (1.2), $\overline{(1.2)}$, (1.3) and $\overline{(1.3)}$ is separated from the unknit by two such basic invariants. These properties do not depend on the choice of the rectangular chart in $\mathbf{T}^2$, and we get the following proposition.

Proposition 8. *The structures* (1.4) *and* $\overline{(1.4)}$ *are separated from one another and from the structures* (1.1)*,* (1.2)*,* $\overline{(1.2)}$*,* (1.3) *and* $\overline{(1.3)}$ *by degree 1 invariants.* □

3.4. *Weaver's knot is 1-nontrivial; weaver's knot is 1-equivalent but not 2-equivalent to the wire netting*

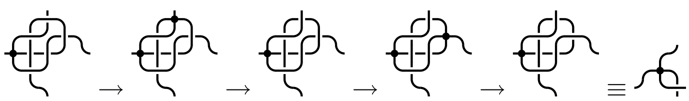
This knot (1.5) can be linked to the trivial one by the surgery



(3.9)

The passport of this surgery is equal to $((0,-1)(1,0))$, and its sign is $+$. Therefore the corresponding Fiedler's invariant separates the weaver's knot (1.5) from the unknit. This invariant does not separate the knot (1.5) from the wire netting (1.4), see §3.3, but fixes the rectangular chart (1.5) as the unique chart in which this knot is 1-equivalent to the wire netting given by the picture (1.4): indeed, any non-trivial $SL(2,\mathbf{Z})$-transformation does not preserve the invariant $((0,-1)(1,0))$.

Let us try to separate (1.5) and (1.4) by second degree invariants. For the path connecting these two structures we take the composition of the path (3.9) and the path (3.8) passed in the opposite direction. This path contains two surgeries. To calculate the difference of their weights in the calculation of degree 2 invariants, we connect them by a generic path inside the discriminant:



(3.10)

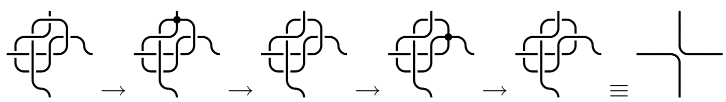
This path contains two second degree surgeries, the first of them with non-crossed chord diagram and passport $((0,-1)(0,0)(1,0))$, and the second with the crossed chord diagram and sign $+$. Therefore we have the following

Proposition 9. *The weaver's knot* (1.5) *and the wire netting* (1.4) *are not separated by degree 1 invariants. They are separated by the invariant* $I_\oplus$*, see §2.3.* □

The mirror images of all these transformations yield the same statement concerning the comparison of $\overline{(1.5)}$ and $\overline{(1.4)}$.

3.5. *Fake weaver's knot is 1-equivalent but not 2-equivalent to the unknit*

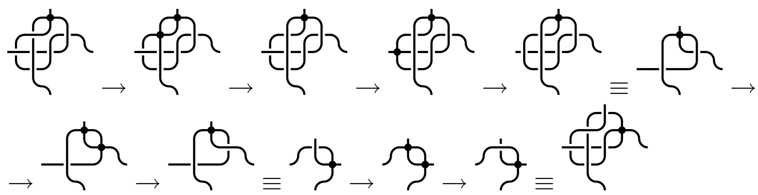
The degree 0 invariant of the structure (1.6) (i.e. the class of the corresponding oriented knot in the group $H_1(\mathbf{T}^2)$) is equal to $(1,-1)$. Therefore we need to compare this structure with the standard unknit in the same class. This unknit is indicated in the right-hand part of the following expression (3.11), representing a path between the structure (1.6) and the trivial knot:



(3.11)

Passports of two surgeries in (3.11) are both equal to $((0,-1)(1,0))$, their signs are equal to $-$ and $+$ respectively. Therefore the fake weaver's knot (1.6) and the unknit (1.1) cannot be separated by invariants of first degree.

Now, let us prove that they can be separated by an invariant of degree 2. To do this, let us connect two singular knots occurring in (3.11) as surgery points, by a generic path inside the discriminant:



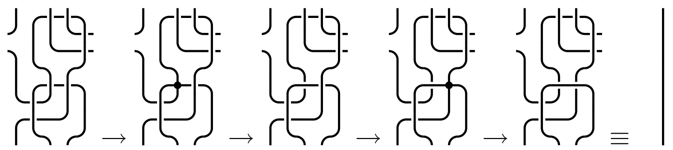
(3.12)

The first and the last surgeries have crossed chord diagrams with signs $-$. Two other surgeries have non-crossed 2-chord diagrams, both with passports equal to $((1,0)(0,0)(0,-1))$. In the same way as in §3.2, this implies the following statement.

Proposition 10. *The invariant* $I_\oplus$ *considered in §2.3 takes value* $+2$ *on the fake weaver's knot* (1.6)*. All basic degree two invariants with the non-crossed chord diagrams (see §2.2) take zero value on this knot.* □

3.6. *Tricot with open loops is 1-equivalent to the unknit but not 2-equivalent to the unknit and to fake weaver's knot*

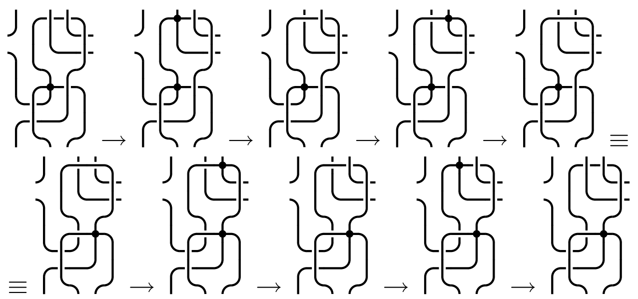
The degree zero invariant of structure (1.7) is equal to $(0, 1) \in H_1(\mathbf{T}^2)$, thus it is necessary to distinguish this structure from the sample knot within the same homology class; this knot is shown on the right-hand side of the next sequence.



(3.13)

This sequence consists of two surgeries connecting the structure (1.7) with the basic unknit. Passports of these two surgeries are both equal to $((-1, 1)(1, 0))$; their signs are equal to $-$ and $+$ respectively. Therefore tricot (1.7) cannot be separated from the unknit by the first degree invariants.

Now, let us prove that these two knots can be separated by a degree 2 invariant. To do this, we connect two singular knots, occurring in (3.13) at surgery points, by a generic path inside the discriminant:



(3.14)

The first and third surgeries have crossed chord diagrams, the second and the fourth surgeries have non-crossed chord diagrams with passports

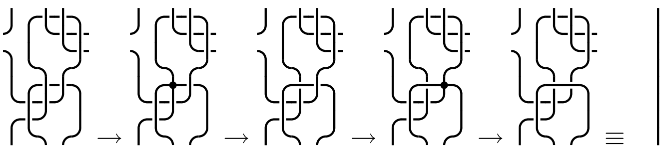
$$((1, 1)(0, -1)(-1, 1)) \text{ and } ((1, 0)(0, 1)(-1, 0)) \tag{3.15}$$

respectively. The signs of all these four surgeries are equal to $-$, $+$, $-$, and $+$. Therefore we get the following statement.

Proposition 11. *Tricot with open loops* (1.7) *is 1-equivalent to the unknit* (1.1)*. Both degree two invariants corresponding to non-crossed chord diagrams and passports* (3.15) *take value* -1 *on the knot* (1.7)*, while no other basic invariants with non-crossed 2-chord diagrams take non-zero values on this knot. The invariant* $I_\oplus$ *(see subsection 2.3) takes value* $+2$ *on this knot. In particular,* (1.7) *is separated from the unknit by any of these three invariants, and from the fake weaver's knot* (1.6) *by any of two invariants with non-crossed chord diagrams and passports* (3.15)*.*
□

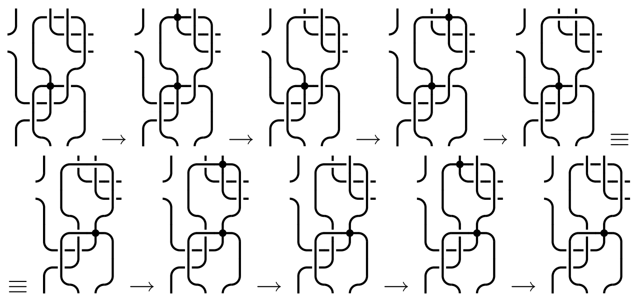
3.7. *Tricot with closed loops is 2-equivalent to tricot with open loops*

Tricot with closed loops can be transformed into the unknit by two surgeries:



(3.16)

Passports of these two moves are both equal to $((-1,1)(1,0))$, their signs are equal to $-$ and $+$ respectively. Therefore the structure (1.8) cannot be separated from the unknit by invariants of degree 1. To separate them by degree 2 invariants, let us connect two singular knots, occurring in (3.16) at surgery points, by a generic path inside the discriminant:



(3.17)

These four surgeries of second order are as follows. The first and the third surgery have crossed chord diagrams, while the other two have non-crossed diagrams. The passports of the second and the fourth surgeries are equal to $(1,0)(0,1)(-1,0)$ and $(1,1)(0,-1)(-1,1)$ respectively. The signs of these four surgeries are equal to $-$, $+$, $-$, and $+$. Therefore we obtain the following proposition.

Proposition 12. *Tricot with closed loops* (1.8) *is 1-equivalent to the unknit* (1.1)*. Both degree two invariants corresponding to non-crossed chord diagrams and passports* (3.15) *take value* -1 *on the knot* (1.8)*, while no other basic invariants with non-crossed 2-chord diagrams take non-zero values on this knot. The invariant* $I_\oplus$ *(see subsection 2.3) takes value* $+2$ *on this knot* (1.8)*. In particular,* (1.8) *is separated from the unknit by any of these three invariants, and from the fake weaver's knot* (1.6) *by any of these two invariants with non-crossed chord diagrams and passports* (3.15)*.* □

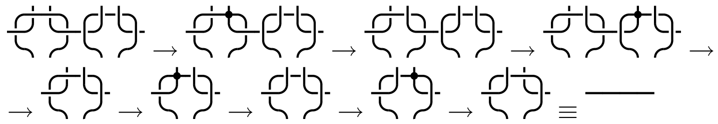
Comparing (3.14) and (3.17), we obtain

Proposition 13. *Tricot with open loops and tricot with closed loops are not separated by the first and second degree invariants.*

Proof. The assertion concerning first degree invariants follows from the first statements of propositions 11 and 12. Further, there is a one-to-one correspondence between the surgeries in (3.14) and (3.17) that preserves both the homotopy types of the corresponding immersed loops with two self-intersections in $\mathbf{T}^2 \times \mathbf{R}^1$ and the signs of these surgeries. Therefore the principal parts of all second degree invariants take equal values on these surgeries corresponding to each other, and provide equal contributions to the values of these invariants at the knots (1.7) and (1.8). □

3.8. *Second degree invariants for 1+1 rib with open loops*

We can connect this knot with the unknit by the path



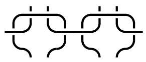
(3.18)

Passports of these moves are equal to

$$((0,-1)(1,1)),\ ((0,1)(1,-1)),\ ((0,1)(1,-1)),\ \text{ and }\ ((0,-1)(1,1)) \qquad (3.19)$$

respectively; their signs are equal to $+,+,-,$ and $-$. Therefore 1+1 rib cannot be distinguished from the unknit shown at the end of (3.18) by invariants of first degree. This gives us the following statement.

Proposition 14. *1+1 rib is 1-equivalent to the unknit, and is not 1-equivalent to the horizontally doubled plain knit*

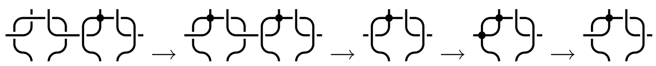


; (3.20)

the latter non-equivalence can be recognized by first degree invariants.

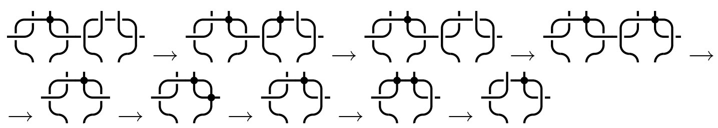
The last assertion of this proposition follows from the calculation of §3.1 repeated twice; this calculation shows us that the values of degree 1 invariants with passports $(0,1)(1,-1))$ and $((0,-1)(1,1))$ on the knot (3.20) are equal to 2 and -2, respectively. □

Now, let us calculate the values of second degree invariants on 1+1 rib knot. To do this, we connect the second surgery in (3.18) with the third one, and the first with the fourth one by generic paths inside the discriminant, see sequences (3.21) and (3.22), respectively.



(3.21)

The first surgery of (3.21) has the non-crossed chord diagram, sign $-$ and passport $((0,1)(1,-2)(0,1))$. The second surgery has the crossed chord diagram and sign $+$.



(3.22)

The first and the last surgeries in (3.22) have non-crossed chord diagrams, passports equal to $((0,-1)(1,0)(0,1))$, but different signs (equal to $+$ and $-$ respectively). The second surgery also has a non-crossed chord diagram; its passport is equal to $((0,-1)(1,2)(0,-1))$ and the sign is equal to $-$. The third surgery has a crossed chord diagram and sign $+$. Thus, in total we have in (3.21) and (3.22) six singular knots with two self-intersections. Considering all corresponding surgeries, we arrive at the following statement.

Proposition 15. *Both degree two invariants defined by non-crossed chord diagrams and passports* $((0,1)(1,-2)(0,1))$ *and* $((0,-1)(1,2)(0,-1))$ *take on the knot* (1.9) *values equal to* -1. *All other basic invariants with non-crossed 2-chord diagrams take zero value on this knot. The invariant* $I_\oplus$ *takes value* $+2$ *on the same knot.* □

Corollary 3. *Any of three invariants indicated in Proposition 15 separates the knot* (1.9) *from the unknit* (1.1). *Both invariants of degree 2, corresponding to non-crossed 2-chord diagrams, indicated in the same proposition, separate* (1.9) *from the fake weaver's knot* (1.6) *and from tricots* (1.7) *and* (1.8).

Proof. The assertions concerning separation of (1.9) from (1.1) and (1.6) are obvious, because all these invariants take zero value on the unknit, and all degree 2 invariants with non-crossed chord diagrams take zero value on the fake weaver's knot. To prove the separation of (1.9) from (1.7), we need to compare the lists of all basic invariants with non-crossed 2-chord diagrams, taking value -1 on the knots (1.9) and (1.7). Namely, these lists consist of invariants with passports $((0,1)(1,-2)(0,1))$ and $((0,-1)(1,2)(0,-1))$ in one case, and invariants with passports $((1,1)(0,-1)(-1,1))$ and $((1,0)(0,1)(-1,0))$ in the other. Since the sample knots (1.7) and (1.9) were considered in the charts where they have homology classes $(0,1)$ and $(1,0)$ respectively, we need to prove the following statement.

Lemma 1. *There does not exist an element of the group* $SL_2(\mathbf{Z})$ *that*

a) moves vector $(0,1)$ *to* $(1,0)$ *and*

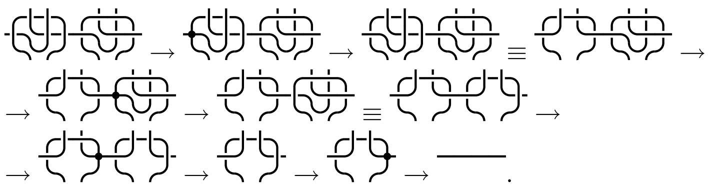
b) either moves the non-ordered triplet $((1,1)(0,-1)(-1,1))$ *of elements of* $\mathbf{Z}^2$ *to* $((0,1)(1,-2)(0,1))$ *and* $((1,0)(0,1)(-1,0))$ *to* $((0,-1)(1,2)(0,-1))$*, or moves the triplet* $((1,1)(0,-1)(-1,1))$ *to* $((0,-1)(1,2)(0,-1))$ *and* $((1,0)(0,1)(-1,0))$ *to* $((0,1)(1,-2)(0,1))$.

This lemma follows easily from the fact that both target triplets do contain coinciding elements, and the source triplets do not. □

Finally, the last assertion of Corollary 3, concerning non-equivalence of (1.9) and (1.8), follows from the 2-equivalence of (1.7) and (1.8), see Proposition 13. □

3.9. *Second degree invariants for 1+1 rib with closed loops*

Let us connect this structure with the unknit by the path



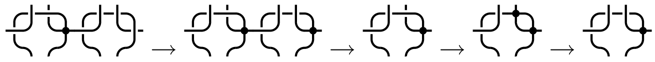
(3.23)

The signs of these four surgeries are equal to $-$, $+$, $+$ and $-$ respectively; their passports are equal to

$$((0,1)(1,-1));\ ((0,1)(1,-1));\ ((0,-1)(1,1))\ \text{ and }\ ((0,-1)(1,1)). \qquad (3.24)$$

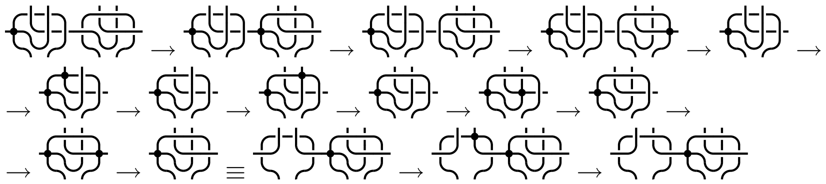
Therefore 1+1 rib with closed loops cannot be distinguished from the unknit by invariants of degree 1.

Now, let us connect the third surgery in (3.23) with the fourth one (see (3.25)), and the first with the second one (3.26).



(3.25)

This path contains two surgeries of second order. The first of them has the non-crossed 2-chord diagram, the passport equal to $(0,-1)(1,2)(0,-1))$, and the sign equal to $-$; the second surgery has the crossed chord diagram and sign equal to $+$.



(3.26)

This path contains seven surgeries. Three of them, namely, the third, the fifth, and the sixth, correspond to crossed 2-chord diagrams, the signs of all of them are equal to $+$. The first, the second, the fourth, and the seventh surgeries have non-crossed chord diagrams; their passports are equal respectively to $((0,1)(1,-2)(1,0))$, $((0,1)(1,0)(1,-1))$, $((1,0)(0,0)(1,-1))$, and $((0,-1)(1,0)(0,1))$; their signs are equal to $+$, $-$, $-$, and $-$. Considering all nine surgeries of (3.25) and (3.26) gives us the following result.

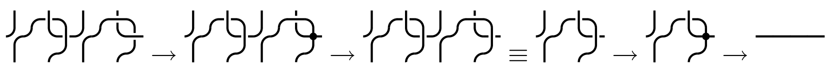
Proposition 16. *1+1 rib knot with closed loops* (1.10) *cannot be distinguished from the unknit by invariants of degree 1. The invariants of degree 2 defined by non-crossed chord diagrams and passports* $((0,1)(1,0)(0,-1)$, $((0,-1)(1,2)(0,-1))$, *and* $((0,1)(1,-2)(0,1))$ *take on this knot values equal to* 2, -1 *and* -1 *respectively. All other basic invariants of degree* 2 *with non-crossed chord diagrams take zero value on this knot. The invariant* $I_{\oplus}$*, defined by the crossed chord diagram, takes value* -2 *on the same knot.* □

Corollary 4. *The 1+1 rib structure with closed loops* (1.10) *is not* 2*-equivalent to any of the knots* (1.1)*,* (1.6)*,* (1.7)*,* (1.8)*, and* (1.9)*.*

This statement follows immediately from the comparison of values of second degree invariants on all these knots. □

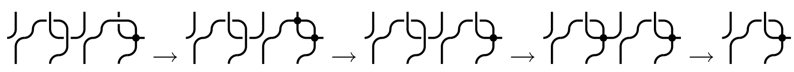
3.10. *Second degree invariants for 1+1 wire netting*

It will be convenient for us to consider the 1+1 wire netting structure in the chart in $\mathbf{T}^2$, in which it will have the form shown in (2.1) on the right. Consider the following path connecting this structure with the unknit:



(3.27)

This path intersects the discriminant twice with signs + and − respectively; passports of both intersections are equal to $((0,-1)(1,1))$. Therefore 1+1 wire netting is 1-equivalent to the unknit (1.1). Further, let us join the corresponding singular knots by a path inside the discriminant:



(3.28)

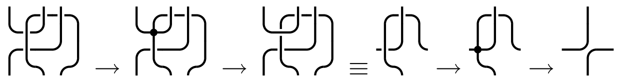
This path traverses twice the self-intersection locus of the discriminant. The first traversing point has the crossed chord diagram $\bigoplus$ and sign +; the second one has the non-crossed 2-chord diagram, passport $((0,-1)(1,2)(0,-1))$, and sign −. Therefore we have the following statement.

Proposition 17. *The 1+1 wire netting* (1.11) *is 1-equivalent to the unknit* (1.1)*. The invariant $I_\oplus$ takes value* +1 *on* (1.11)*. The degree 2 invariant with non-crossed chord diagram and passport* $((0,-1)(1,2)(0,-1))$ *takes value* −1 *on* (1.11)*. All other basic degree 2 invariants with non-crossed chord diagram take zero value on* (1.11)*. In particular, the invariant $I_\oplus$ separates this structure* (1.11) *from any of knots* (1.1)*,* (1.6)—(1.10) *and their mirror images.* □

4. Proof of Theorem 3

The group $\pi_1(\mathbf{K}^2)$ is generated by two elements a, b with unique basic relation $a = bab$, in particular any of its elements can be reduced to the normal form $a^p b^q$, where p and q are integers. An element of this group, represented by a word in the letters a, b, a^{-1} and b^{-1}, violates the orientation of $\mathbf{K}^2$ if and only if the total number of letters a and a^{-1} in the word is odd. In our pictures (1.12), (1.13) and all pictures of the present section, we choose the left-hand bottom corner of the picture frame for the basepoint; generators a and b of $\pi_1(\mathbf{K}^2)$ are represented by the vertical and horizontal boundary segments originating from this corner and oriented up and to the right, respectively. The group $H_1(\mathbf{K}^2, \mathbf{Z})$ is equal to $\mathbf{Z} \oplus \mathbf{Z}_2$, its free part is generated by the class $\{a\}$ of the loop a.

Let us connect the structure (1.12) in $\mathbf{K}^2 \times \mathbf{R}^1$ with the unknit by the sequence



(4.1)

This sequence contains two surgeries; their passports are equal to

$$(ba^{-1}, aba) \text{ and } (ab^{-1}, b) \tag{4.2}$$

Their homology classes are different; in particular, they define different irreducible components of the discriminant variety.

It follows from Proposition 4, that both of these discriminant components have global transversal orientations. Indeed, the simultaneous conjugation with any element of $\pi_1(\mathbf{K}^2)$ violating the orientation of $\mathbf{K}^2$ cannot preserve the passport (ab^{-1}, b), because the conjugation with a word in letters a, b, a^{-1}, b^{-1} sends b to $(-1)^k b$, where k is the total number of letters a and a^{-1} in this word, and this total number should be odd. Also, the simultaneous conjugation with such a word cannot permute the words ab^{-1} and b because it should preserve the corresponding homology classes in $H_1(\mathbf{K}^2)$. The proof for the passport (ba^{-1}, aba) is the same.

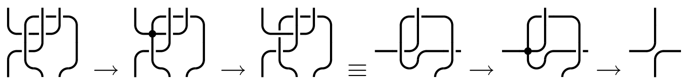
Further, we need to prove the homological condition that the 1-cohomology classes in $C^\infty(S^1, \mathbf{K}^2 \times \mathbf{R}^1)$, defined by intersection indices with these (arbitrarily oriented) components of the discriminant are equal to 0, see §2.4. This condition follows immediately from the following lemma. For any $h \in \pi_1(M^3)$, let $\Omega_h(M^3)$ be the connected component of $C^\infty(S^1, M^3)$ consisting of maps sending the fundamental cycle of S^1 to a loop homotopic to h.

Lemma 2. *Any element of $H_1(\Omega_a(\mathbf{K}^2 \times \mathbf{R}^1))$ can be represented by a 1-cycle not meeting the discriminant subvariety.*

Proof. Let S^1_1 and S^1_2 be two circles with coordinates x and $\lambda \in \mathbf{R}/2\pi\mathbf{Z}$. Any 1-homology class in $\Omega_a(M^3)$, $M^3 = \mathbf{K}^2 \times \mathbf{R}^1$, can be represented by a smooth map $\Theta : S^1_1 \times S^1_2 \to M^3$ considered as a family of maps $\theta_\lambda : S^1_1 \to M^3$ depending smoothly on the parameter $\lambda \in S^1_2$, where $\Theta(0,0)$ is the basepoint in M^3, and the homotopy class of the loop $\Theta(\cdot, 0) : S^1_1 \to M^3$ in $\pi_1(M^3)$ is equal to a. We can assume that this loop is a smooth knot in M^3. The class of the loop $\Theta(0, \cdot) : S^1_2 \to M^3$ in $\pi_1(M^3)$ should commute with the element a. It is easy to calculate that it is possible only if this class is equal to a^k for some integer k. There is an obvious map $\Theta : S^1_1 \times S^1_2 \to M^3$ with such homotopy classes of restrictions on coordinate circles S^1_1, S^1_2: it is the family of maps θ_λ obtained from one another by sliding along the source circle: $\theta_\lambda(x) \equiv \theta_0(x - k\lambda)$. On the other hand, all maps $S^1 \times S^1 \to \mathbf{K}^2 \times \mathbf{R}^1$ with homotopic restrictions on the coordinate cross $(S^1 \times 0) \cup (0 \times S^1)$ are homotopic to one another since $\pi_2(\mathbf{K}^2) = 0$. □

Therefore both components of the discriminant with passports (4.2) define dual knot invariants of degree 1. These invariants separate the structure (1.12) from the trivial knot shown on the left of (4.1), and also from the mirror image $\overline{(1.12)}$ of this structure.

In a similar way, we connect the structure (1.13) with the trivial one by the path



(4.3)

These two surgeries have the same passports (4.2) as the surgeries of (4.1), hence the structure (1.13) is separated from the trivial knot and from $\overline{(1.13)}$ by the same two independent degree 1 knot invariants.

It is easy to see that the first (respectively, the second) surgeries in (4.1) and (4.3) cross these components of Σ in different directions, therefore the corresponding invariants separate also (1.12) from (1.13) and $\overline{(1.12)}$ from $\overline{(1.13)}$, but do not separate (1.12) from $\overline{(1.13)}$ or $\overline{(1.12)}$ from (1.13).

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