Fiedler type combinatorial formulas for generalized Fiedler type invariants of knots in $M^2 \times \mathbf{R}^1$

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Abstract

We construct combinatorial formulas of Fiedler type (i.e. composed of oriented Gauss diagrams arranged by homotopy classes of loops in the base manifold, see [4], [7]) for an infinite family of finite type invariants of knots in $M^2 \times \mathbf{R}^1$ (M^2 orientable), introduced in [5].

Keywords: knot invariant, weight system, Gauss diagram, combinatorial formula

1 Introduction

In [5], we have constructed an infinite family of weight systems (i.e. potential knot invariants) for knots in an arbitrary oriented manifold M^3 with non-trivial first homology group. Namely, any unordered collection $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{k+1}\}$, $\gamma_i \in H_1(M^3)\setminus\{0\}$, of $k+1$ non-zero 1-homology classes defines a weight system I_Γ of degree k for knots in M^3. All these weight systems defined by different collections Γ are linearly independent. By a theorem of [1], in the case $M^3 = M^2 \times \mathbf{R}^1$ any weight system can be integrated to a knot invariant of the same degree. In particular, these weight systems I_Γ define implicitly (i.e. up to the choice of this integration) a series of finite type invariants of knots in such manifolds. In the case of degrees $k = 1$ and 2 such invariants (with a minor exception for $k = 2$) were previously explicitly constructed by T. Fiedler [3], [4] in the terms of Gauss diagrams arranged by 1-homology classes of M^2. In fact, in both constructions instead of 1-homology classes γ_i we can use the *loop homotopy classes* of M^3, i.e. the elements of the set $h_1(M^3)$ of conjugacy classes in $\pi_1(M^3)$. This provides us with a more ample family of weight systems and invariants if $\pi_1(M^3)$ is non-commutative. Below, we generalize the Fiedler's construction to arbitrary degree k and obtain a large family of explicitly defined and easily calculable invariants $\Phi_{\mathbf{\Gamma}}$ of knots in all manifolds $M^3 = M^2 \times \mathbf{R}^1$, M^2 orientable with $\pi_1(M^2) \neq \{1\}$, whose principal parts coincide with appropriate weight systems I_Γ. These invariants $\Phi_{\mathbf{\Gamma}}$ are parameterized by **ordered** collections $\mathbf{\Gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{k+1})$ of classes $\gamma_i \in h_1(M^3)\setminus\{1\}$, no more than two of which can coincide. If all these classes are different, then their ordering in $\mathbf{\Gamma}$ is arbitrary, and if some two of them coincide then they should be the first and the last elements of the collection $\mathbf{\Gamma}$. All $(k+1)!$ (in the first case) or $(k-1)!$ (in the second one) invariants $\Phi_{\mathbf{\Gamma}'}$, where $\mathbf{\Gamma}'$ is obtained from $\mathbf{\Gamma}$ by some permissible permutation of elements, coincide with $\Phi_{\mathbf{\Gamma}}$ up to invariants of degree lover than k: the principal parts of all these invariants coincide with one another and with the weight system I_Γ, where Γ is obtained from $\mathbf{\Gamma}$ by forgetting the order of its elements.

In §2 we define functions $\Phi_{\mathbf{\Gamma}}$ on knots in $M^2 \times \mathbf{R}^1$ with generic projections to M^2, in §3 we prove their invariance under Reidemeister moves, and in §4 we present two knots in

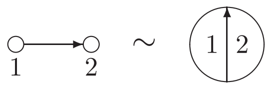


Fig. 1: Universal planar chain of degree 1

$\mathbf{T}^2 \times \mathbf{R}^1$ not separated by any finite type invariants of degree ≤ 2 but separated by some our invariant $\Phi_{\mathbf{\Gamma}}$ of degree 3.

2 Basic construction and main result

Recall that a k-*chord diagram* consists of an oriented circle (which we shall identify with the unit circle in $\mathbf{R}^2$ oriented counterclockwise) and k chords of this circle, all whose $2k$ endpoints are different. A k-*arrow diagram* is a k-chord diagram, all whose chords are oriented, see [7]. Two chord or arrow diagrams are *equivalent*, if they can be transformed one into the other by an orientation-preserving homeomorphism of the circle. We shall consider only *planar* chord and arrow diagrams, i.e. those with non-intersecting chords. Obviously, any planar k-chord diagram cuts the unit disc into $k+1$ convex domains. A *marking* of a planar chord (or arrow) diagram is any labelling of these domains by numbers $1, \ldots, k+1$. Marked k-chord (respectively, k-arrow) diagrams, considered up to equivalence, are in the obvious one-to-one correspondence with equivalence classes of non-oriented (respectively, oriented) planar trees with $k+1$ vertices marked by the same numbers and considered up to plane isotopies. Specifically, given a planar marked arrow diagram, we can choose for the ith vertex of the corresponding tree an arbitrary point in the ith domain of the unit disc; if two domains with numbers i and j are separated by a single arrow, then we connect the vertices i and j by a segment oriented in such a way that the pair of orientations of the segment and the arrow defines the standard orientation of the plane.

Definition 1 The *universal degree k planar chain* is either

a) the formal sum of all equivalence classes (up to planar isotopies) of marked planar trees with $k+1$ vertices, supplied with the canonical orientation of all edges, from the smaller number to the greater one, or

b) the formal sum of equivalence classes of all marked k-arrow diagrams, corresponding to the summands of (a) by the standard correspondence.

For example, the universal degree k planar chains with $k = 1, 2$ and 3 are shown in Figs. 1, 2 and 3 respectively. As S. Lando has explained to us, for any natural k the number $\nu(k)$ of summands of the universal degree k chain is equal to $(2k-1)!/k!$.

For any connected surface M^2, denote by $h_1(M^2)$ the set of conjugacy classes of $\pi_1(M^2)$, i.e. of free (non-punctured) homotopy classes of loops $S^1 \to M^2$; denote by $\mathbf{1} \in h_1(M^2)$ the class of constant loops.

Definition 2 Given any ordered collection $\mathbf{\Gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{k+1}), \gamma_i \in h_1(M^2)$ (some of which can coincide), the chain $\Phi_{\mathbf{\Gamma}}$ is defined as the formal sum of all planar k-arrow diagrams, whose $k+1$ domains are labelled by these elements γ_i: we take the universal degree k planar chain and in any its summand replace any label i by γ_i.

Definition 3 The ordered collection $\mathbf{\Gamma}$ is *unambiguous*, if all $\nu(k)$ summands of the chain $\Phi_{\mathbf{\Gamma}}$ are different.

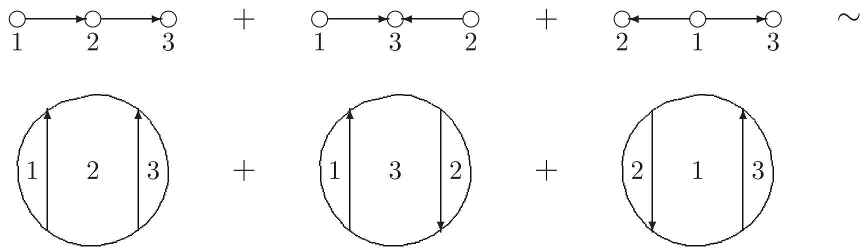


Fig. 2: Universal planar chain of degree 2

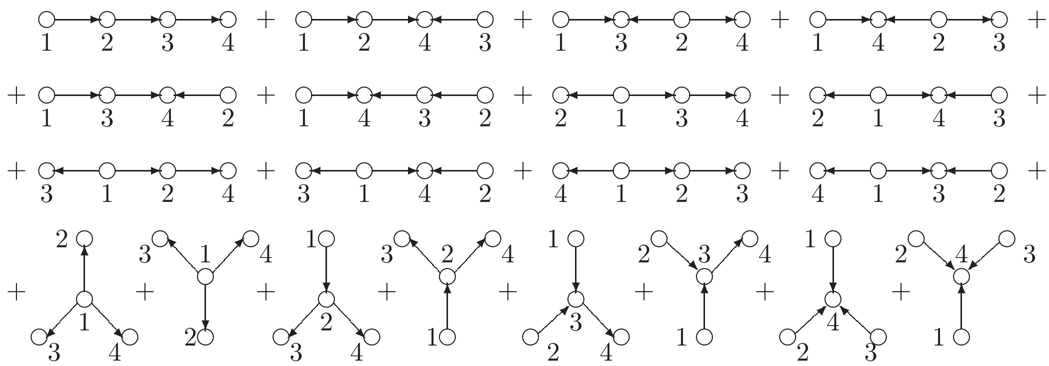


Fig. 3: Universal planar chain of degree 3 (graph presentation only)

Lemma 1 $\mathbf{\Gamma}$ *is an unambiguous ordered collection if and only if one of three conditions is satisfied:*

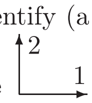
a) all its elements γ_i *are different* (*and the ordering is arbitrary*)*;*

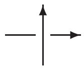
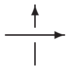
b) some two of these elements coincide, namely the last and the first one: $\gamma_1 = \gamma_{k+1}$ (*and the remaining* $k-1$ *elements are again ordered arbitrarily*)*;*

c) $k = 2$.

Proof. The part "if" of the proposition can be checked immediately, let us prove "only if". Suppose that there are numbers $i, j, m \in \{1, 2, \dots, k+1\}$ such that $\gamma_i = \gamma_j$ and either $i > m < j$ or $i < m > j$. Consider a marked planar tree with two roots (i) and (j) connected with the vertex (m) in such a way that the segments $[i, m]$ and $[j, m]$ are neighbors among all segments issuing from the vertex m. Permuting the roots i and j we obtain a different (unless $k = 2$) tree, defining the same summand of $\Phi_{\mathbf{\Gamma}}$. □

2.1 Marked arrow diagrams as functions on generic knots

Let M^2 be a connected oriented 2-dimensional manifold. Let $f : S^1 \to M^2 \times \mathbf{R}^1$ be a knot in $M^2 \times \mathbf{R}^1$, generic with respect to the standard projection $p : M^2 \times \mathbf{R}^1 \to M^2$, i.e. $p \circ f$ is an immersion with transverse crossing points only. In our pictures, we identify (a piece of) the oriented factor M^2 with the "blackboard plane", oriented by the frame 

, and the factor $\mathbf{R}^1$ with the line orthogonal to it and oriented "to us". Recall that the *local writhe* of a crossing point looking like  (respectively, ) is equal to -1 (respectively, $+1$).

Let A be a planar k-arrow diagram, whose $k+1$ domains are labelled by elements $\gamma_i \in h_1(M^2) \setminus \{\mathbf{1}\}$, $i = 1, 2, \dots, k+1$, and $\mathring{A}$ the basic circle of this arrow diagram.

Definition 4 (cf. [4], [7]) A *representation* of A in the generic knot $f : S^1 \to M^2 \times \mathbf{R}^1$ is any orientation-preserving homeomorphism $r : \mathring{A} \to S^1$ such that

a) for any chord of A, connecting some points $x, y \in \mathring{A}$ and oriented from y to x, the images of x and y under the map $p \circ f \circ r : \mathring{A} \to M^2$ coincide, and moreover the point $f \circ r(x) \in M^2 \times \mathbf{R}^1$ lies above $f \circ r(y)$ in the sense of the standard orientation of the factor $\mathbf{R}^1$; in particular for any domain of the diagram A, the class in $h_1(M^2)$ of the canonically oriented boundary of this domain under the map $p \circ f \circ r$ is well-defined;

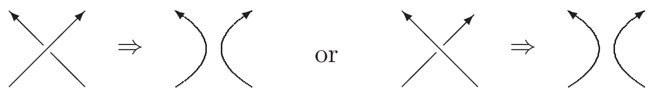
b) for any domain of the diagram A, the latter class in $h_1(M^2)$ is equal to the element γ_i marking this domain.

The *sign* of such a representation is equal to the product of k local writhes of our knot over all k crossing points of $p \circ f(S^1)$ corresponding to the chords of A via this representation.

Two representations of A in one and the same knot f are *equivalent* if they coincide on all endpoints of chords. Obviously, equivalent representation have equal signs, so the *sign of* any *equivalence class of representations of A in f* is well-defined.

The *value* $A(f)$ of the planar labelled arrow diagram A on the generic knot f is equal to the sum of signs over all equivalence classes of representations of A in f. The value on f of a formal linear combination of arrow diagrams (like $\Phi_{\mathbf{\Gamma}}$) is defined by linearity.

Given a representation of a planar arrow diagram in a knot, the homotopy classes of boundaries of domains separated by arrows can be realized as follows: we replace in the diagram of the knot all crossing points corresponding to arrows of this diagram by the rule



(1)

and consider the homotopy classes of oriented curves into which these surgeries split our knot; see e.g. Figs. 4, 5.

Theorem 1 *For any unambiguous ordered collection of elements $\gamma_1, \dots, \gamma_{k+1} \in h_1(M^2) \setminus \{\mathbf{1}\}$, the value of the corresponding chain $\Phi_{\mathbf{\Gamma}}$ is an invariant of knots in $M^2 \times \mathbf{R}^1$.*

Corollary 1 *The similar statement holds, in which the set $h_1(M^2) \setminus \{\mathbf{1}\}$ is replaced by the set $H_1(M^2) \setminus \{0\}$ of non-zero 1-homology classes.*

Indeed, for the corresponding invariant $\bar{\Phi}_{\bar{\mathbf{\Gamma}}}$, $\bar{\mathbf{\Gamma}} \in (H_1(M^2))^{k+1}$, we can take the sum of invariants $\Phi_{\mathbf{\Gamma}}$ with $\mathbf{\Gamma}$ running over the pre-image of $\bar{\mathbf{\Gamma}}$ under the obvious map converting homotopy classes into homology classes.

If $k = 2$ and the set $\bar{\mathbf{\Gamma}}$ satisfies one of conditions a) or b) of Lemma 1, then the invariant $\bar{\Phi}_{\bar{\mathbf{\Gamma}}}$ defined in this corollary coincides with some invariant I_3^K from Theorem 2.10 of [4].

Remark 1 In [4], the representations were defined for all (not necessarily planar) arrow diagrams, with homology classes associated with the arrows, and not with pieces of the disc. In the case of planar arrows, the homology class associated in [4] with an arrow is equal to the sum of homology classes of boundaries of all domains of D^2 on a certain side of this arrow. However, in our particular case the current notation is much more convenient.

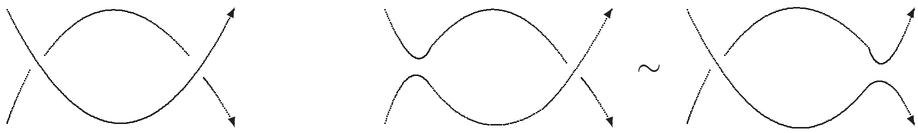


Fig. 4: Second move (+), see Fig. 3 of [3]

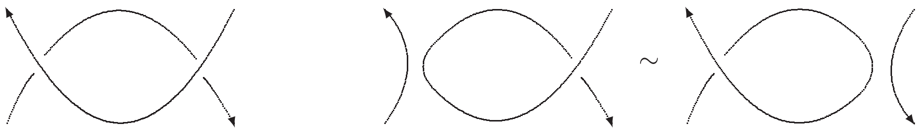


Fig. 5: Second move (−)

Remark 2 The invariants described in Theorem 1 are closely related to but are not reduced to the ones from [5], [6]. It is easy to see that for any unambiguously ordered set $\mathbf{\Gamma}$ satisfying the condition a) or b) of Lemma 1, the weight system of the corresponding invariant $\Phi_{\mathbf{\Gamma}}$ is equal to I_Γ (see [5]), where Γ is obtained from $\mathbf{\Gamma}$ by forgetting the ordering. On the other hand, if we subtract two our functions $\Phi_{\mathbf{\Gamma}}$, $\Phi_{\mathbf{\Gamma}'}$, where $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ differ only by the order of elements γ_i, then we obtain an invariant of smaller degree, and we do not see why the weight system of this invariant should be of the same type.

3 Proof of Theorem 1

We need to prove that the value $\Phi_{\mathbf{\Gamma}}(f)$ of the sum $\Phi_{\mathbf{\Gamma}}$ is invariant under all Reidemeister moves of the generic knot f, cf. [3].

First move. The crossing point arising/perishing at this move cannot contribute to the calculation of any function $\Phi_{\mathbf{\Gamma}}$ because the smoothing (1) at such a point provides a loop in M^2 not satisfying the condition $\gamma_i \neq \mathbf{1}$.

Second move. There are four different situations to consider. First, the projections of participating branches of the knot can have coinciding (+) or opposite (−) directions at the instant of the surgery, see the left-hand parts of Figs. 4 and 5 (and also Fig. 4 of [3]). Also, we need to count the contributions to the value $\Phi_{\mathbf{\Gamma}}(f)$ arising from the representations of marked plane arrow diagrams such that both (II) or only one (I) of crossing points vanishing at this move correspond to some arrows of these diagrams.

In the case (II+) there are no such representations of planar diagrams, because already the corresponding two chords cross one another.

In the case (II−) there also are no such representations of diagrams participating in the chain $\Phi_{\mathbf{\Gamma}}$, because the image in M^2 of the boundary of the domain, placed in the disc between these two chords, is homotopic to a constant loop.

In cases (I+) and (I−) any relevant representation of a marked diagram is matched by another representation of the same arrow diagram, counted with the opposite sign, see the right-hand parts of Figs. 4 and 5 respectively.

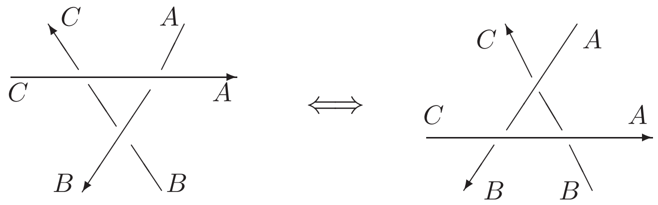


Fig. 6: Third Reidemeister move (Δ)

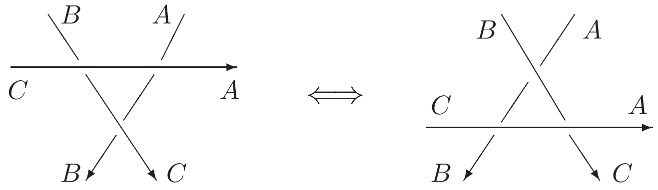


Fig. 7: Third Reidemeister move (∇)

Third move. We need to consider six different cases. Indeed, there are (up to reflections) exactly two topologically different arrangements of orientations of participating branches of a knot: Δ and ∇ shown respectively in Figs. 6 and 7 (and in Fig. 4 of [3]). The letters in these pictures indicate the way in which the endpoints of the visible (i.e. shown in the picture) part of the knot are connected through the exterior of the picture. Also, for both cases Δ and ∇ we need to consider and compare the representations of arrow diagrams in which exactly one, two or three different chords correspond to the crossing points shown in our picture.

In the first two cases, Δ_{I} and ∇_{I}, any such representation in a knot, whose part is shown in one of four pictures of Figs. 6, 7, is matched by a unique representation of the same arrow diagram in the knot shown on the other side on the same picture. Indeed, such a representation deals with only one arrow corresponding to the crossing point of some two local branches of the knot in our picture; our move does not change the mutual disposition of any such two branches. These matched representations have equal signs, so their contributions to the value $\Phi_{\mathbf{\Gamma}}(f)$ are equal to one another.

In the case Δ_{III}, every splitting (1) of the knot diagram at all three crossing points shown in any side of Fig. 6 provides a contractible component; therefore there are no representations of arrow diagrams participating in the sum $\Phi_{\mathbf{\Gamma}}$ and realizing this case Δ_{III}. In the case ∇_{III} at least two of three arrows corresponding to crossing points in any side of Fig. 7 cross one another, so again such representations do not contribute to the value $\Phi_{\mathbf{\Gamma}}(f)$.

In the case Δ_{II} (respectively, ∇_{II}) the possible splittings (1) at some two of three crossing points of a picture in Fig. 6 (respectively, 7) provide diagrams shown in Fig. 8 (respectively, 9). In the case Δ_{II}, only three of these six splittings, namely all the left-hand ones in Fig. 8, are related with non-crossing pairs of arrows. The product of local writhes of corresponding two crossing points in any of these three cases is equal to -1, -1 and $+1$ respectively. The arrow diagrams composed by only these pairs of arrows are as shown in Fig. 12, where A, B and C indicate the labels of endpoints of visible parts of boundaries of corresponding domains in Fig. 8.

Lemma 2 *For any unambiguous collection* $\mathbf{\Gamma} = (\gamma_1, \dots, \gamma_{k+1})$ *and any generic knot* f *in*

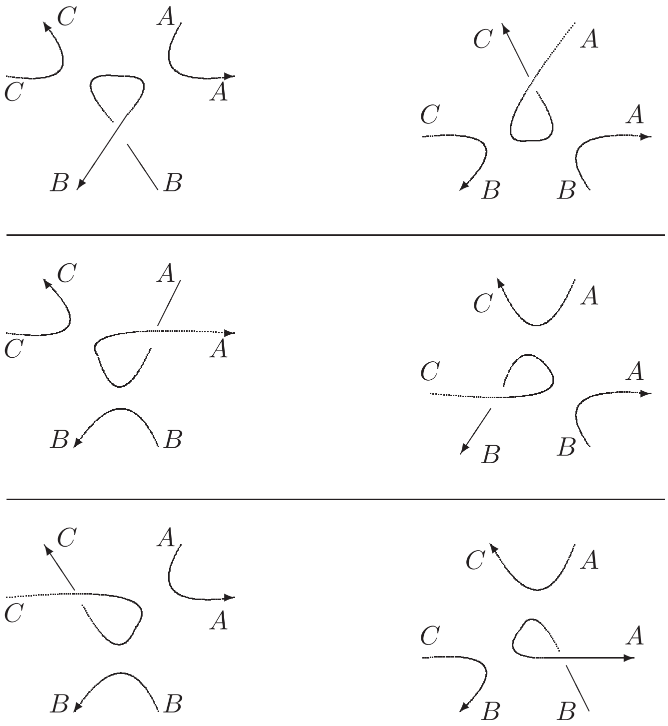


Fig. 8: 2-splittings at the third Reidemeister move (Δ_{II})

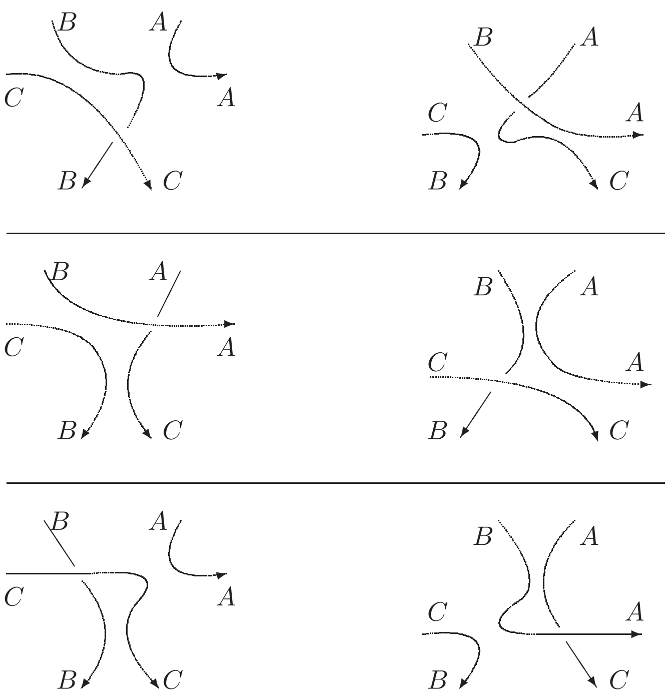


Fig. 9: 2-splittings at the third Reidemeister move (∇_{II})

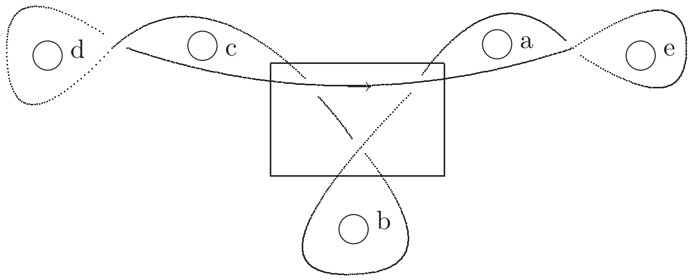


Fig. 10: An example

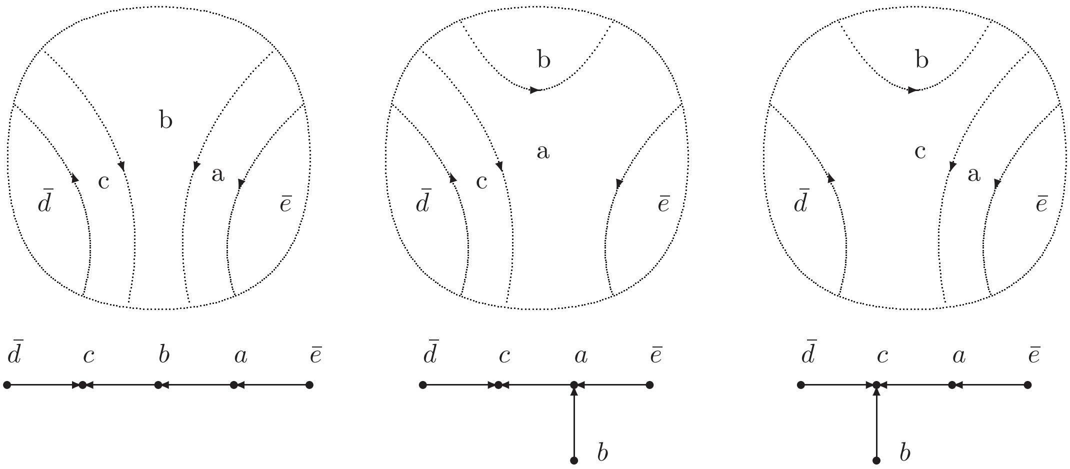


Fig. 11: Possible representations in the knot of Fig. 10

$M^2 \times \mathbf{R}^1$, whose local part is shown in the left-hand picture of Fig. 6, there is a natural matching on the set of pairs (φ, r) consisting of a summand φ of the formal sum $\Phi_{\mathbf{\Gamma}}$ and an equivalence class r of representations of this summand in our knot f, such that exactly two arrows of the arrow diagram φ correspond to crossing points of this picture. For any two matched pairs (φ_1, r_1) and (φ_2, r_2) the signs of representations r_1 and r_2 are opposite.

Example 1 Suppose that M^2 is the plane $\mathbf{R}^2$ with five discs removed, and our knot is as shown in Fig. 10, with its central part (distinguished by the rectangular frame) coinciding with the left-hand picture of Fig. 6. Denote by a, b, c, d and e the classes in $h_1(M^2)$ of boundaries of these removed discs oriented counterclockwise, see Fig. 10, and by $\bar{a}, \bar{b}$ etc the classes of these boundaries with opposite orientations. There are only three equivalence classes of planar arrow diagrams arranged by classes $a, b, c, \bar{d}$ and $\bar{e}$ and having representations in this knot, see Fig. 11. Namely, any of these three diagrams has exactly one such representation, sending the endpoints of some two arrows into some two of three crossing points of the boxed central part of our knot. For the first (respectively, the second, the third) from the left diagram in Fig. 11 it are the crossing points, the splitting at which is shown in the upper (respectively, the middle, the bottom) left-hand picture of Fig. 8.

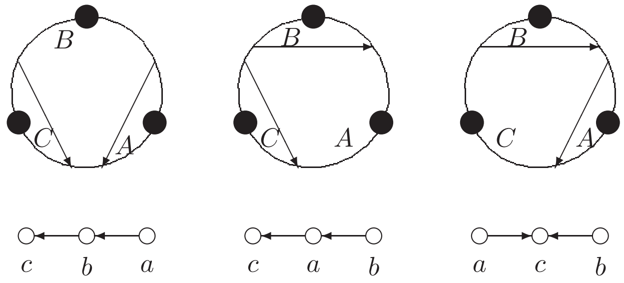


Fig. 12: Arrow diagrams with maximal splitting for Δ_{II}

Let $\mathbf{\Gamma}$ be the collection of classes $a, b, c, \bar{d}, \bar{e}$ ordered somehow. If at least one of three arrow diagrams of Fig. 11 participates as a summand in the chain $\Phi_{\mathbf{\Gamma}}$, then in this ordering we necessarily have $\bar{e} < a < c$ and $\bar{d} < c > b$. Namely, if additionally we have $a < b$, then exactly the left-hand and right-hand diagrams of Fig. 11 participate in $\Phi_{\mathbf{\Gamma}}$, and if $a > b$ then exactly the middle and the right-hand diagrams do. In both cases the signs of two corresponding representations are opposite, so all functions $\Phi_{\mathbf{\Gamma}}$ defined by such orderings take zero value on our knot.

Proof of Lemma 2. Let $f : S^1 \to M^2 \times \mathbf{R}^1$ be a knot, whose diagram is partially shown in the left-hand part of Fig. 6; let W be a marked arrow diagram participating in the chain $\Phi_{\mathbf{\Gamma}}$, and $r : \overset{\circ}{W} \to S^1$ some representation of W in f having exactly two arrows corresponding to the crossing points shown in Fig. 6. Depending on these points, the corresponding splitting of the knot provides one of three pictures of the left-hand part of Fig. 8, and the arrow diagram W looks like one of three diagrams of Fig. 12, maybe with some additional non-intersecting arrows inside the area covered by black discs. Namely, two arrows in this diagram of Fig. 12 should coincide with arrows of W corresponding to crossing points in question, and the letters A, B and C indicate the arcs of $\overset{\circ}{W}$ not shown in Fig. 6 and connecting the correspondingly marked endpoints of segments of this figure.

Consider the link in $M^2 \times \mathbf{R}^1$, obtained from $f(S^1)$ by the desingularization (1) at crossing points corresponding to all arrows of the diagram W via the representation r. By definition of $\Phi_{\mathbf{\Gamma}}$, this link consists of $k+1$ components, whose classes in $h_1(M^2)$, $\gamma_1, \dots, \gamma_{k+1}$, constitute the set $\mathbf{\Gamma}$. Some three of these components, with some homotopy classes γ_a, γ_b, and γ_c, intersect our picture of Fig. 8; here the indices a, b, c correspond to the capital letters labelling the endpoints of intersections of these components with this picture. Since $\mathbf{\Gamma}$ is an unambiguous ordered set, the numbers $a, b, c \in \{1, 2, \dots, k+1\}$ are uniquely defined. It follows from (the lower part of) Fig. 12 that we necessarily have $c < a$ and $c < b$. Further, our representation r of W is matched by exactly one other representation of an arrow diagram participating in the sum $\Phi_{\mathbf{\Gamma}}$, also having two arrows corresponding to crossing points of the left-hand picture of Fig. 6. Namely, this pair of matched diagrams always contains the right-hand diagram of Fig. 12, and exactly one of other two diagrams, depending on whether $a < b$ or $a > b$. These two matched representations always have opposite signs, so their contributions to the value $\Phi_{\mathbf{\Gamma}}(f)$ annihilate one another. □

In the case ∇_{II} the following three splittings of the knots shown in Fig. 7 can participate in the calculation of the values of some chains $\Phi_{\mathbf{\Gamma}}$ on these knots: the upper and the lower splittings in the left-hand part of Fig. 9, and the middle one in the right-hand part of the same figure. The products of local writhes of corresponding arrows are equal to $+1$ in all

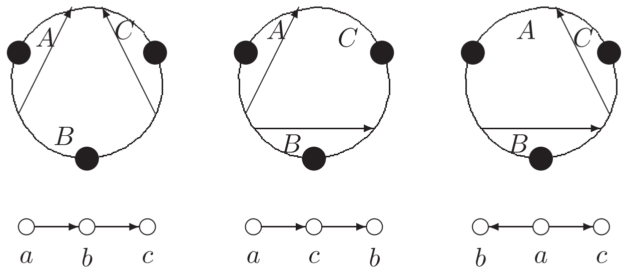


Fig. 13: Arrow diagrams with maximal splitting for $\nabla_{\rm II}$

three cases.

Lemma 3 *Suppose that* $\mathbf{\Gamma} = (\gamma_1, \dots, \gamma_{k+1})$ *is an unambiguous collection, and* f_1, f_2 *are two generic knots in* $M^2 \times \mathbf{R}^1$*, coinciding respectively with the left- and right-hand pictures of Fig. 7 in some small domain in* $M^2 \times \mathbf{R}^1$*, and with one another outside of this small domain. Then there is a natural one-to-one correspondence between*

1) the set of pairs (φ, r) *consisting of a summand* φ *of the formal sum* $\Phi_{\mathbf{\Gamma}}$ *and an equivalence class* r *of representations of this summand in the knot* f_1*, such that exactly two arrows of the arrow diagram* φ *correspond to crossing points of the left-hand picture of Fig. 7, and*

2) the set of similar pairs (ψ, ρ) *where* ρ *is a representation of* ψ *in* f_2*, also with exactly two arrows of* ψ *corresponding to crossing points of the right-hand picture of Fig. 7.*

For any two pairs (φ, r) *and* (ψ, ρ)*, related via this correspondence, the signs of representations* r *and* ρ *are opposite.*

This lemma can be proved very similarly to the proof of Lemma 2 and terminates the proof of the fact that the third Reidemeister move of type ∇ also does not change the value $\Phi_{\mathbf{\Gamma}}(f)$.

If we consider the Reidemeister move that is a mirror image of Fig. 6 (respectively, 7) with respect to the blackboard plane, then instead of C dominated by (respectively, A dominating) both other elements we need only to consider the opposite condition. □

4 Realization of invariants $\Phi_{\mathbf{\Gamma}}$

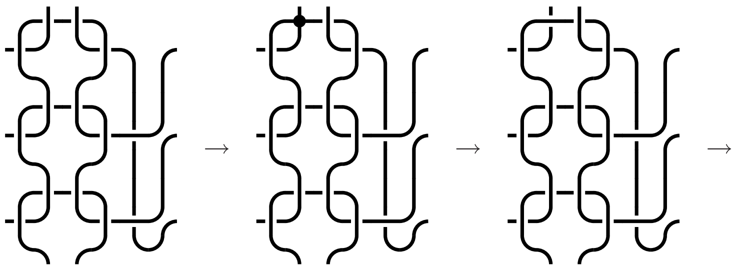
In this section we demonstrate two knots in $\mathbf{T}^2 \times \mathbf{R}^1$ which cannot be separated by any finite type invariants of degree 1 or 2, but are separated by some invariant $\Phi_{\mathbf{\Gamma}}$ of degree 3.

We depict knots in $\mathbf{T}^2 \times \mathbf{R}^1$ by diagrams in rectangular domains of $\mathbf{R}^2$. The torus $\mathbf{T}^2$ is considered as the factor of such a domain by the identification of its opposite margins, and the knot diagrams may have paired endpoints on these margins. 1-dimensional homology classes in $\mathbf{T}^2$ will be expressed by pairs of integers (a, b), where the class $(1, 0)$ is the horizontal generator oriented from the left to the right, and $(0, 1)$ is the vertical generator oriented to the top of the page.

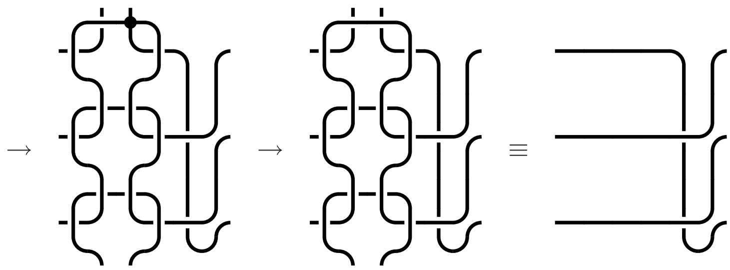
Theorem 2 *Two knots* (2) *in* $\mathbf{T}^2 \times \mathbf{R}^1$ *cannot be separated by any degree 1 and 2 invariants, but are separated by our invariant* $\Phi_{\mathbf{\Gamma}}$ *with* $\mathbf{\Gamma} = ((1, 0), (1, -1), (0, 1), (1, 0))$.

(2)

Proof. We orient our knots so that their classes in $H_1(\mathbf{T}^2 \times \mathbf{R}^1)$ are equal to $(3, 0)$. These knots are connected in the space of maps $S^1 \to \mathbf{T}^2 \times \mathbf{R}^1$ by the path (3–4).

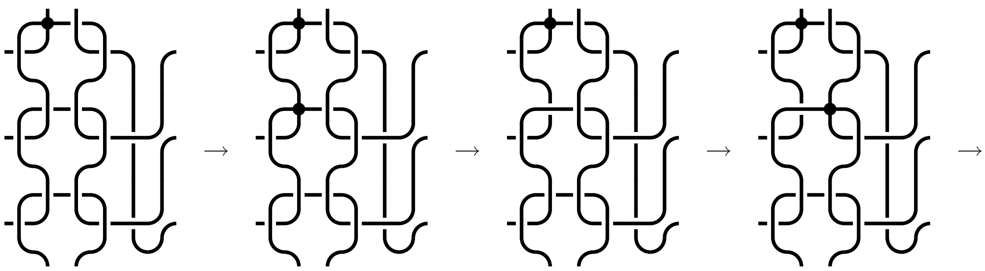


(3)

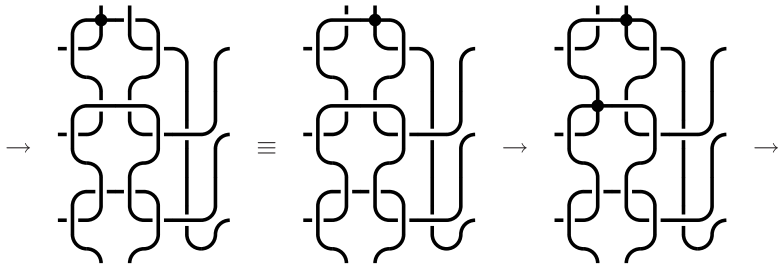


(4)

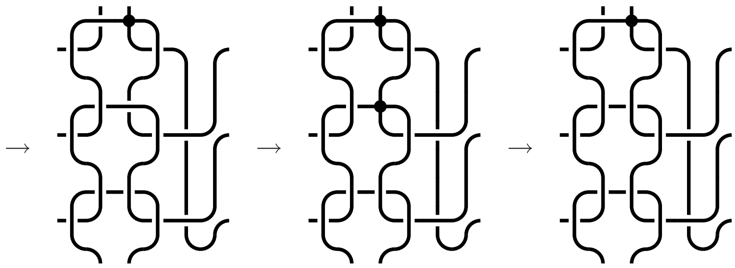
This path contains two surgeries, their signs are equal to $+$ and $-$ respectively. These surgeries represent homotopic singular knots: a generic homotopy connecting them in the space of singular knots is realized by the path (5–7). Thus our knots (2) are not separated by degree 1 invariants.



(5)



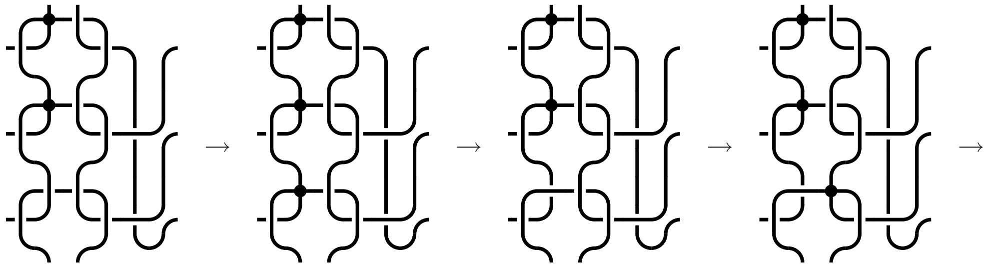
(6)



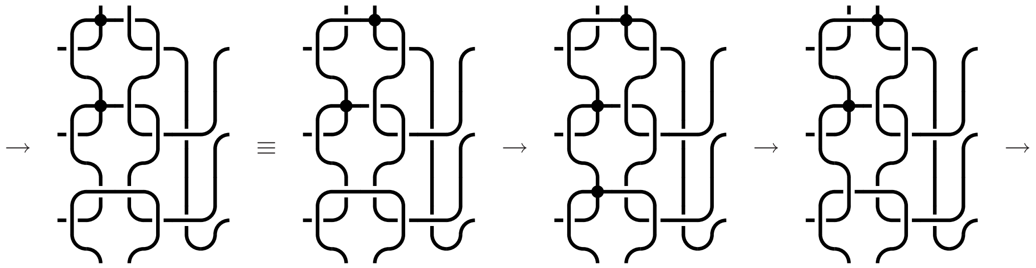
(7)

Let us prove that all degree 2 finite type invariants also do not separate these knots. Given such an invariant, the difference of its values at the knots (2) is equal to the difference of values of first order indices of this invariant on two surgery points in the path (3–4). This difference is equal to the sum of values of the principal part of our invariant at all second order surgeries in the path (5–7) multiplied by the signs of these surgeries.

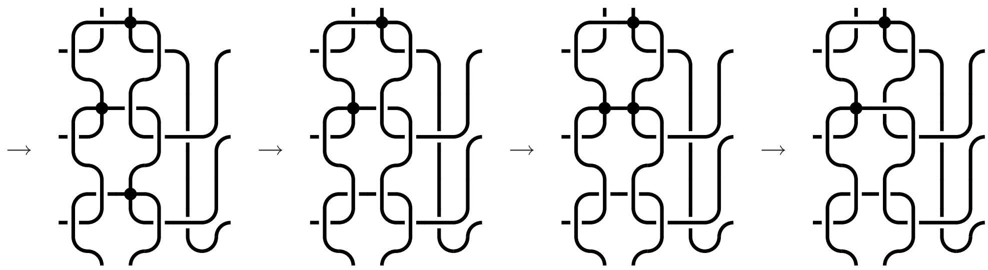
There are four such surgeries of order 2 with signs equal to $+$, $-$, $-$ and $+$ respectively. The first and the third (respectively, the second and the fourth) surgeries can be connected by a smooth path inside the self-intersection locus of the discriminant variety, namely, by the path (8–10) (respectively, (11–13)) below. Therefore the principal parts of all degree 2 invariants take equal values on these surgeries, in particular all such invariants take equal values on initial knots (2).



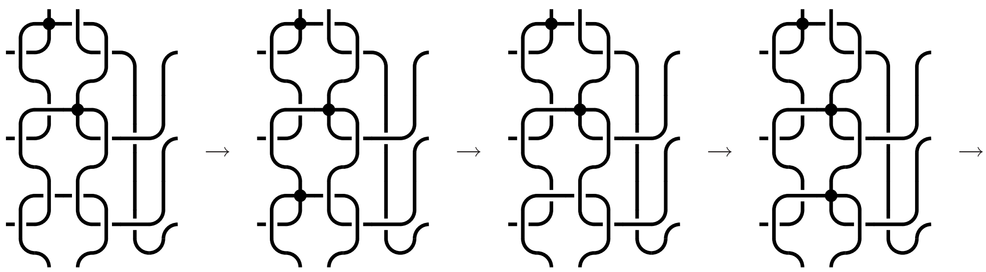
(8)



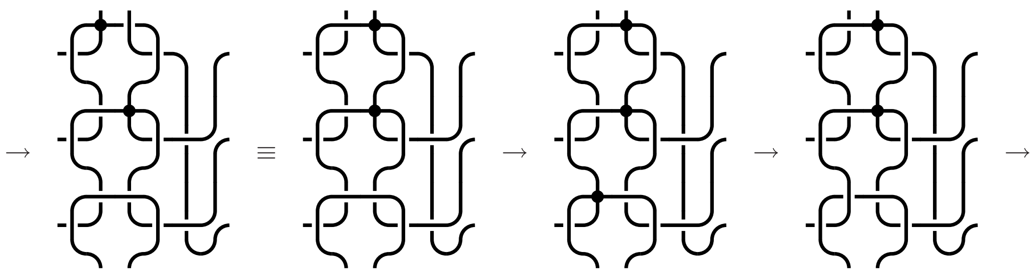
(9)



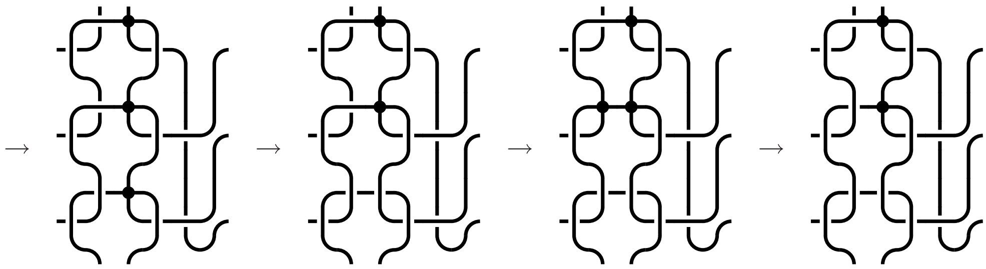
(10)



(11)

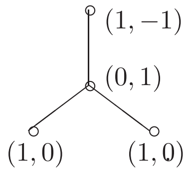


(12)



(13)

Two paths (8–10) and (11–13) contain in total ten surgeries of third order, only one of which (the very first one) has a chord diagram with non-intersecting chords. The corresponding marked planar chord diagram is as follows:



Therefore any of two invariants $\Phi_{\mathbf{\Gamma}}$ with $\mathbf{\Gamma} = ((1,0),(0,1),(1,-1),(1,0))$ or $((1,0),(1,-1),(0,1),(1,0))$ takes a non-zero value on our knot, and Theorem 2 is proved.

Remark 3 The degree 2 invariants proposed in [4] were extended in [2] to isotopy invariants of smooth embedded tori in certain 4-manifolds. We reproduce here the question of the Referee, whether our generalizations of invariants from [4] also can be extended to new invariants of smooth embeddings of surfaces into M^4.

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