HOMOLOGY OF SPACES OF HOMOGENEOUS POLYNOMIALS IN $\mathbb{R}^2$ WITHOUT MULTIPLE ZEROS

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1. Introduction

For any natural $d \geq k \geq 2$ we calculate the cohomology groups of the space of homogeneous polynomials $\mathbb{R}^2 \to \mathbb{R}$ of degree d, which do not vanish with multiplicity $\geq k$ on real lines. For $k = 2$ this problem provides the simplest example of the situation, when the "finite-order" invariants of nonsingular objects are not a complete system of invariants.

The "affine" version of this problem (the calculation of the homology group of the space of polynomials $\mathbb{R}^1 \to \mathbb{R}^1$ with leading term x^d and without $\geq k$-fold roots) was solved by V. I. Arnold in [2], see also [3]. As in these works, our present calculation is based on the study of the discriminant set, i.e. of the set of polynomials with forbidden multiple zeros.

The problem solved below turns out to be more complicated, because an essential contribution to the homology group comes from the neighborhood of the "infinitely degenerate" polynomial equal identically to 0. By this reason, the method of simplicial resolutions of the discriminant set, solving immediately the "affine" problem, is replaced by its continuous analog: conical resolution, used previously in [5].

Here we have the simplest situation, when the invariants of "finite order" of the space of nonsingular objects do not constitute a complete system of invariants. Indeed, our spaces of nonsingular polynomials can be considered as finite-dimensional approximations of the space $\mathcal{F} \setminus \Sigma_k$ of smooth functions $S^1 \to \mathbb{R}^1$ without k-fold zeros (if d is odd, then with values in a nontrivial line bundle); here S^1 is realized as a half of the unit circle in $\mathbb{R}^2$. By analogy with [4], the cohomology classes of "finite order" of the space $\mathcal{F} \setminus \Sigma_k$ are exactly those, which are obtained by a natural stabilization of such cohomology groups for approximating spaces. It turns out, that for $k = 2$ and even d all such 0-dimensional

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cohomology classes (i.e. the invariants of degree d polynomial functions $S^1 \to \mathbb{R}^1$ without multiple zeros) are polynomials of the number of zeros of these functions, in particular they cannot separate everywhere positive functions from the everywhere negative ones. For any finite even d such polynomial functions can be distinguished by a certain 0-dimensional cohomology class, arising from the construction of the conical resolution, but this class is not stable.

Note that a similar example, proving that the system of all finite-order invariants of knots in $\mathbb{R}^3$ (see [4]) is not complete, is not constructed yet.

2. Main result

Notation. Denote by HP_d the space of homogeneous polynomials $\mathbb{R}^2 \to \mathbb{R}^1$ of degree d, and by $\Sigma_k \subset HP_d$ the set of polynomials taking zero value with multiplicity $\geq k$, $k \geq 2$, on some line in $\mathbb{R}^2$. For any topological space X, $B(X, j)$ denotes its j-th configuration space (i.e. the space of j-point subsets in X, supplied with a natural topology).

Main Theorem. *1. If k is even, then the group $\tilde{H}^*(HP_d \setminus \Sigma_k)$ is free Abelian of rank $2[d/k]+1$, and its free generators have dimensions $k-2, 2(k-2), \ldots, [d/k](k-2)$, $k-1, 2(k-2)+1, \ldots, [d/k](k-2)+1$ and $d-2[d/k]$.*

2. If k is odd and d is not a multiple of k, then the group $\tilde{H}^*(HP_d \setminus \Sigma_k)$ is the direct product of following groups:

(a) for any $p = 1, 2, \ldots, [d/k]$ such that $d-p\cdot k$ is odd, $\mathbb{Z}$ in dimension $p(k-2)$ and $\mathbb{Z}$ in dimension $p(k-2)+1$;

(b) for any $p = 1, 2, \ldots, [d/k]$ such that $d - p \cdot k$ is even, $\mathbb{Z}_2$ in dimension $p(k-2)+1$;

(c) $\mathbb{Z}$ in dimension $d-2[d/k]$.

3. If k is odd and d is a multiple of k, the answer is almost the same as in the case 2, only the summand $\mathbb{Z}_2$ in dimension $d-2(d/k)+1$ vanishes.

Main example. Let d be even and $k = 2$. Then the space $HP_d \setminus \Sigma_k$ consists of $d/2+2$ connected components, two of which (corresponding to everywhere positive and everywhere negative functions) are contractible, and all the other are homotopy equivalent to a circle. This homotopy equivalence is a composition of two: the first maps any polynomial to the collection of its zero lines (i.e. to an element of the configuration space $B(\mathbb{R}P^1, 2p)$ with appropriate $p \in [1, d/2]$), and the second is the arrow in the following well-known statement.

Lemma 1. *For any j, there is a locally trivial fibre bundle $B(S^1, j) \to S^1$, whose fiber is homeomorphic to an open $(j-1)$-dimensional disc. This fibre bundle is trivial if j is odd and is nonorientable if j is even.* $\square$

Caratheodory's theorem. In the proof of the main theorem we use the following fact.

Suppose that a manifold (or finite CW-complex) M is embedded generically in the space $\mathbb{R}^N$ of a very large dimension, and denote by M^{*r} the union of all $(r-1)$-dimensional simplices in $\mathbb{R}^N$, whose vertices lie on this embedded manifold (and the "genericity" of the embedding means that if two such simplices intersect at a certain point, then their minimal faces, containing this point, coincide). If M is a semialgebraic variety, then by the Tarsky—Zaidenberg lemma also M^{*r} is, in particular it has a triangulation.

Proposition 1 (C. Caratheodory, see also [7]). *The space $(S^1)^{*r}$ is PL-homeomorphic to S^{2r-1}.* $\square$

3. Proof of the main theorem

Following [1], we use the Alexander duality

$$H^l(HP_d \setminus \Sigma_k) \simeq \bar{H}_{d-l}(\Sigma_k), \qquad (1)$$

where $\bar{H}_*$ is the notation for the Borel—Moore homology, i.e. the homology of the one-point compactification modulo the added point.

To calculate the right—hand group in (1) we construct the *conical resolution* of the space Σ_k. Let us embed the projective line $\mathbb{R}P^1$ generically in the space $\mathbb{R}^N$ of a very large dimension, and for any function $f \in \Sigma_k$, not equal identically to zero, consider the simplex $\Delta(f)$ in $\mathbb{R}^N$, spanned by the images of all points $x_i \in \mathbb{R}P^1$, corresponding to all possible lines, on which f takes zero value with multiplicity $\geq k$. (The maximal possible number of such lines is obviously equal to $[d/k]$.) In the direct product $HP_d \times \mathbb{R}^N$ consider the union of all simplices of the form $f \times \Delta(f)$, $f \in \Sigma_l \setminus 0$. This union is not closed: the set of its limit points, not belonging to it, is the product of the point $0 \in HP_d$ and the union of all simplices in $\mathbb{R}^N$, spanned by the images of no more than $[d/k]$ different points of the line $\mathbb{R}P^1$. By the Caratheodory's theorem, the latter union is homeomorphic to the sphere $S^{2[d/k]-1}$. We can assume that our embedding $\mathbb{R}P^1 \to \mathbb{R}^N$ is algebraic, and hence this sphere is semialgebraic. Take a $2[d/k]$-dimensional semialgebraic disc in $\mathbb{R}^N$ with boundary at this sphere (e.g., the union of segments connecting the points of this sphere with a generic point in $\mathbb{R}^N$) and add to the previous union of simplices in $HP_d \times \mathbb{R}^N$ the product of the point $0 \in HP_d$ and this disc. The obtained set will be denoted by σ.

Lemma 2. *The obvious projection $\sigma \to \Sigma_k$ is proper, and the corresponding map of one-point compactifications of these spaces is a homotopy equivalence.*

This follows easy from the fact that this projection is a stratified map of semialgebraic spaces, and the preimage of any point $\bar{\Sigma}_k$ is contractible, cf. [6], [7]. $\square$

The space σ has a natural increasing filtration: its term F_p, $p \leq [d/k]$, is the union of all $\leq (p-1)$-dimensional faces of all simplices, participating in our construction (or, which is the same, the closure of the union of all simplices of the form $f \times \Delta(f)$ over all polynomials f having no more than p forbidden multiple lines), and $F_{[d/k]+1} = \sigma$.

Lemma 3. *For any $p = 1, \ldots, [d/k]$, the term $F_p \setminus F_{p-1}$ of our filtration is the space of a locally trivial fiber bundle over the configuration space $B(\mathbb{R}P^1, p)$, whose fiber is the direct product of an $(p-1)$-dimensional open simplex and an $(d+1-pk)$-dimensional real space. The corresponding bundle of open simplices is orientable if and only if p is odd (i.e. exactly when the base configuration space is orientable), and the bundle of $(d+1-pk)$-dimensional spaces is orientable if and only if the number $k(d+1-pk)$ is even.*

The last term $F_{[d/k]+1} \setminus F_{[d/k]}$ of this filtration is homeomorphic to the open $2[d/k]$-dimensional disc.

Indeed, to any configuration $(x_1, \ldots, x_p) \subset \mathbb{R}P^1$ there corresponds the direct product of the interior part of the simplex in $\mathbb{R}^N$, spanned by the images of points of this configuration under our embedding $\mathbb{R}P^1 \to \mathbb{R}^N$, and the subspace in HP_d, consisting of polynomials, having k-fold zeros on corresponding p lines in $\mathbb{R}^2$. The assertion concerning the orientations can be checked elementary. $\square$

Consider the spectral sequence $E^r_{p,q}$, calculating the group $\bar{H}_*(\Sigma_k)$ and generated by this filtration. Its term $E^1_{p,q}$ is canonically isomorphic to the group $\bar{H}_{p+q}(F_p \setminus F_{p-1})$.

Corollary. *The term $E^1_{p,q}$ of our spectral sequence is as in Fig. 1 for even k and as in Fig. 2 is k is odd, i.e.:*

if k is even or $d-p$ is odd, $1 \leq p \leq [d/k]$, then in the column $E^1_{p,*}$ only the cells $E^1_{p,q}$ with $q = d - p(k-1)$ and $q = d - p(k-1) - 1$ are nontrivial; these two cells are isomorphic to $\mathbb{Z}$;

if the number $k(d-p-1)$ is odd, $1 \leq p \leq [d/k]$, then the column $E^1_{p,*}$ contains unique nontrivial cell $E^1_{p,d-p(k-1)-1} \simeq \mathbb{Z}_2$;

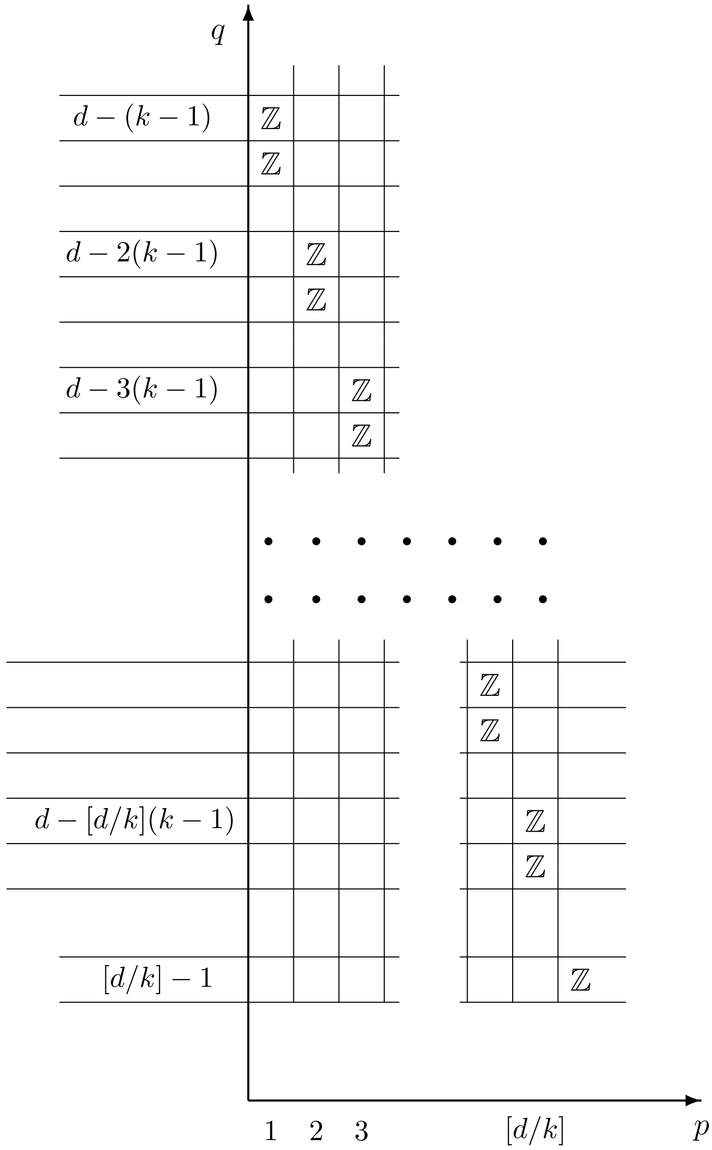

FIG. 1. First term of the spectral sequence for even k

for any k and d the unique nontrivial cell of the column $E^1_{[d/k]+1,*}$ is $E^1_{[d/k]+1,[d/k]-1} \simeq \mathbb{Z}$;
all other columns of the spectral sequence are trivial. $\square$

REMARK. The situation shown in columns $p = [d/k]$ and $p = [d/k]-1$ of Fig. 2 appears when $d - [d/k]$ is even. In fact the opposite also can happen, but in the most interesting case, when d is a multiple of k, the situation is exactly as in the picture. In this last case the unique

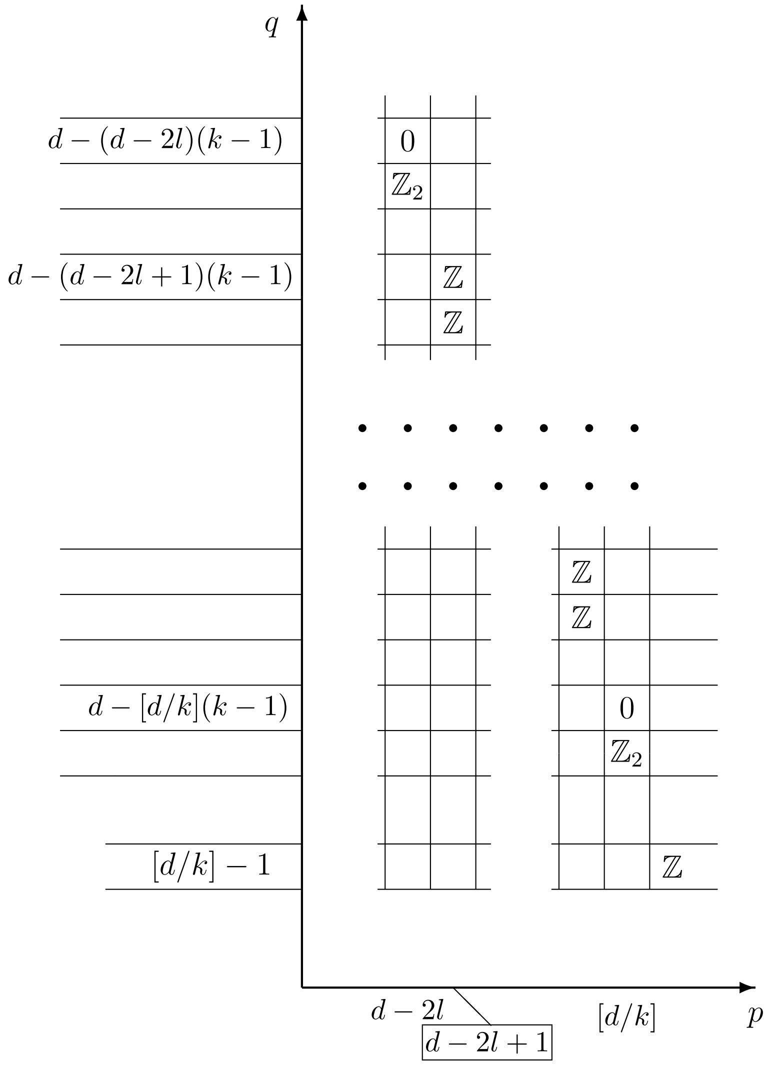

FIG. 2. Term E^1 for odd k

nontrivial cell $E^1_{[d/k]+1,[d/k]-1} \simeq \mathbb{Z}$ of the column $p = [d/k] + 1$ lies in the same horizontal row as the unique nontrivial cell $E^1_{[d/k],[d/k]-1} \simeq \mathbb{Z}_2$ of the column $p = [d/k]$.

PROPOSITION 2. *For any d, k, except for the case when k is odd and d is a multiple of k, our spectral sequence degenerates in the term E_1, i.e. $E_1 = E_\infty$. In the exceptional case there is unique nontrivial operator d_1, acting from the cell $E^1_{d/k+1,d/k-1} \sim \mathbb{Z}$ to the cell $E^1_{d/k,d/k-1} \sim \mathbb{Z}_2$, killing the latter cell, after which the spectral sequence also degenerates.*

Main theorem follows immediately from this proposition.

Proof of Proposition 2. For almost all d and k the assertion follows immediately from the explicit form of the term E_1. The only two cases, when nontrivial differentials could occur, are the case $k = 2$ (a half of which is already studied, see Main example in § 2, and the remaining case of odd d is even so easy) and the case when d is a multiple of k. In the latter case (for $k > 2$) the unique nontrivial differential can act from the cell $E^1_{d/k+1,d/k-1} \sim \mathbb{Z}$ to the cell $E^1_{d/k,d/k-1}$. If k is even, then the latter cell is isomorphic to $\mathbb{Z}$, and the corresponding term $F_{d/k} \setminus F_{d/k-1}$ of our filtration is the oriented bundle with fiber $\mathbb{R}^1$, whose base also is oriented and is, in its turn, the space of a fibre bundle with base $B(S^1, d/k)$ and the fiber equal to an $(d/k-1)$-dimensional open simplex. The boundary of the disc $F_{d/k+1} \setminus F_{d/k}$ in this term of filtration coincides with the zero section of the former (line) bundle, therefore it is the boundary (modulo lower terms of the filtration) of the space of the bundle with fiber $\mathbb{R}_+$.

In the case of odd k similar line bundle is nonorientable, therefore when we try to span this zero section by a similar chain over a maximal simple-connected domain in the base, then we construct a homology between the image of this section and a cycle generating the group $H_{2d/k-1}(F_{d/k} \setminus F_{d/k-1}) \sim \mathbb{Z}_2$. (An adequate picture here is the Möbius band, whose equator circle is the boundary of the disc $F_{d/k+1} \setminus F_{d/k}$.) All the further differentials are trivial by the dimensional reasons, and main theorem is completely proved.

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