THERE ARE NO ALGEBRAICALLY INTEGRABLE OVALS IN EVEN-DIMENSIONAL SPACES

V.A. VASSILIEV

To the memory of Egbert Brieskorn

Abstract. We prove that there are no bounded domains with smooth boundaries in even-dimensional Euclidean spaces, such that the volumes cut off from them by affine hyperplanes depend algebraically on these hyperplanes. For convex ovals in $\mathbb{R}^2$, this is the Newton's Lemma XXVIII, see [11], [14], [2], [3].

1. Introduction

Sir Isaac Newton has proved (see [11], Lemma XXVIII of Book 1) that the areas cut off by different lines from a convex bounded domain with infinitely smooth boundary in $\mathbb{R}^2$ never define an algebraic function on the space of lines[1].

This fact contrasts to the Archimedes' theorem stating that the volume cut off by a plane from a ball in $\mathbb{R}^3$ depends algebraically on the plane; it is easy to check that this theorem holds also for arbitrary ellipsoids in odd-dimensional spaces.

In 1987, in connection with the anniversary of the Newton's Book, V.I. Arnold has asked whether his result is true in the case of other dimensions and general domains with smooth boundaries, see [6], problems 1987-14, 1988-13 and 1990-27.

In 1988, solving this problem, I have extended the Newton's result to convex domains in even-dimensional spaces and to arbitrary bounded domains with smooth boundaries in $\mathbb{R}^2$, see [3], [12]. In the present paper, the same statement is proved for arbitrary bounded domains with smooth boundaries in even-dimensional spaces. The proof is based on the Picard-Lefschetz theory and elementary facts on finite reflection groups.

1.1. **Definitions and main theorem.** Denote by P_n the space of all affine hyperplanes in $\mathbb{R}^n$. It can be considered as $\mathbb{R}P^n$ with one point removed, in particular as an algebraic manifold. Given a compact domain $D \subset \mathbb{R}^n$, the corresponding two-valued *volume function* $P_n \to \mathbb{R}$ associates with any hyperplane $L \in P_n$ the volumes of both parts cut off from D by this hyperplane. The domain D is called *algebraically integrable* if this function is algebraic, i.e. there exists a non-trivial polynomial Φ in $n + 2$ variables such that for any real numbers $a_1, \dots, a_n, b$ and

1991 *Mathematics Subject Classification.* 14D05, 44A99; 20F55.

Key words and phrases. Integral geometry, Picard-Lefschetz theory, reflection group, algebraic function, monodromy, Newton's lemma 28.

Supported by RFBR grant 13-1-00383 and the program "Leading scientific schools", grant No. NSh-4850.2012.1.

[1] "There is no oval figure whose area, cut off by right lines at pleasure, can be universally found by means of equations of any number of finite terms and dimensions", originally "Nulla extat figura Ovalis cujus area, rectis pro lubitu abscissam possit per aequationes numero terminorum ac dimensionum finitas generaliter inveniri"

any value V_i, $i = 1, 2$, of the volume function on the plane $L \in P_n$ defined by the equation $a_1 x_1 + \cdots + a_n x_n = b$, we have $\Phi(a_1, \dots, a_n, b, V_i) = 0$.

Theorem 1. *If n is even, then there is no algebraically integrable bounded domain with C^∞-smooth boundary in $\mathbb{R}^n$.*

Remark 1. 1. For ovals (i.e. convex bounded domains) in $\mathbb{R}^2$ this theorem was proved in [11], see also [14], [3].

2. If D is a bounded algebraically integrable domain with C^∞-smooth boundary in $\mathbb{R}^n$, then by projective duality and Tarski–Seidenberg lemma this boundary ∂D is semialgebraic and hence analytic, see [2], [3]. Therefore it is enough to consider only the bodies with regular semialgebraic boundaries in $\mathbb{R}^n$, i.e. to assume that ∂D consists of several smooth connected components of the zero locus of a real polynomial. Also, it is enough to prove our theorem for any connected component of D separately, so we will assume that D is connected. On the other hand, for any finite m there exist algebraically integrable domains in $\mathbb{R}^{2k}$ with C^m-smooth boundaries, see [4]. Therefore the condition of C^∞-smoothness in Theorem 1 cannot be reduced.

3. By an Archimedes' theorem, the volume cut off by a plane from a ball in $\mathbb{R}^3$ depends polynomially on the distance of the plane from the center, i.e. on a two-valued algebraic function in P_3. We need also take into account all planes not intersecting the ball, therefore we add two single-valued functions $P_3 \to \mathbb{R}^1$ equal identically to 0 and to the volume of the ball, and obtain a four-valued algebraic function proving the algebraic integrability of the ball in $\mathbb{R}^3$. Moreover, the Archimedes' theorem can be easily extended to balls and ellipsoids in any odd-dimensional spaces. It seems likely that there are no other examples of irreducible integrable domains with smooth boundaries in odd-dimensional spaces, but this conjecture is not yet proved.

2. Two main examples

2.1. **Convex domains in $\mathbb{R}^{2k}$.** Let n be an even number, and D a convex domain in $\mathbb{R}^n$ bounded by a compact semialgebraic variety ∂D without singular points. Choose a linear function $l : \mathbb{R}^n \to \mathbb{R}^1$, whose restriction to ∂D is a Morse function (such functions exist by the Sard's lemma applied to the Gauss map $\partial D \to \mathbb{R}P^{n-1}$). Let $m < M$ be both critical values of this restriction. Denote by A the complexification of ∂D (i.e. the hypersurface in $\mathbb{C}^n$ distinguished by the same polynomial equation). Also we can and will consider l as the restriction to $\mathbb{R}^n$ of a linear function $(\mathbb{C}^n, \mathbb{R}^n) \to (\mathbb{C}^1, \mathbb{R}^1)$, which we'll denote also by l.

For any $t \in (m, M)$ define $V(t)$ as the volume of the set $D \cap l^{-1}((-\infty, t])$. If D is algebraically integrable, then V is an algebraic function, in particular its analytic continuation to $\mathbb{C}^1$ is finite-valued.

Further, denote by $\mathbb{C}P_n$ the space of all complex affine hyperplanes in $\mathbb{C}^n$. For any $X \in \mathbb{C}P_n$, consider the group

$$H_n(\mathbb{C}^n, A \cup X). \quad (1)$$

Lemma 1. *There is a well-defined linear function $H_n(\mathbb{C}^n, A \cup X) \to \mathbb{C}^1$, whose value on a relative homology class is equal to the integral of the volume form*

$$dx_1 \wedge \cdots \wedge dx_n \quad (2)$$

along an arbitrary piecewise-smooth relative cycle representing this class.

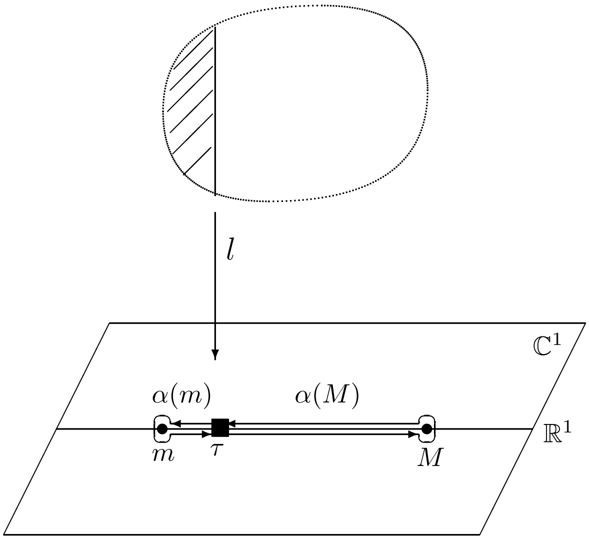


FIGURE 1. Picard-Lefschetz monodromy of integration cycles for a convex domain in even-dimensional space

Proof. This follows from the Stokes' theorem applied to the holomorphic form (2), and from the fact that the integral of this form along any singular n-chain in the $(n-1)$-dimensional complex variety $A \cup X$ is equal to zero. $\square$

There is a Zariski open subset $\text{Reg} \subset \mathbb{C}P_n$ such that all planes $X \in \text{Reg}$ are transversal to (maybe singular) variety A: indeed, for any stratum of any algebraic Whitney stratification of A the set of planes X not transversal to this stratum is a semialgebraic subvariety of positive codimension in $\mathbb{C}P_n$. Then by Thom's isotopy lemma (see e.g. [8]) there is a locally trivial fiber bundle over Reg, whose fiber over the point $\{X\}$ is the pair $(\mathbb{C}^n, A \cup X)$. Consider the associated homological bundle over Reg, whose fiber over $\{X\}$ is the group (1).

This fiber bundle is locally trivialized, i.e. it carries the flat *Gauss-Manin connection* (see e.g. [5]) defined by continuous shifts of cycles into the neighboring fibers in correspondence with any local trivialization of the initial fiber bundle of pairs $(\mathbb{C}^n, A \cup X)$. This connection is well-defined because the homological classes of these moved cycles do not depend on the exact choice of this local trivialization. In particular, if we fix a point $\{X_0\} \in \text{Reg}$ and a class $\gamma \in H_n(\mathbb{C}^n, A \cup X_0)$, then a function $\beta_\gamma(X)$ arises in any simply-connected neighborhood of $\{X_0\}$ in Reg: its value at the point X is equal to the integral of the form (2) along the cycle obtained from γ by the Gauss-Manin connection over any path connecting X_0 and X in our neighborhood. This function is analytic in this neighborhood, and can be continued to an analytic function on entire manifold Reg. The ramification of this function along paths in Reg depends on the monodromy action of the group $\pi_1(\text{Reg}, \{X_0\})$ on $H_n(\mathbb{C}^n, A \cup X_0)$.

For instance, let $X_0 = X(\tau) \in \text{Reg}$ be the hyperplane $l^{-1}(\tau) \subset \mathbb{C}^n$, $\tau \in (m, M) \subset \mathbb{R}^1$, and $\gamma = \gamma(\tau) \in H_n(\mathbb{C}^n, A \cup X(\tau))$ the class of the figure $D \cap l^{-1}((-\infty, \tau])$ oriented by the form (2). Denote by $\mathcal{R}eg$ the set of all $t \in \mathbb{C}^1$ such that $X(t) \equiv$

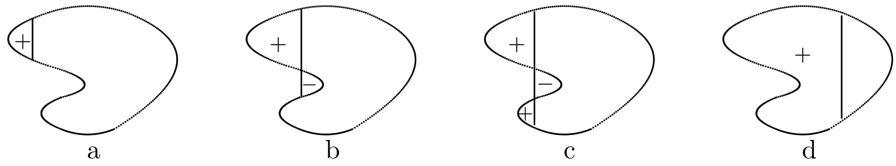


FIGURE 2. Transformations of the integration cycle in $\mathbb{R}^2$

$l^{-1}(t) \in$ Reg. For all $t \in \mathcal{R}eg$ close to τ, the function $\beta_\gamma(X(t))$ coincides with the volume function $V(t)$, hence their analytic continuations to entire $\mathcal{R}eg$ also coincide. So, in order to investigate the ramification of the volume function, let us study the orbit of this element γ under the monodromy action of $\pi_1(\mathcal{R}eg)$.

It follows easily from the Picard-Lefschetz formula (see e.g. [12]) that the loop $\alpha(m) \in \pi_1(\mathcal{R}eg)$ (see Fig. 1) moves the element γ to $-\gamma$, and the loop $\alpha(M)$ moves γ to $2[D] - \gamma$, where $[D]$ is the homology class of the entire domain D. Hence the composition of these two operators moves γ to $\gamma + 2[D]$. The relative cycle $[D]$ does not depend on t and is invariant under the action of the group $\pi_1(\mathcal{R}eg)$, hence iterating the operator $\alpha(M) \circ \alpha(m)$ we obtain consecutively $\gamma + 4[D]$, $\gamma + 6[D]$, etc. But the volume of D is positive, hence our analytic function takes infinitely many different values at one and the same point τ, and cannot be algebraic.

Remark 2. Picard-Lefschetz operators act differently in spaces $\mathbb{R}^n$ of different parities, because the intersection form in the middle homology group of an $(n-2)$-dimensional complex manifold is symmetric if n is even and antisymmetric if n is odd. In particular, if n is odd, then the loops $\alpha(m)$ and $\alpha(M)$ act trivially on $\gamma(t)$, which makes the Archimedes' example possible.

2.2. **General compact domain with smooth algebraic boundary in $\mathbb{R}^2$.** Consider first the sample domain D shown in Fig. 2. Let $l : \mathbb{R}^2 \to \mathbb{R}^1$ be the projection downwards in the page, where the target line $\mathbb{R}^1$ is oriented to the right. The restriction of this function to ∂D is a strictly Morse function. Let $\tau \in \mathbb{R}^1$ be a value greater than the global minimum of l on ∂D, but lower than all other its critical values. For real $t \approx \tau$, let $\gamma(t) \in H_2(\mathbb{C}^2, A \cup X(t))$ be the class of the positively oriented domain $D \cap l^{-1}((-\infty, t])$, see Fig. 2a. Let us increase t until the right-hand boundary segment of this domain meets a critical point of $l|_{\partial D}$, shortly before this meeting move t into the complex domain in $\mathbb{C}^1$ and go around the corresponding critical value. The Gauss-Manin connection over entire this path moves $\gamma(t)$ into the homology class in $H_2(\mathbb{C}^2, A \cup X(t))$ of the sum of two domains as shown in Fig. 2b; notice that these two domains should be taken with opposite orientations, so that their common boundary in $X(t)$ is a single oriented segment. Further, we decrease t until one of endpoints of this segment becomes again a critical point of $l|_{\partial D}$, turn t around the corresponding critical value (see Fig. 2c), etc. It is important that when t passes for the second time the critical value at the local (but not global) maximum of l on ∂D (between Figs. 2c and 2d), the corresponding leaf of the integral of $dx_1 \wedge dx_2$ along $\gamma(t)$ has a regular point: indeed, the derivative of this integral over t is equal to the length of the boundary segment in $X(t)$. Therefore we do not need to make a whole circle around this critical value: instead, we miss this critical value by a half-circle in either half-plane of $\mathbb{C}^1$ and continue to increase t along the real line, see Fig. 2d.

Finally our boundary segment in $X(t)$ shrinks to a point, namely to the global maximum point of l on ∂D. At the instant of this degeneration the cycle obtained from $\gamma(t)$ by the Gauss-Manin connection over our path would coincide with entire domain D. However we do not let t to meet the maximum value, but shortly before this rotate t again in the complex domain, and then go back along the same path. When t returns to the initial position τ, the Gauss–Manin connection over the obtained loop in $\mathcal{R}eg$ moves the cycle $\gamma(\tau)$ to $2[D] - \gamma(\tau)$. The rest of the proof repeats that in the previous example.

In the case of a general connected domain $D \subset \mathbb{R}^2$ with smooth boundary, we proceed in the similar way. Namely, we take the exterior component $\overline{\partial} D$ of ∂D (which is homeomorphic to S^1) and a linear function $l : (\mathbb{C}^n, \mathbb{R}^n) \to (\mathbb{C}, \mathbb{R})$, whose restriction to $\overline{\partial} D$ is Morse. Denote by $m < M$ the global extremal values of l on $\overline{\partial} D$. The corresponding global extremal points of l break $\overline{\partial} D$ into two segments. Consider the set of pairs $(a, b) \in \overline{\partial} D \times \overline{\partial} D$ such that a belongs to one such part and b to the other, and $l(a) = l(b)$. It is easy to see that this set is a smooth one-dimensional submanifold with boundary, and this boundary consists of two points (a, a) where a is either the global minimum or the global maximum point of l on ∂D. Although this submanifold can be not connected, its boundary points should belong to one and the same its connected component diffeomorphic to a segment. So, we start from its point (a, b) where $\{a, b\} = \partial D \cap l^{-1}(\tau)$, $\tau \approx m$, which is close to one endpoint of this segment, and go along this segment towards the other endpoint, watching the corresponding common values of l and making appropriate rotations in $\mathcal{R}eg$ each time when a or b approaches a critical point of $l|_{\overline{\partial} D}$. This gives us a path in $\mathcal{R}eg$. Gauss-Manin connection over this path moves the cycle $\gamma(\tau)$ into the entire domain bounded by $\overline{\partial} D$; in particular the value of the analytic continuation of the volume function $V(t)$ along this path close to its endpoint is almost equal to the area of this domain. Then we turn in $\mathbb{C}^1$ around the maximal value M of l on ∂D and come back to τ along the same path. The monodromy along this path in $\mathcal{R}eg$ moves the homology class $\gamma(\tau)$ into $2[D] - \gamma(\tau)$. The rest of the construction is the same as above.

Remark 3. To be rigorous, there can be other points of $\mathbb{C}^1 \setminus \mathcal{R}eg$ on the segment $[m, M]$ apart from the critical values of the restriction of l to $\overline{\partial} D$. Therefore, constructing our path in $\mathcal{R}eg$, we need to avoid these points along small arcs in the complex domain. This does not affect our consideration, because our integration cycle (and its area function) behave regularly when t passes these additional points.

3. Proof of the main theorem

3.1. **On finite reflection groups.** We use the following well-known facts about finite reflection groups, see e.g. [7], [5].

Proposition 1. *Let $\mathbb{Z}^N$ be an integer lattice with integer-valued symmetric bilinear form $\langle \cdot, \cdot \rangle$; let $\{e_j\} \subset \mathbb{Z}^N$ be a finite collection of elements of length $\sqrt{2}$ (i.e. $\langle e_j, e_j \rangle = 2$ for each j) generating the entire $\mathbb{Z}^N$ as a $\mathbb{Z}$-module. Let G be the subgroup in $SL(N, \mathbb{Z})$ generated by reflections corresponding to these elements e_j, acting on $\mathbb{Z}^N$ by the formula*

$$R_j : a \mapsto a - \langle e_j, a \rangle e_j. \qquad (3)$$

Suppose that the orbits in $\mathbb{Z}^N$ of all generating elements e_j under the action of the group G are finite. Then

1. *The group G is finite.*
2. *The form $\langle \cdot, \cdot \rangle$ is non-degenerate*[2] *: if $\langle e_j, a\rangle = 0$ for any j, then $a = 0$.*

3.2. **Reflection group related with a smooth semialgebraic domain in $\mathbb{R}^{2k}$.** Suppose that n is even. Let D be a bounded connected domain with semialgebraic non-singular boundary in $\mathbb{R}^n$. This boundary can consist of several connected components, but exactly one of them separates the entire D and all other components of ∂D from the infinity. Let us denote this component by $\overline{\partial} D$. Let A be the complexification of $\overline{\partial} D$: this is a hypersurface in $\mathbb{C}^n$, which can have singularities apart from a neighborhood of D. Let us fix some complex semialgebraic Whitney stratification of A (see e.g. [8]), all whose strata of dimension $< n-1$ do not meet ∂D.

Definition 1. A linear function $l : (\mathbb{C}^n, \mathbb{R}^n) \to (\mathbb{C}^1, \mathbb{R}^1)$ with real coefficients is *generic* with respect to the hypersurface A if

1. Its restriction to ∂D is a strictly Morse function;
2. For any critical value $t \in \mathbb{R}^1$ of this restriction, the corresponding complex hyperplane $X(t) = l^{-1}(t) \subset \mathbb{C}^n$ is transversal to all strata of A apart from the corresponding critical point of this restriction.

Generic linear functions are dense in the space of all linear functions; let l be one of them. Recall the notation $\mathcal{R}eg$ for the set of all values $t \in \mathbb{R}^1$ such that the plane $X(t)$ is transversal to A.

Let $a_j \in \overline{\partial} D$ be a critical point of $l|_{\overline{\partial} D}$, and t_j the corresponding critical value $l(a_j)$. For any sufficiently small $\varepsilon > 0$ and any $\nu > 0$ sufficiently small with respect to ε, denote by $B_\varepsilon(a_j)$ the ball of radius ε in $\mathbb{C}^n$ centered at a_j, and suppose that $t \in \mathbb{C}^1$ is an arbitrary point in the punctured ν-neighborhood of t_j. Then by Milnor's theorem (see e.g. [5]) the (reduced modulo a point) homology group $\tilde{H}_{n-2}(B_\varepsilon(a_j) \cap A \cap X(t))$ is isomorphic to $\mathbb{Z}$. The chain of isomorphisms

$$\begin{aligned} &H_n(B_\varepsilon(a_j), B_\varepsilon(a_j) \cap (A \cup X(t))) \to H_{n-1}(B_\varepsilon(a_j) \cap (A \cup X(t))) \to \\ &\to H_{n-1}(B_\varepsilon(a_j) \cap X(t), B_\varepsilon(a_j) \cap A \cap X(t)) \to \tilde{H}_{n-2}(B_\varepsilon(a_j) \cap A \cap X(t)) \end{aligned} \tag{4}$$

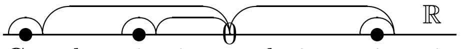
(in which the first and the third arrows are boundary homomorphisms, and the second one is a fragment of the exact sequence of the pair $(B_\varepsilon(a_j) \cap (A \cup X(t)), B_\varepsilon(a_j) \cap A))$ proves the same for its left-hand group

$$H_n(B_\varepsilon(a_j), B_\varepsilon(a_j) \cap (A \cup X(t))). \tag{5}$$

The monodromy action on these groups, defined by the rotation of t around the value t_j in its ν-neighborhood, commutes with all these isomorphisms and takes one generator of this group (5) into the other (i.e. it is multiplication by -1).

Let us choose an arbitrary such generator δ_j for $t = t_j + \nu/2$, extend this choice by the Gauss-Manin connection to similar groups (5) for all $t \in (t_j, t_j + \nu)$, and define the function $\varphi_j : (t_j, t_j + \nu) \to \mathbb{C}^1$, whose value $\varphi_j(t)$ is equal to the integral of the form (2) along this generator of (5). It is easy to calculate that this function is real or purely imaginary depending on the parity of the Morse index of $l|_{\partial D}$ at a_j, and its absolute value vanishes asymptotically as $(t - t_j)^{(n+1)/2}$ when t tends to t_j. Also, it is analytic and can be extended to a neighborhood of the point $t_j + \nu/2$ in $\mathbb{C}^1$. Let again $[m, M]$ be the segment of values $l(\overline{\partial} D) \subset \mathbb{R}^1$. Since the set $\mathbb{C}^1 \setminus \mathcal{R}eg$ is finite, there is a neighborhood $U([m, M])$ of this segment, such that

[2] and even positive definite, which is less important for us now

all points of this set in $U([m,M])$ are real; they include the set Σ of critical values of $l|_{\overline{\partial D}}$. We can and will assume that $0 \in [m,M] \cap \mathcal{R}eg$. Let us connect the point 0 with all points $t_j + \nu/2$ by arbitrary paths in the domain $U([m,M]) \cap \mathcal{R}eg \cap \{t : Im(t) > 0\}$: 

. Define the function germs $\Phi_j : (\mathbb{C}^1, 0) \to \mathbb{C}$ at the point 0 as analytic continuations of functions φ_j along these paths. Their values at 0 are equal to the integrals of (2) along the relative cycles $\tilde{\Delta}_j$ in $\mathbb{C}^n$ mod $(A \cup X(0))$ obtained from the chosen generators of (5) by the Gauss-Manin connection over these paths. If we realise this Gauss-Manin connection by the local trivialization of the fiber bundle of pairs $(\mathbb{C}^n, A \cup X(t))$ following from the Thom's isotopy lemma, then the closures of these cycles $\tilde{\Delta}_j$ do not meet the singular locus of A. Denote by $\overset{\circ}{A}$ the non-singular part of A, then the cycles obtained in this way define certain elements $\Delta_j \in H_n(\mathbb{C}^n \setminus \mathrm{sing}(A), \overset{\circ}{A} \cup X(0))$.

Let $\mathfrak{F}$ be the subgroup of the space of all complex-valued function germs at 0, consisting of integer linear combinations of our function germs Φ_j. This subgroup is the image of a homomorphism of an integer lattice (consisting of all integer linear combinations of symbols Φ_j) into the space of germs, hence also an integer lattice. For any i, j define the scalar product $\langle \Phi_i, \Phi_j \rangle$ as follows: we take two $(n-2)$-dimensional cycles in $\overset{\circ}{A} \cap X(0)$ obtained from Δ_i and Δ_j by the composite homomorphism

$$\begin{aligned} H_n(\mathbb{C}^n \setminus \mathrm{sing}(A), \overset{\circ}{A} \cup X(0)) \to H_{n-1}(\overset{\circ}{A} \cup X(0)) \to \\ \to H_{n-1}(X(0), \overset{\circ}{A} \cap X(0)) \to H_{n-2}(\overset{\circ}{A} \cap X(0)) \end{aligned} \tag{6}$$

similar to (4), calculate the intersection index of these $(n-2)$-dimensional cycles in the complex variety $\overset{\circ}{A} \cap X(0)$, and multiply this intersection index by $(-1)^{1+n/2}$. Since n is even, $\langle \Phi_i, \Phi_j \rangle = \langle \Phi_j, \Phi_i \rangle$ for any i, j. The scalar squares $\langle \Phi_j, \Phi_j \rangle$ of all generating elements Φ_j are equal to 2 (see e.g. [5], §II.1.3).

Lemma 2. *The scalar products $\langle \Phi_i, \Phi_j \rangle$ can be extended by linearity (in a unique way) to a symmetric bilinear form on the lattice $\mathfrak{F}$.*

Proof. Suppose that the linear combination $c_1\Phi_1 + \cdots + c_r\Phi_r$ defines the identically zero function germ. Consider the loop in $U([m,M]) \cap \mathcal{R}eg$, consisting of the distinguished path from 0 to $t_j + \nu/2$, rotation around t_j along the circle of radius $\nu/2$, and return to 0 by our distinguished path. By the Picard-Lefschetz formula, the *analytic continuation of our zero germ* along this path turns it into the germ Φ_j taken with the coefficient $\pm\langle (c_1\Phi_1 + \cdots + c_r\Phi_r), \Phi_j \rangle$. The germ Φ_j is not equal to identical zero, hence this coefficient should vanish for any j. □

Each basic element Φ_j defines a reflection in the space $\mathfrak{F}$ acting by

$$\Phi \mapsto \Phi - \langle \Phi_j, \Phi \rangle \Phi_j; \tag{7}$$

by Picard–Lefschetz formula it describes the analytic continuation of an arbitrary germ $\Phi \in \mathfrak{F}$ along the loop constructed in the proof of lemma 2. These reflections preserve the scalar product $\langle \cdot, \cdot \rangle$ and thus generate a subgroup of the group of automorphisms of $(\mathfrak{F}; \langle \cdot, \cdot \rangle)$.

Proposition 2. *If the domain D is algebraically integrable, then the subgroup generated by operators (7) in the group of automorphisms of the lattice $\mathfrak{F}$ is finite.*

Proof. The set of all complex hyperplanes $X(t) = l^{-1}(t)$, $t \in \mathbb{C}^1$, forms a line in the space $\mathbb{C}P_n$. We can assume that ν is small enough, so that $(m, m+\nu) \subset \mathcal{R}eg$. Let $\tau = m + \nu/2$, and denote by $\gamma(X(\tau))$ the class in $H_n(\mathbb{C}^n, A \cup X(\tau))$ of the positively oriented domain $D \cap l^{-1}((-\infty, \tau])$. Consider also the analytic function Vol on Reg $\subset \mathbb{C}P_n$ defined by integrals of the form (2) along the similar cycles $\gamma(X) \in H_n(\mathbb{C}^n, A \cup X)$ obtained from $\gamma(X(\tau))$ by Gauss-Manin connection over different paths in Reg connecting $X(\tau)$ and X.

Lemma 3. *The set of restrictions of different leaves of the function* Vol *to neighborhoods of the point* 0 *in the line* $\mathbb{C}^1 \subset \mathbb{C}P_n$ *of planes* $X(t)$ *contains all function germs* $\Phi_j : (\mathbb{C}^1, 0) \to \mathbb{C}^1$ *corresponding to all critical points of the restriction of* l *to* $\overline{\partial}D$.

Proof of lemma 3. It is enough to prove that for any j the integral functions $\varphi_j : (t_j, t_j + \nu) \to \mathbb{C}^1$ defined above appears among restrictions of the function Vol to the interval $(t_j, t_j + \nu) \subset \mathbb{C}^1$. Consider the set of all non-singular points of the hypersurface A, at which the second fundamental form of this hypersurface is non-degenerate. Its complement has complex codimension 1 in A, hence this set is path-connected within any irreducible component of A. Since $l|_{\overline{\partial}D}$ is a Morse function, this set contains all its critical points (and all of them obviously belong to one and the same component of A). Choose a smooth path in this set, connecting the critical points with values m and t_j. The tangent hyperplanes at these points define a path in $\mathbb{C}P_n$. Take a path in the set Reg $\subset \mathbb{C}P_n$ escorting this one in its thin neighborhood and connecting the planes $X(\tau)$ and $X(t_j + \nu/2)$ in such a way that for any point $\{X\}$ of this path the element of the group (1) obtained from γ by the Gauss-Manin connection over this path is realised by a cycle inside a small ball centered at the neighboring tangency point of A and a hyperplane parallel to X, in particular this cycle for the endpoint of this path generates the group (5). This terminal cycle can coincide with the integration cycle defining the function φ_j or be opposite to it. In the first case φ_j is defined by the analytic continuation of the function Vol along this path, in the second one we need additionally go once around the critical value t_j in the set $\mathcal{R}eg$ of generic hyperplanes $X(t)$. □

Remark 4. In this lemma, we do not state that any germ Φ_j can be obtained from the initial function at the point τ by the analytic continuation along a path inside our line $\mathbb{C}^1$. However, if we add to genericity conditions of Definition 1 the claim that the embedding $\mathbb{C}^1 \cap \text{Reg} \to \text{Reg}$ induces an epimorphism of fundamental groups (which we can do by Zariski theorem), then this fact will also be true (although the path defining this continuation can partly run far away from the domain $U([m, M])$). In particular, the reflection group on $\mathfrak{F}$ generated by operators (7) (defined by paths inside this domain only) can be reducible, see subsection 3.3 below.

Remark 5. The path in $\mathbb{C}P_n$ used in this proof can be chosen in an arbitrarily small neighborhood of the set of (complexifications of) real hyperplanes. Indeed, let us connect our critical points of $l|_{\overline{\partial}D}$ by a generic path inside our component of $\overline{\partial}D$; then to avoid the set of parabolic points of A we can move this path slightly into the complex domain in arbitrarily small neighbourhoods of points at which it crosses this set.

Further, the set of analytic continuations of all function germs Φ_j contains all germs obtained from them by reflections (7). Therefore if the domain D is algebraically integrable then the orbit of any element Φ_j under the reflection group generated by operators (7) is finite. By proposition 1 the entire this reflection group is then finite, in particular the symmetric form $\langle\cdot,\cdot\rangle$ on $\mathfrak{F}$ is non-degenerate. Proposition 2 is proved. □

Proposition 3. *The class* $[\overline{D}] \in H_n(\mathbb{C}^n, A \cup X(0))$ *of the entire domain in* $\mathbb{R}^n$ *bounded by* $\overline{\partial}D$ *is equal to the sum of classes* Δ_j *(taken with appropriate signs) over all critical points of* $l|_{\overline{\partial}D}$*. In particular, the corresponding sum of function germs* $\pm\Phi_j$ *is the constant function equal identically to the volume bounded by* $\overline{\partial}D$*.*

This fact follows immediately from the following one (Lemma 4 below). For any critical value t_j of $l|_{\overline{\partial}D}$ we have two elements in the group $H_n(\mathbb{C}^n, A \cup X(t_j + \nu/2))$: one (let us call it ∇_+) given by the positively oriented part of the half-space $\mathbb{R}^n \cap l^{-1}((-\infty, t_j+\nu/2])$ bounded by $\overline{\partial}D$, and the other, $J(\nabla_-)$, obtained from the similar class $\nabla_- \in H_n(\mathbb{C}^n, A \cup X(t_j - \nu/2))$ by the Gauss–Manin connection over the arc of radius $\nu/2$ in the upper half-plane of $\mathbb{C}^1$.

Consider also absolute homology classes $\Pi_\pm \in H_{n-2}(A \cap X(t_j \pm \nu/2))$ represented by naturally oriented real manifolds $\overline{\partial}D \cap X(t_j \pm \nu/2)$ respectively. These classes are obtained from $\nabla_\pm$ by maps similar to the composition (6).

Lemma 4 (see [12], Lemma 3.3 on page 121)**.** *1. The difference of two relative homology classes* ∇_+ *and* $J(\nabla_-)$ *in* $H_n(\mathbb{C}^n, A \cup X(t_j+\nu/2))$ *is equal to the image of the vanishing cycle* $\delta_j \in H_n(B_\varepsilon(a_j), A \cup X(t_j+\nu/2))$ *under the identical embedding* $B_\varepsilon(a_j) \hookrightarrow \mathbb{C}^n$*.*

2. The difference in the group $H_{n-2}(A \cap X(t_j + \nu/2))$ *of the class* Π_+ *and the element obtained from* Π_- *by the Gauss–Manin connection over the arc of radius* $\nu/2$ *in the upper half-plane of* $\mathbb{C}^1$ *can be realized by a vanishing cycle generating the group* $H_{n-2}(B_\varepsilon(a_j) \cap A \cap X(t_j + \nu/2)) \simeq \mathbb{Z}$*.* □ □

Remark 6. The earliest (very technical) proof of item 2 known to me is given in [10]; for another proof, deducing it directly from the Picard-Lefschetz formula, see [13] and §V.3 of [12].

Since the volume bounded by $\overline{\partial}D$ is positive, the sum $\sum_j \pm\Phi_j$ mentioned in proposition 3 is a non-zero element of the lattice $\mathfrak{F}$. On the other hand, the image of the corresponding cycle $[\overline{D}]$ under the map (6) is equal to zero, because the image of this cycle under the first arrow in (6) is a cycle in A, and the second arrow in (6) is the reduction modulo A. Therefore this non-zero element of $\mathfrak{F}$ belongs to the kernel of the scalar product $\langle\cdot,\cdot\rangle$. Thus by proposition 1 the group G cannot be finite. Theorem 1 is completely proved. □

3.3. **Reducibility of the reflection group.** The reflection group

$$(\mathfrak{F}; \langle\cdot,\cdot\rangle; \{R_j\}) \tag{8}$$

can be reducible, for instance this is the case if D is a thin tubular neighborhood of the standard circle embedded into $\mathbb{R}^4$. In this case we can choose the linear function $l : \mathbb{R}^4 \to \mathbb{R}$ in such a way that its restriction to ∂D has four critical points a_j, $j = 1, 2, 3, 4$ with Morse indices $0, 1, 2$ and 3 respectively, and the critical values t_2 and t_3 at the points a_2 and a_3 almost coincide. Then we have the following calculation.

Proposition 4. *For appropriate choice of orientations of vanishing cycles* $\beta_i \in H_2(B_\varepsilon(a_j) \cap A \cap X(t_j + \nu/2)) \simeq \mathbb{Z}$*, the intersection matrix in* $H_2(A \cap X(0))$ *of cycles obtained from them by the Gauss-Manin connection over any paths in the upper half of* $U([m, M]) \cap \mathcal{R}eg$ *(see picture in page 7) is equal to*

$$\begin{vmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{vmatrix}.$$

In particular, in this case the entire action of the reflection group (8) splits into the direct sum of two-dimensional ones, each of which is isomorphic to one arising in the example of subsection 2.1.

Sketch of the proof. $\langle \beta_2, \beta_3 \rangle = 0$ because the corresponding critical points are distant, but their critical values can be made arbitrarily close to one another by bending the function l (which does not affect the intersection index). Further, we can assume that 0 is slightly above the critical values t_2 and t_3. In this case the cycle in the variety $A \cap X(0)$ obtained by the Gauss–Manin connection from β_4 is presented by the set $\partial D \cap X(0)$ of all its real points. Its intersection with the vanishing cycles β_3 and β_2 can be calculated in local terms of Morse critical points a_2 and a_3 (see [9], [1], [12]). Replacing 0 by a value slightly below t_2 and t_3 (which does not affect the intersection indices of cycles obtained by Gauss–Manin connection in the upper half-plane) we obtain similar indices for cycles obtained from β_1, β_2 and β_3.

Finally, we know from lemma 4 that the cycle of real points in $A \cap X(t_4 - \nu/2)$ can be realised as the sum of three cycles obtained by Gauss-Manin connection in the upper half-plane from β_1, β_2 and β_3. Hence, by the previous calculations, the cycle obtained from β_4 has with this sum the same intersection index, as with β_3 only. This gives us zero for the remaining corner elements $(1, 4) = (4, 1)$ of the intersection matrix. □

In general, if the reflection group (8) is reducible, then the set of all cycles Δ_j splits into collections of cycles generating germs Φ_j that belong to different irreducible components. It is easy to see that the sum of all function germs corresponding to all cycles Δ_j from any such collection belongs to the kernel of the form $\langle \cdot, \cdot \rangle$. Indeed, this sum for one such collection is obviously orthogonal to all generators Φ_j from the other collections. On the other hand, this sum is equal to the difference of the sum of all Φ_j over all collections and the sum of such sums over all collections except for this one; both sums are orthogonal to all Φ_j from our collection.

References

[1] N. A'Campo, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes. I*, Ann. Math. 213:1 (1975), 1-32.

[2] V.I. Arnold, *Kepler's second law and the topology of Abelian integrals*, Kvant, 1987, 12, 17–21 (Russian). Eng. translation in Kvant Selecta: Algebra and Analysis II, Math. World, vol. 15. Amer. Math. Soc., Providence, RI, 1999, 131-140.

[3] V.I. Arnold, V.A. Vassiliev, *Newton's Principia read 300 years later*, Notices AMS, **36**:9 (1989), 1148-1154.

[4] V.I. Arnold, V.A. Vassiliev, *Addendum to* [3], Notices AMS, **37**:1, 144.

[5] V.I. Arnold, V.V. Goryunov, O.V. Lyashko, V.A. Vassiliev, *Singularity Theory 1. Local and Global Theory.* Encycl. of Math. Sci., Vol. 6, Springer-Verlag, 1993.
[6] V.I. Arnold a.o., *Arnold's Problems*, Springer-Verlag, 2004.
[7] N. Bourbaki, *Groupes et algébres de Lie. Ch. IV, V, VI*, Hermann, Paris, 1968.
[8] M. Goresky, R. MacPherson, *Stratified Morse Theory*, Springer-Verlag, 1988.
[9] S.M. Gusein-Zade, *Intersection matrices for some singularities of functions of two variables,* Funct. Anal. and its Appl., 8 (1974), 10-13.
[10] J. Leray, *Un prolongement de la transformation de Laplace qui transforme la solution unitaire d'un opérateur hyperbolique en sa solution élémentaire (Probleme de Cauchy, IV).* Bull Soc. Math. France, 1962, V. 90, 39-156.
[11] I. Newton, *Philosophiae Naturalis Principia Mathematica*, London, 1687.
[12] V.A. Vassiliev, *Applied Picard-Lefschetz Theory*, Amer. Math. Soc., Providence, RI,, 2002.
[13] V.A. Vassiliev, *Sharpness and the local Petrovskii condition for hyperbolic equations with constant coefficients*, Mat. USSR Izvestiya **28**:2 (1987), 233-273.
[14] H. Brougham and E.J. Routh, Analytical View of Sir Isaac Newton's Principia, London, 1855.

Steklov Mathematical Institute and National Research University Higher School of Economics, Moscow, Russia
E-mail address: vva@mi.ras.ru