Stratified Picard-Lefschetz Theory with Twisted Coefficients

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To V. I. Arnold on the occasion of his 60-th anniversary (1997)

Abstract

The monodromy action in the homology (generally with twisted coefficients) of complements of stratified complex analytic varieties depending on parameters is studied. For a wide class of local degenerations of such families (stratified Morse singularities) local monodromy and variation operators are reduced to similar operators acting in the transversal slices of corresponding strata. These results imply a main part of generalized Picard–Lefschetz formulae of [Ph1] and (in the case of constant coefficients) similar reduction theorems of [V1], [V2].

Introduction

Let M^n be a complex analytic manifold and Λ_τ a family of subvarieties in M^n depending analytically on the parameter τ. For almost all values of τ the corresponding pairs (M^n, Λ_τ) are homeomorphic to one another, and the fundamental group of the set of such generic τ acts on different homology groups related to such pairs. This action is responsible for the ramification and qualitative analytic features of all known functions given by integral representations (such as Fourier and Radon transforms, fundamental solutions of hyperbolic and parabolic equations, Newton–Coulomb potentials, hypergeometric functions, Feynman integrals etc.). Explicit formulae expressing this action are called (generalized) Picard–Lefschetz formulae, see e.g. [AVG 1, 2], [AVGL 1, 2], [Ph1, Ph2], [Giv], [V1].

The classical Picard–Lefschetz formula describes the case when Λ_τ is the family of irreducible hypersurfaces in M^n, non-singular for generic τ and having a Morse singularity for τ running over a hypersurface in the parameter space, the investigated loop is a small circle embracing this hypersurface, and the homology group in question is $H_{n-1}(\Lambda_\tau)$.

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Similar formulae for many other possible degenerations of the variety Λ_τ and other homology groups were studied in [Ph1] (see also [Giv]) and in [V2]. In the present paper we consider the most general situation: as in [V1], [V2] we investigate the *stratified Morse singularities* of Λ_τ (which include all degenerations of [Ph1] as special cases), and as in [Ph1], [Ph2] we consider the homology groups of $M^n \setminus \Lambda_\tau$ (with twisted coefficients, and with both closed and compact supports), to which the study of almost all other homology groups related with the pair (M^n, Λ_τ) can be reduced (except may be for the intersection homology of Λ_τ studied in the last part of [V2]).

The Arnold's "complexification" functor (see e.g. [A]), establishing an informal analogy between objects of "real" and "complex" worlds, maps the Morse theory into the Picard–Lefschetz theory; certainly, the analogue of the stratified Morse theory of [GM] should be a stratified Picard–Lefschetz theory. On the other hand, recently it became clear that instead of the homology groups of (sub)varieties, usually considered in all these theories, in the "complex" situation it is natural to deal with the homology, generally with coefficients in non-constant local systems, of their complements, cf. [Ph2], [GVZ], [Giv]. Thus the matter of our paper seems to be the most adequate "complexification" of that of [GM].

Agreements. In what follows we assume that $M^n = \mathbf{C}^n$ (which is not restricting because all our considerations are local), and families Λ_τ (which appear later as $A \cup X_t$) depend on one parameter. We consider only homology groups reduced modulo a point in the case of absolute homology and modulo the fundamental cycle in the case of relative homology of a complex analytic variety. We often use a short notation of type $H_*(X, Y)$ instead of a more rigorous $H_*(X, Y \cap X)$ or $H_*(X \cup Y, Y)$. The sign $\Box$ denotes the end or absence of a proof.

1 Main characters and stating the problem

Let A be a complex analytic subvariety in $\mathbf{C}^n$ with a fixed analytic Whitney stratification (see e.g. [GM], [V1]), let $\sigma \subset A$ be a stratum of dimension k, a a point of σ, and $B \equiv B_\varepsilon$ a small closed disc in $\mathbf{C}^n$ centred at a, so that B has nonempty intersections only with the strata adjoining σ (or σ itself) and ∂B is transversal to the stratification. In particular (see [Che], [GM]) the induced partition of the pair $(B, \partial B)$ is again a Whitney stratification of B.

Let $f : (B, a) \to (\mathbf{C}, 0)$ be a holomorphic function such that $df \neq 0$ in B.

Definition 1 (cf. [GM]). The function f has a *Morse singularity* on A at the point a if its restriction on the manifold σ has a Morse singularity at a, and for any stratum $\tau \neq \sigma$ and any sequence of points $b_i \in \tau$ converging to a and such that the planes tangent to τ at b_i (considered as the points in the associated Grassmann bundle $G^{\dim \tau}(T_*\mathbf{C}^n)$) converge to a plane in $T_a\mathbf{C}^n$, this limit plane does not lie in the hyperplane $\{df|_a = 0\} \subset T_a\mathbf{C}^n$. If this condition is satisfied, we

Figure 1: A Morse function on a stratified variety

say that the one-parametric family of varieties $A \cup f^{-1}(t)$, $t \in \mathbf{C}$, has a *stratified Morse singularity* at a.

Notation. For any $t \in \mathbf{C}$ denote by X_t the set $f^{-1}(t)$ and by AX_t the set $A \cup X_t$.

For example, in Fig. 1b (the complexification of) the linear function defining the plane X has a stratified Morse singularity on (the complexification of) A.

Let L_α be the linear (i.e. with fibre $\mathbf{C}^1$) local system on $\mathbf{C}^n \setminus AX_t$ with the set $\alpha = (\alpha_1, \dots, \alpha_\nu)$ of ramification indices: the cardinality ν of this set is equal to the number of irreducible $(n-1)$-dimensional components of AX_t, and any small loop going around a smooth piece of the i-th component in the positive direction (with respect to the natural complex structure in the normal bundle) acts on the fibre as multiplication by α_i. We assume that X_t is the first component of AX_t, so that α_1 is responsible for the rotations around X_t.

The dual local system will be denoted by L_{α^*}, $\alpha^* = (\alpha_1^{-1}, \dots, \alpha_\nu^{-1})$. For all $t \in \mathbf{C} \setminus 0$ sufficiently close to 0 (say, satisfying the condition $|t| \leq \delta$ with sufficiently small δ) all corresponding sets $\mathbf{C}^n \setminus AX_t$ are homeomorphic to one another and form a locally trivial bundle over the set of such t.

Denote by C the loop $\{\delta e^{i\tau}\}$, $\tau \in [0, 2\pi]$, generating the group $\pi_1(\mathbf{C}^1 \setminus 0)$; the inverse loop $\{\delta e^{-i\tau}\}$ is denoted by C^*, and the closed disc bounded by any of these loops by D_δ.

The monodromy action of the loop C (and C^* as well) in two groups

$$H_*(\mathbf{C}^n \setminus AX_t, L_\alpha) \quad \textit{and} \quad H_*^{lf}(\mathbf{C}^n \setminus AX_t, L_\alpha)$$

is well defined, where the letters lf in the last expression denote the homology groups of locally finite chains. The study of this action is our main aim.

These monodromy operators can be localized in the standard way, see e.g. [Ph2], [V1]. We suppose that a is the unique point of nontransversality of the manifold X_0 and the stratified variety A, and the number δ participating in the definition of the loop C is so small that for any $t \in D_\delta \setminus 0$ the manifold X_t is smooth and transversal to the naturally stratified variety $A \cup \partial B$.

We choose $\{\delta\}$ as the distinguished point in $D_\delta \setminus 0$ and redenote X_δ and AX_δ simply by X and AX. For any non-negative integer i we consider four groups

$$\begin{array}{rclrcl} \bar{\mathcal{H}}_{i,\alpha} & \equiv & H_i^{lf}(B \setminus AX, \partial B; L_\alpha), & \mathcal{H}_{i,\alpha} & \equiv & H_i^{lf}(B \setminus AX, L_\alpha), \\ \bar{\chi}_{i,\alpha} & \equiv & H_i(B \setminus AX, \partial B; L_\alpha), & \chi_{i,\alpha} & \equiv & H_i(B \setminus AX, L_\alpha). \end{array} \quad (1)$$

There are obvious homomorphisms

$$\begin{array}{rclrcl} \tilde{i}_\alpha : \mathcal{H}_{*,\alpha} & \to & H_*^{lf}(\mathbf{C}^n \setminus AX, L_\alpha), & \tilde{j}_\alpha : H_*^{lf}(\mathbf{C}^n \setminus AX, L_\alpha) & \to & \bar{\mathcal{H}}_{*,\alpha}, \\ i_\alpha : \chi_{*,\alpha} & \to & H_*(\mathbf{C}^n \setminus AX, L_\alpha), & j_\alpha : H_*(\mathbf{C}^n \setminus AX, L_\alpha) & \to & \bar{\chi}_{*,\alpha}. \end{array} \quad (2)$$

Also, any loop $\lambda \in \pi_1(D_\delta \setminus 0)$ defines in the standard way (see [AVGL 1, 2], [V1]) the *local variation operators*

$$\widetilde{Var}_{(\lambda)} : \bar{\mathcal{H}}_{*,\alpha} \to \mathcal{H}_{*,\alpha}\,, \quad Var_{(\lambda)} : \bar{\chi}_{*,\alpha} \to \chi_{*,\alpha}\,, \quad (3)$$

so that the monodromy action of the loop λ in the group $H_*^{lf}(\mathbf{C}^n \setminus AX, L_\alpha)$ (respectively, $H_*(\mathbf{C}^n \setminus AX, L_\alpha)$) is equal to $\mathrm{Id} + \tilde{i}_\alpha \circ \widetilde{Var}_{(\lambda)} \circ \tilde{j}_\alpha$ (respectively, $\mathrm{Id} + i_\alpha \circ Var_{(\lambda)} \circ j_\alpha$).

The composition operators $\tilde{J}_\alpha \equiv \tilde{j}_\alpha \circ \tilde{i}_\alpha : \mathcal{H}_{*,\alpha} \to \bar{\mathcal{H}}_{*,\alpha}$ and $J_\alpha \equiv j_\alpha \circ i_\alpha : \chi_{*,\alpha} \to \bar{\chi}_{*,\alpha}$ allow us to express the *local monodromy* action of the loop λ on four groups $\bar{\mathcal{H}}_{*,\alpha}, \bar{\chi}_{*,\alpha}, \mathcal{H}_{*,\alpha}, \chi_{*,\alpha}$ as

$$\begin{array}{ll} Id + \tilde{J}_\alpha \circ \widetilde{Var}_{(\lambda)}, & Id + J_\alpha \circ Var_{(\lambda)}, \\ Id + \widetilde{Var}_{(\lambda)} \circ \tilde{J}_\alpha, & Id + Var_{(\lambda)} \circ J_\alpha \end{array} \quad (4)$$

respectively.

The groups (1) are related by nondegenerate Poincaré–Lefschetz pairings

$$\begin{array}{rcl} \bar{\mathcal{H}}_{i,\alpha} \otimes \chi_{2n-i,\alpha^*} & \to & \mathbf{C}, \\ \mathcal{H}_{i,\alpha} \otimes \bar{\chi}_{2n-i,\alpha^*} & \to & \mathbf{C}. \end{array} \quad (5)$$

Proposition 1. *For any i and α, the operators*

$$\begin{array}{rccl} \widetilde{Var}_{(C)} : & \bar{\mathcal{H}}_{i,\alpha} & \to & \mathcal{H}_{i,\alpha}\,, \\ Var_{(C^*)} : & \bar{\chi}_{2n-i,\alpha^*} & \to & \chi_{2n-i,\alpha^*} \end{array} \quad (6)$$

are conjugate with respect to the pairings (5), i.e., for any elements $x \in \bar{\mathcal{H}}_{i,\alpha}$ and $y \in \bar{\chi}_{2n-i,\alpha^*}$ we have

$$\langle x, Var_{(C^*)} y \rangle = \langle \widetilde{Var}_{(C)} x, y \rangle.$$

Proof (cf. [GZ]). Let x and y be some elements of the groups $\bar{\mathcal{H}}_{i,\alpha}$ and $\bar{\chi}_{2n-i,\alpha^*}$ respectively, and $\mathbf{x}$ and $\mathbf{y}$ some relative cycles representing them and intersecting generically in B (in particular having no common points in ∂B). Then the intersection number $\langle \mathbf{x}, \mathbf{y}\rangle$ is well defined (although it is not an invariant of the homology classes x and y). Let $h_C\mathbf{x}$ and $h_{C^*}\mathbf{y}$ be two relative cycles in $B \setminus AX$ obtained from $\mathbf{x}$ and $\mathbf{y}$ by the monodromy over the loops C and C^* as in the construction of variation operators (i.e., fixed close to the boundary of B). Then $\langle \widetilde{Var}_{(C)}(x), y\rangle - \langle x, Var_{(C^*)}(y)\rangle = \langle h_C\mathbf{x} - \mathbf{x}, \mathbf{y}\rangle - \langle \mathbf{x}, h_{C^*}\mathbf{y} - \mathbf{y}\rangle = \langle h_C\mathbf{x}, \mathbf{y}\rangle - \langle \mathbf{x}, h_{C^*}\mathbf{y}\rangle = \langle h_C\mathbf{x}, \mathbf{y}\rangle - \langle h_{C^*}h_C\mathbf{x}, h_{C^*}\mathbf{y}\rangle = 0$ because these intersection pairings are invariant under the monodromy action. $\square$

Set $k = \dim \sigma$, $m = n - k$, and let T be the m-dimensional complex plane through a transversal to σ. Then, replacing the sets B, A and X_t by their intersections with T, we get all structures as above (homology groups, local variation and monodromy action, intersection indices and maps $\tilde{J}_\alpha$, J_α) placed in the space of reduced dimension m. In the next section we show how, knowing all these objects for this reduced space, we can reconstruct them for the initial objects in entire space $\mathbf{C}^n$. This reconstruction will de done by induction over a flag of planes connecting T and $\mathbf{C}^n$.

2 Main results

Consider the flag of complex planes (or, more generally, smooth complex submanifolds)

$$T \equiv T^m \subset T^{m+1} \subset \cdots \subset T^n \equiv \mathbf{C}^n, \tag{7}$$

of dimensions $m, m+1, \ldots, n$ respectively, all of which intersect σ transversally at the point a. Then the intersections with the strata of A define a Whitney stratification on any of varieties $A \cap B \cap T^r$. If the flag (7) is generic, then for any $r = m, \ldots, n$ the restriction of f on $A \cap T^r$ is a Morse function in B; we shall suppose that this condition is satisfied for all r. Any loop λ in $D_\delta \setminus 0$, considered as family of planes $X_t \cap T^r$, $t \in \lambda$, defines then the variation operators

$$\widetilde{\mathrm{Var}}_{(\lambda),r} : H_i^{lf}(B \cap T^r \setminus AX, \partial B; L_\alpha) \to H_i^{lf}(B \cap T^r \setminus AX, L_\alpha), \tag{8}$$

$$\mathrm{Var}_{(\lambda),r} : H_i(B \cap T^r \setminus AX, \partial B; L_\alpha) \to H_i(B \cap T^r \setminus AX, L_\alpha), \tag{9}$$

in particular $\widetilde{\mathrm{Var}}_{(\lambda),n} = \widetilde{\mathrm{Var}}_{(\lambda)}$, $\mathrm{Var}_{(\lambda),n} = \mathrm{Var}_{(\lambda)}$.

Everywhere below we consider only operators (8), (9) defined by the loop $\lambda = C$ and denote these operators simply by Var_r.

Denote the groups participating in these operators as follows:

$$\begin{array}{ll} \bar{\mathcal{H}}_{i,\alpha}(r) \equiv H_i^{lf}(B \cap T^r \setminus AX, \partial B; L_\alpha), & \mathcal{H}_{i,\alpha}(r) \equiv H_i^{lf}(B \cap T^r \setminus AX, L_\alpha), \\ \bar{\chi}_{i,\alpha}(r) \equiv H_i(B \cap T^r \setminus AX, \partial B; L_\alpha), & \chi_{i,\alpha}(r) \equiv H_i(B \cap T^r \setminus AX, L_\alpha), \end{array} \tag{10}$$

and local monodromy operators defined by the loop C on four groups $\bar{\mathcal{H}}_{*,\alpha}(r)$, $\mathcal{H}_{*,\alpha}(r)$, $\bar{\chi}_{*,\alpha}(r)$, $\chi_{*,\alpha}(r)$ by $\bar{M}_r, M_r, \bar{\mu}_r, \mu_r$ respectively.

Of course, the analogues of the Poincaré duality (5) and Proposition 1 are valid for any r.

Theorem 1. *For any $l = 0, \dots, n-m$, and any α, there are almost canonical isomorphisms*

$$\begin{array}{rlcl} \Sigma^l : & \mathcal{H}_{i,\alpha}(m) & \to & \mathcal{H}_{i+l,\alpha}(m+l), \\ \downarrow^l : & \bar{\chi}_{i,\alpha}(m) & \to & \bar{\chi}_{i+l,\alpha}(m+l). \end{array} \tag{11}$$

The construction of these maps will be described in the next section. For the explanation of the word "almost" see Remark in § 3.3.

The stabilization of groups $\bar{\mathcal{H}}_{i+l,\alpha}(m+l)$ and $\chi_{i+l,\alpha}(m+l)$ depends very much on the fact, is the coefficient α_1 (responsible for the ramification of L_α close to the plane X_t) equal to 1 or not. To unify the corresponding formulae, set $\Phi_{i,\alpha} \equiv \psi_{i,\alpha} \equiv 0$ if $\alpha_1 \neq 1$; if $\alpha_1 = 1$, we consider the local system $L_{\hat{\alpha}}$ on $B \setminus A$, isomorphic to L_α in a small closed disc $b \subset B \setminus X_t$ with center a (i.e. having the same monodromy coefficients $\alpha_2, \dots, \alpha_\nu$ related to all components of A) and set $\Phi_{i,\alpha} \equiv H_i^{lf}(B \cap T^m \setminus A, \partial B; L_{\hat{\alpha}}) \simeq H_{i+2l}^{lf}(B \cap T^{m+l} \setminus A, \partial B; L_{\hat{\alpha}})$, $\psi_{i,\alpha} \equiv H_i(B \cap T^m \setminus A, L_{\hat{\alpha}}) \simeq H_i(B \cap T^{m+l} \setminus A, L_{\hat{\alpha}})$.

Theorem 1′. *For any $l = 1, \dots, n-m$, and any α, there are almost canonical isomorphisms*

$$\begin{array}{rcl} \bar{\mathcal{H}}_{i+l,\alpha}(m+l) & \simeq & \mathcal{H}_{i,\alpha}(m) \oplus \Phi_{i-l,\alpha} \oplus \Phi_{i-l+1,\alpha}, \\ \chi_{i+l,\alpha}(m+l) & \simeq & \bar{\chi}_{i,\alpha}(m) \oplus \psi_{i+l,\alpha} \oplus \psi_{i+l-1,\alpha}. \end{array} \tag{12}$$

For any $l = 1, \dots, n-m$, we denote by $\rightleftharpoons_l$ (respectively, ∞_l) the injection $\mathcal{H}_{i,\alpha}(m) \to \bar{\mathcal{H}}_{i+l,\alpha}(m+l)$ (respectively, $\bar{\chi}_{i,\alpha}(m) \to \chi_{i+l,\alpha}(m+l)$) defined by the first summands in the right-hand parts of (12).

Theorem 2. *For any $l = 1, \dots, n-m$,*

a) the obvious homomorphism $\tilde{J}_{\alpha,m+l} \equiv \tilde{j}_\alpha \circ \tilde{i}_\alpha : \mathcal{H}_{i+l,\alpha}(m+l) \to \bar{\mathcal{H}}_{i+l,\alpha}(m+l)$ (i.e., the reduction mod ∂B) maps any element $\Sigma^l(x)$, $x \in \mathcal{H}_{i,\alpha}(m)$, to

$\rightleftharpoons_l \circ \widetilde{Var}_m \circ \tilde{J}_{\alpha,m}(x) \equiv \rightleftharpoons_l \circ (M_m - Id)(x)$ *if l is even and to*

$- \rightleftharpoons_l (2x + \widetilde{Var}_m \circ \tilde{J}_{\alpha,m}(x)) \equiv - \rightleftharpoons_l \circ (M_m + Id)(x)$ *if l is odd.*

b) the similar homomorphism $J_{\alpha,m+l} \equiv j_\alpha \circ i_\alpha : \chi_{i+l,\alpha}(m+l) \to \bar{\chi}_{i+l,\alpha}(m+l)$ is equal to zero on the last two summands in (12) and maps any element $\infty_l(y)$, $y \in \bar{\chi}_{i,\alpha}(m)$, to

$\downarrow^l \circ J_{\alpha,m} \circ Var_m(y) \equiv \downarrow^l \circ (\bar{\mu}_m - Id)(y)$ *if l is even and to*

$- \downarrow^l (2y + J_{\alpha,m} \circ Var_m(y)) \equiv - \downarrow^l \circ (\bar{\mu}_m + Id)(y)$ *if l is odd.*

Theorem 3. *For any α and any $l = 1, \dots, n-m$,*

a) the operator $\widetilde{Var}_{m+l} : \bar{\mathcal{H}}_{i+l,\alpha}(m+l) \to \mathcal{H}_{i+l,\alpha}(m+l)$ is equal to zero on two last summands in the first row of (12) and maps the element $\rightleftharpoons_l (x)$, $x \in \mathcal{H}_{i,\alpha}(m)$, of the first summand to $\Sigma^l(x)$;

b) the operator $Var_{m+l} : \bar{\chi}_{i+l,\alpha}(m+l) \to \chi_{i+l,\alpha}(m+l)$ *maps any element* $\downarrow^l (y)$, $y \in \bar{\chi}_{i,\alpha}(m)$, *to* $\infty_l(y)$.

Corollary. *The local monodromy operators* $M_{m+l}, \bar{M}_{m+l}, \bar{\mu}_{m+l}$ *and* μ_{m+l} *respectively map the elements* $\Sigma^l(x), \rightleftharpoons_l (x), \downarrow^l (y)$ *and* $\infty_l(y)$ *(where* $x \in \mathcal{H}_{*,\alpha}(m)$, $y \in \bar{\chi}_{*,\alpha}(m)$*) into the elements* $(-1)^l\Sigma^l(M_m(x))$, $(-1)^l \rightleftharpoons_l (M_m(x))$, $(-1)^l \downarrow^l (\bar{\mu}_m(y))$ *and* $(-1)^l\infty_l(\bar{\mu}_m(y))$ *respectively. Operators* $\bar{M}_{m+l}$ *and* μ_{m+l} *act trivially (i.e. as the identity operators) on the last two summands in both formulae (12).*

Theorem 4. *For any* $l = 1, \ldots, n-m$,
A) for any $x \in \mathcal{H}_{i,\alpha}(m)$ *and* $y^* \in \bar{\chi}_{i,\alpha^*}(m)$,

$$\langle \Sigma^l(x), \downarrow^l (y^*) \rangle = (-1)^{l(i+1+(l-1)/2)} \langle x, y^* \rangle;$$

B) for any $\xi \in \bar{\mathcal{H}}_{i+l,\alpha}(m+l)$ *and* $\zeta^* \in \chi_{2m-i+l,\alpha^*}(m+l)$,

1. $\langle \xi, \zeta^* \rangle = 0$ *if* ξ *and* ζ^* *belong to summands in the right-hand part of (12) placed not one over the other;*
2. *If* $\xi = \rightleftharpoons_l (x)$, $x \in \mathcal{H}_{i,\alpha}(m)$, *and* $\zeta^* = \infty_l(y^*)$, $y^* \in \bar{\chi}_{i,\alpha^*}(m)$, *then* $\langle \xi, \zeta^* \rangle = (-1)^{il+1+l(l-1)/2} \langle x, \bar{\mu}_m(y^*) \rangle$.

Corollary (periodicity theorem). *Suppose that the groups* $\Phi_{*,\alpha}$, $\psi_{*,\alpha}$ *are trivial (e.g.* $\alpha_1 \neq 1$*). Then all maps* $\rightleftharpoons_l$, ∞_l *are isomorphisms, and the entire structure depending on* A, α *and* r *and consisting of four graded groups* $\bar{\mathcal{H}}_{*,\alpha}(r)$, $\mathcal{H}_{*,\alpha}(r)$, $\bar{\chi}_{*,\alpha}(r)$, $\chi_{*,\alpha}(r)$, *similar four groups with* α *replaced by* α^*, *all possible intersection pairings between them, and the operators* $\tilde{J}_\alpha$, J_α, $\widetilde{Var}_r$, Var_r, $M_r, \bar{M}_r, \mu_r$ *and* $\bar{\mu}_r$ *is periodic in* r *with period 2.*

The ideas of proofs of some of these results are essentially contained in [V2], however the *answers* are sometimes different and should have been written explicitly: this it the main purpose of the present article.

We shall often use the following well-known fact (see e.g. [GM], [M]).

Proposition 2. *For any point* a *of an analytic subset* $A \subset \mathbf{C}^n$ *there exists a small disc* B *centred at* a *such that the pair* $(A \cap B, A \cap \partial B)$ *is homeomorphic to the cone over* $A \cap \partial B$, *and this homeomorphism is identical on* $A \cap \partial B$ *and maps* a *into the vertex of the cone. Moreover, the same is true for any smaller concentric disc.* $\square$

3 Realization of formulae (11), (12)

The stabilization of groups (10) will be constructed by induction over the flag of planes T^{m+l}. Namely, for any $r = m, \dots, n-1$ we construct the maps

$$\begin{array}{rccl} \Sigma: & \mathcal{H}_{j,\alpha}(r) & \to & \mathcal{H}_{j+1,\alpha}(r+1), \\ \downarrow: & \bar{\chi}_{j,\alpha}(r) & \to & \bar{\chi}_{j+1,\alpha}(r+1), \\ \rightleftharpoons: & \mathcal{H}_{j,\alpha}(r) & \to & \bar{\mathcal{H}}_{j+1,\alpha}(r+1), \\ \infty: & \bar{\chi}_{j,\alpha}(r) & \to & \chi_{j+1,\alpha}(r+1). \end{array} \tag{13}$$

Two first of them (and also two last if $\Phi_{*,\alpha} = \psi_{*,\alpha} = 0$) are isomorphisms.

Here are some preliminary constructions and reductions, cf. [V2].

3.1 Adapted coordinates and polydisc B'

Let $A, a, \sigma, B, f, D_\delta$ and X_t be the same as in the previous sections.

Definition. A local analytic coordinate system $\{z_1, \dots, z_n\}$ in $\mathbf{C}^n$ with origin at a is called *adapted* if $z_n \equiv f$, the tangent space to σ at a is spanned by the vectors $\partial/\partial z_1, \dots, \partial/\partial z_k$ (so that the restrictions of the functions $z_1, \dots, z_k$ constitute a local coordinate system on σ), and in restriction to σ

$$z_n \equiv z_1^2 + \cdots + z_k^2.$$

In Fig. 1b a real version of this situation is shown, where $n = 3$, the plane $X \equiv X_\delta$ is given by the equation $z_3 \equiv \delta$, and z_1 is the coordinate along the stratum σ. The transversal slice of this picture by the plane $\{z_1 = 0\}$ is shown in Fig. 1a.

By the Morse lemma, adapted coordinates always exist; let us fix such a coordinate system. Without loss of generality we can define the flag (7) by the conditions $T^r = \{z \mid z_1 = \cdots = z_{n-r} = 0\}$.

We can assume that $B \equiv B_\varepsilon$ is a closed disc of radius ε with respect to the standard Hermitian metric defined by these coordinates $z_1, \dots, z_n$. Moreover, in our considerations we can replace it by a closed polydisc defined by these coordinates. Namely, let $B' \subset B$ be the polydisc $\{z \mid |z_i| \leq \varepsilon/n \text{ for all } i\}$ and suppose that the number δ (participating in the definition of the disc $D \equiv D_\delta \subset \mathbf{C}^1$) is sufficiently small with respect to ε and ε^2.

Lemma 1. *For every surface $X_\lambda \subset B$ defined by the equation $z_n \equiv \lambda$, $\lambda \in D$, the pair $(B' \setminus AX_\lambda, \partial B' \setminus AX_\lambda)$ is homeomorphic to the pair $(B \setminus AX_\lambda, \partial B \setminus AX_\lambda)$. These homeomorphisms depend continuously on the parameter λ, and the induced isomorphisms of all groups (1) onto similar groups in whose definition B is replaced by B' coincide with the morphisms induced by the identical embedding $B' \setminus AX_\lambda \to B \setminus AX_\lambda$ in the case of groups $\mathcal{H}_{i,\alpha}, \chi_{i,\alpha}$ and to the morphisms induced*

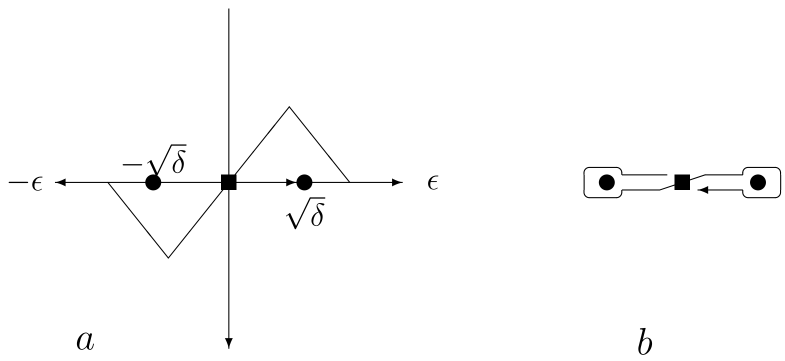

Figure 2: Base Ω of the fibre bundle z_{n-r}

by the obvious map $(B \setminus AX_\lambda)/(\partial B \setminus AX_\lambda) \to (B' \setminus AX_\lambda)/(\partial B' \setminus AX_\lambda)$ *(reduction mod* $B \setminus B'$*) in the case of groups* $\bar{\mathcal{H}}_{i,\alpha}, \bar{\chi}_{i,\alpha}$.

Moreover, for every $r = m, \ldots, n$ *and every* $\lambda \in D$*, all these statements remain valid if we replace both* B *and* B' *by* $B \cap T^r$ *and* $B' \cap T^r$ *respectively.*

The proof repeats the proof of Proposition 5 in [V2].

Thus, everywhere in the proof of Theorems 1–4 we can replace the disc B by the polydisc B'.

3.2 Fibre bundle z_{n-r}

For any $r = m, \ldots, n$ denote the $2r$-dimensional polydisc $B' \cap T^r$ by $B^{(r)}$ and the characteristic radius ε/n of all such polydiscs by ϵ. Remember the notation $X \equiv X_\delta$ and $AX \equiv A \cup X_\delta$. Denote the ϵ-disc $\{w \in \mathbf{C}^1 \mid |w| \leq \epsilon\}$ by Ω. For arbitrary $r < n$ consider the projection

$$z_{n-r} : (B^{(r+1)} \setminus AX) \to \Omega. \tag{14}$$

For any $t \in \Omega$ denote by $\mathcal{F}_t$ the fibre $(B^{(r+1)} \setminus AX) \cap \{z | z_{n-r} = t\}$ of this projection over the point t, and by $\partial \mathcal{F}_t$ the boundary $\mathcal{F}_t \cap \{z | \max(|z_{n-r+1}|, \ldots, |z_n|) = \delta\} \subset \partial B^{(r+1)}$ of this fibre. In particular, $\mathcal{F}_0 \equiv B^{(r)} \setminus AX$, $H_*^{lf}(\mathcal{F}_0, \partial \mathcal{F}_0; L_\alpha) \simeq \tilde{\mathcal{H}}_{*,\alpha}(r)$, $H_*^{lf}(\mathcal{F}_0, L_\alpha) \simeq \mathcal{H}_{*,\alpha}(r)$, $H_*(\mathcal{F}_0, \partial \mathcal{F}_0; L_\alpha) \simeq \tilde{\chi}_{*,\alpha}(r)$, and $H_*(\mathcal{F}_0, L_\alpha) \simeq \chi_{*,\alpha}(r)$.

Any of two planes $\{z | z_{n-r} = \pm\sqrt{\delta}\}$ contains a distinguished point $\Delta^\pm$, the critical point of the restriction of z_{n-r} to the manifold $\sigma \cap X \cap T^{r+1}$.

Lemma 2. *If the radius* ϵ *of the polydisc* B' *is sufficiently small and* δ *is sufficiently small with respect to* ϵ^2*, then* *a) the projection (14) defines a locally trivial fibre bundle over the disc* Ω *with two points* $\pm\sqrt{\delta}$ *removed, the standard fibre* $(\mathcal{F}_t, \partial\mathcal{F}_t)$ *of which is homeomorphic to* $(\mathcal{F}_0, \partial\mathcal{F}_0)$*;*

b) the fibres over the exceptional points $\sqrt{\delta}$ *and* $-\sqrt{\delta}$ *are homeomorphic to the direct product* $\partial\mathcal{F}_0 \times (0,1]$ *(or, more transparently, to the disc* $B^{(r)}$ *from which a cone over* $\partial B^{(r)} \setminus \partial\mathcal{F}_0$ *is removed);*

c) the restriction of this projection on $\partial B^{(r+1)}$ *defines a trivializable bundle with typical fibre* $\partial B^{(r)} \setminus AX$ *over the interior part of* Ω*.*

This lemma follows directly from the construction, from Thom's isotopy theorem and from Proposition 2, cf. [V2]. □

Consider two variation operators $\tilde{V}_{+,-} : H_*^{lf}(\mathcal{F}_0, \partial\mathcal{F}_0; L_\alpha) \to H_*^{lf}(\mathcal{F}_0, L_\alpha)$ and two similar operators $V_{+,-} : H_*(\mathcal{F}_0, \partial\mathcal{F}_0; L_\alpha) \to H_*(\mathcal{F}_0, L_\alpha)$ defined by the simple loops in Ω corresponding to the segments $[0, \sqrt{\delta}]$ and $[0, -\sqrt{\delta}]$. (For instance the ∞-shaped loop in Fig. 2b is a composition of the second of these loops and the loop inverse to the first of them.)

Lemma 3. *The operators* $\tilde{V}_+$*,* $\tilde{V}_-$ *are equal to one another and to the operator* $\widetilde{Var}_r : \bar{\mathcal{H}}_*(r) \to \mathcal{H}_*(r)$*. The operators* V_+*,* V_- *are equal to one another and to the operator* $Var_r : \bar{\chi}_*(r) \to \chi_*(r)$*.*

Indeed, any of these operators is defined by a loop in the space of pairs of complex hyperplanes in T^{r+1} in general position with A: for V_+ and V_- the first plane of this pair is fixed and coincides with $X_\delta \cap T^{r+1}$, while the second is distinguished by the condition $z_{n-r} = t$ where t runs over the corresponding simple loop in Ω; for Var_r the *second* plane is fixed and coincides with the one distinguished by $z_{n-r} = 0$, and the first moves and coincides with the planes $X_\lambda \cap T^{r+1}$ where λ runs over the circle $C \equiv \partial D_\delta$.

It is easy to see that all these three loops are homotopic to one another in the space of pairs of hyperplanes generic with respect to A, and Lemma 3 is proved. □

3.3 Construction of maps (13).

Let x be any element of the group $\mathcal{H}_{j,\alpha}(r)$. The class $\Sigma(x) \in \mathcal{H}_{j+1,\alpha}(r+1)$ realizing the first isomorphism in (13) is obtained from x by a sort of the suspension operation. Namely, using the fibre bundle structure (14), we transport the realizing x cycle $\mathbf{x} \subset \mathcal{F}_0$ over the segment $[-\sqrt{\delta}, \sqrt{\delta}]$. This one-parametric family of cycles sweeps out a $(j+1)$-dimensional chain in $B^{(r+1)} \setminus AX$ oriented by the pair of orientations, the first of which is the orientation of the base segment chosen as in Fig. 2a, and the second is induced by the original orientation of $\mathbf{x}$. The boundary of this chain lies in the marginal fibres over the endpoints $-\sqrt{\delta}$ and $\sqrt{\delta}$. By the statement b) of Lemma 2 the homology groups with closed supports of these fibres are trivial, thus we can contract these boundaries inside these fibres and get a cycle; $\Sigma(x)$ is defined as its homology class.

To construct the cycle $\rightleftharpoons(x)$ we first transport the cycle $\mathbf{x}$ over the S-shaped path in Fig. 2a and get similar cycles in the fibres over some two interior points of the segments $[\sqrt{\delta}, \epsilon]$ and $[-\epsilon, -\sqrt{\delta}]$, then sweep out some $(j+1)$-dimensional chains over these segments oriented as is shown in the same Fig. 2a, and again span the boundaries of the obtained chains inside the fibres over the points $\pm\sqrt{\delta}$. The class $\rightleftharpoons(x) \in \bar{\mathcal{H}}_{j+1,\alpha}(r+1)$ corresponding to x is defined by half the difference of these two cycles.

Given a homology class $y \in \bar{\chi}_{j,\alpha}(r)$, the corresponding cycle $\downarrow(y) \in \bar{\chi}_{j+1,\alpha}(r+1)$ is swept out by the similar one-parametric family of cycles obtained from a realizing y compact cycle $\mathbf{y}$ transported over the axis $\{Re\ z_{n-r} = 0\}$ oriented downwards.

To obtain the class $\infty(y) \in \chi_{j+1,\alpha}(r+1)$ we transport the same cycle $\mathbf{y}$ along the ∞-shaped path in Fig. 2b. The boundary of the $(j+1)$-dimensional chain swept out by it belongs to $\partial B'$ and is homeomorphic there (via the trivializing homeomorphism from statement c) of Lemma 2) to the direct product of $\partial\mathbf{y}$ and this path. In particular, it is the boundary of the $(j+1)$-dimensional chain in $\partial B'$, homeomorphic to the direct product of $\partial\mathbf{y}$ and the 2-chain in $\mathbf{C}^1$ bounded by this path. Thus the difference of these two $(j+1)$-dimensional chains is an absolute compact cycle in $B^{(r+1)} \setminus AX$; $\infty(y)$ is defined as its homology class.

Operations Σ^l and $\downarrow^l$ participating in Theorem 1 are just the l-fold iterations of these homomorphisms Σ and $\downarrow$. The homomorphisms $\rightleftharpoons_l$ and ∞_l defining the first summands in (12) are defined as $\rightleftharpoons \circ\Sigma^{l-1}$ and $\infty\circ\downarrow^{l-1}$ respectively.

Remark. If we replace the coordinate z_{n-r} by $-z_{n-r}$, all homomorphisms Σ, $\downarrow$, $\rightleftharpoons$ and ∞ will be multiplied by -1. The operations Σ^l, $\downarrow^l$, $\rightleftharpoons_l$ and ∞_l are thus defined only up to a (common) sign switching with any involution $z_q \to z_{-q}$, $q = k, k-1, \ldots, k-l+1$.

Conjecture. *There is a continuous involution of the pair $(B^{(r+1)}, AX)$ commuting with the involution $z_{n-r} \to -z_{n-r}$ of Ω and identical on the fibre $\{z|z_{n-r} = 0\}$ over the fixed point of the latter involution.*

Although we do not prove this conjecture, in the next section we use its "homological shadow", i.e. the involution in the homology groups, existing (as we shall see) independently on this conjecture.

Finally, we realize two last summands in both equations (12). Let $\Delta \in B^{(m+l)}$ be any intersection point of the plane X_δ and the stratum σ of A. Let $\beta \subset B^{(m+l)}$ be a very small closed disc centred at Δ. Then by the Künneth formula for arbitrary α

$$H_*(\beta \setminus AX, L_\alpha) \simeq \psi_{*,\alpha} \otimes H_*(\mathbf{C}^1 \setminus 0, L_{\alpha_1}), \tag{15}$$

$$H_*^{lf}(\beta \setminus AX, \partial\beta; L_\alpha) \simeq \Phi_{*-2l,\alpha} \otimes H_*^{lf}(\mathbf{C}^1 \setminus 0, L_{\alpha_1}) \tag{16}$$

(where the groups $H_*(\mathbf{C}^1\setminus 0, L_{\alpha_1})$ and $H_*^{lf}(\mathbf{C}^1\setminus 0, L_{\alpha_1})$ are trivial if $\alpha_1 \neq 1$ and are isomorphic to $H_*(S^1)$ and $H_{*-1}(S^1)$ respectively if $\alpha_1 = 1$, and Φ_{*-2l} is the graded group obtained from Φ_* by the shift of grading). The sum of two last summands in the second row of (12) is isomorphic to the $(i+l)$-dimensional component of the group (15) and is realized as follows.

Lemma 4. *The homomorphism*

$$H_*(\beta \setminus AX, L_\alpha) \to \chi_{*,\alpha}(m+l) \tag{17}$$

induced by the identical embedding is injective.

If $l > 1$ (so that the set $B^{(m+l)} \cap X_\delta \cap \sigma$ of possible points Δ is path-connected) then the sum of two last summands in the second row of (12) coincides with the image of this injection. If $l = 1$ and this set consists of two points Δ^+ and Δ^- (see § 3.2), then the corresponding groups $H_*(\beta^+ \setminus AX, L_\alpha)$, $H_*(\beta^- \setminus AX, L_\alpha)$ are naturally identified to one another. (This identification is transparent on the dual cohomological level, because both cohomology groups dual to two factors in (15) are induced from groups $H^*(B^{(m+l)} \setminus A)$ and $H^*(B^{(m+l)} \setminus X)$ respectively.)

Consider the similar homomorphism $H_*(\beta^+ \setminus AX, L_\alpha) \oplus H_*(\beta^- \setminus AX, L_\alpha) \to \chi_{*,\alpha}(m+l)$ defined by the embedding $(\beta^+ \cup \beta^-) \to B^{(m+l)}$ and take the subgroup of its image invariant under this identification; this subgroup is again isomorphic to either of $H_*(\beta^\pm \setminus AX, L_\alpha)$ and realizes the last summands in the second row of (12). Two last summands in the first row are Poincaré dual to them, let us describe them explicitly.

The pair $(B^{(m+l)}, A)$ is homeomorphic to the product $(B^{(m)}, B^{(m)} \cap A) \times \mathbf{C}^l$, therefore for any cycle $w \in \Phi_{i-l}$ the cycle $w \times \mathbf{C}^l$ defines an element of the group $H^{lf}_{i+l}(B^{(m+l)} \setminus A, \partial B; L_{\hat\alpha})$ and, since $\alpha_1 = 1$, also of the group $H^{lf}_{i+l}(B^{(m+l)} \setminus AX, \partial B; L_\alpha) \equiv \bar{\mathcal{H}}_{i+l,\alpha}(m+l)$. We choose this cycle in general position with a cycle generating the group $H^{lf}_{2(m+l)-1}(B^{(m+l)} \setminus X, \partial B)$ (say, with the cycle given by the condition $z_n < \delta$), then the intersection of these two cycles gives us also an element of the group $\bar{\mathcal{H}}_{i+l-1,\alpha}(m+l)$; this element corresponds to the element z of the third summand of the decomposition (12) of this group.

4 Proof of main theorems

Lemma 5. *For any $x \in \mathcal{H}_{i,\alpha}(r)$, the homomorphism $\tilde J_{\alpha,r+1} \equiv \tilde j_\alpha \circ \tilde i_\alpha : \mathcal{H}_{j+1,\alpha}(r+1) \to \bar{\mathcal{H}}_{j+1,\alpha}(r+1)$ (the reduction mod $\partial B^{(r+1)}$) maps any element $\Sigma(x)$, $x \in \mathcal{H}_{j,\alpha}(r)$, to $- \rightleftharpoons (2x + \widetilde{Var}_r \circ \tilde J_{\alpha,r}(x)) \equiv - \rightleftharpoons \circ (M_r + Id)(x)$.*

The similar homomorphism $J_{\alpha,r+1} \equiv j_\alpha \circ i_\alpha : \chi_{j+1,\alpha}(r+1) \to \bar\chi_{j+1,\alpha}(r+1)$ maps any element $\infty(y)$, $y \in \bar\chi_{j,\alpha}(r)$, to $- \downarrow (2y + J_{\alpha,r} \circ Var_r(y)) \equiv - \downarrow \circ (\bar\mu_r + Id)(y)$.

Proof. See Figs. 3a), b).

Lemma 6. *For any $r = m, \dots, n-1$, and any elements $x \in \mathcal{H}_{j,\alpha}(r)$* and *$y \in \bar\chi_{j,\alpha}(r)$, $\widetilde{Var}_{r+1}(\rightleftharpoons (x)) = \Sigma(x)$, $Var_{r+1}(\downarrow (y)) = \infty(y)$.*

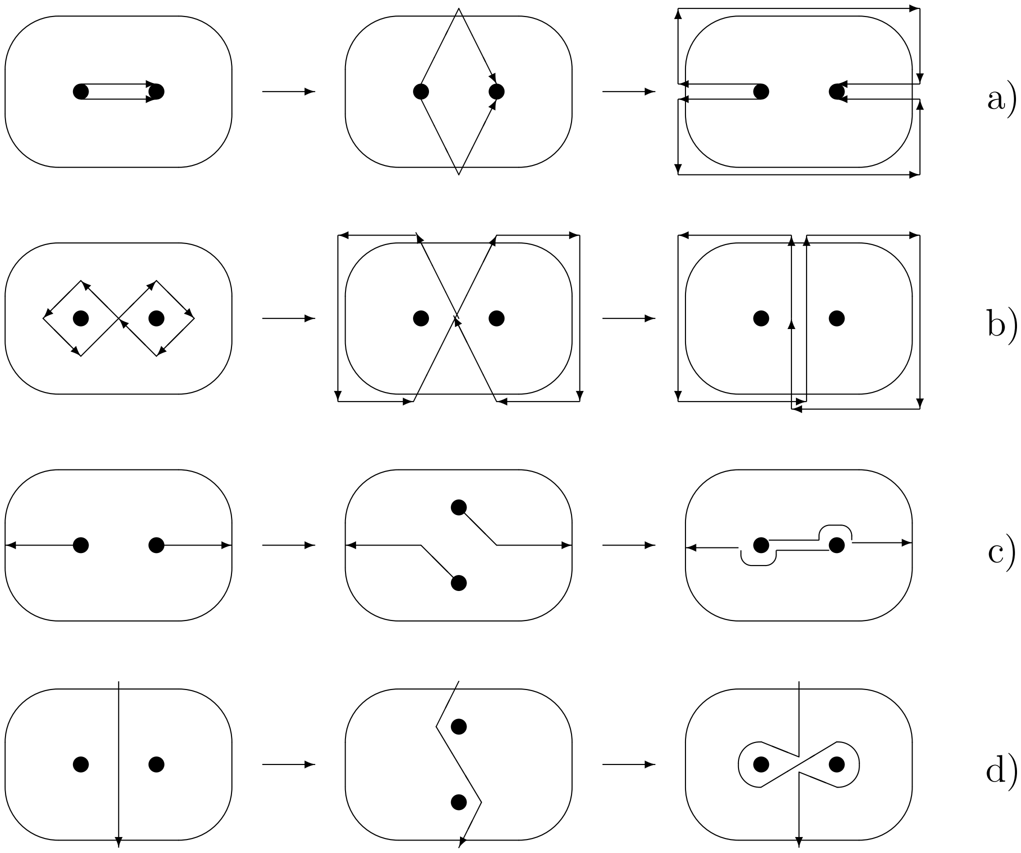

Figure 3: Proofs of Lemmas 5 and 6

Proof. When ξ moves along the circle $\delta \cdot e^{i\tau}, \tau \in [0, 2\pi]$, the ramification points $\pm\sqrt{\xi} \subset \Omega$ of the fibre bundle $z_{n-r} : (B^{(r+1)}, AX_\xi) \to \Omega$ move as is shown in Fig. 3 c). We move the segments $[-\epsilon, -\sqrt{\xi}]$, $[\sqrt{\xi}, \epsilon]$ connecting the ramification points with "infinity" in such a way that they coincide with the parts of the real axis close to the boundary of Ω. Let us also move the cycle realizing $\rightleftharpoons (\alpha)$ in such a way that at any instant of this deformation it forms a fibre bundle with standard fibre $\alpha/2$ over the union of two corresponding segments (except for their endpoints). At the final instant of the monodromy each of these two parts of $\rightleftharpoons (\alpha)$ will increase by the cycle $\Sigma(\alpha)/2$ swept by the formal half of the initial cycle α in transport over the added part of the resulting segment, so that $\widetilde{Var}_{r+1}(\rightleftharpoons (\alpha)) = \Sigma(\alpha)$. For the similar proof of the formula for the variation of $\downarrow (y)$, see Fig. 3d).

This lemma proves statement b) of Theorem 3 and assertion of statement a) concerning the variation of cycles $\rightleftharpoons_l (x)$.

Statement a) of Theorem 2 and assertion of statement b) concerning the action on elements $\infty_l(y)$ follow from Lemmas 5 and 6 by induction over r.

Corollary. *The local monodromy operators* $M_{r+1}, \bar{M}_{r+1}, \bar{\mu}_{r+1}$ *and* μ_{r+1} *respectively map the elements* $\Sigma(x), \rightleftharpoons (x), \downarrow (y)$ *and* $\infty(y)$ *(where* $x \in \mathcal{H}_{*,\alpha}(r), y \in \bar{\chi}_{*,\alpha}(r)$*) into* $-\Sigma(M_r(x))$, $- \rightleftharpoons (M_r(x))$, $- \downarrow (\bar{\mu}_r(y))$ *and* $-\infty(\bar{\mu}_r(y))$ *respectively.*

Lemma 7. *For any* $x \in \mathcal{H}_{j,\alpha}(r)$ *and* $y^* \in \bar{\chi}_{2r-j,\alpha^*}(r)$,

$$\langle \Sigma(x), \downarrow (y^*) \rangle = (-1)^{1+j} \langle x, y^* \rangle,$$

$$\langle \rightleftharpoons (x), \infty(y^*) \rangle = (-1)^{1+j} \langle x, \bar{\mu}_r(y^*) \rangle.$$

This lemma follows immediately from constructions. Statements A) and B2) of Theorem 4 follow from it by induction over l.

Proposition 3. *The homomorphisms* $\Sigma : \mathcal{H}_{j,\alpha}(r) \to \mathcal{H}_{j+1,\alpha}(r+1)$ *and* $\downarrow$: $\bar{\chi}_{j,\alpha}(r) \to \bar{\chi}_{j+1,\alpha}(r+1)$ *are isomorphisms for any* α *and* $r = m, \ldots, n-1$.

Proof. Consider any smooth deformation contracting the disc Ω onto the segment $[-\sqrt{\delta}, \sqrt{\delta}]$. Using the fibre bundle (14), we can lift this deformation to the space $B^{(r+1)} \setminus AX$. This lifted field allows us to realize any element of the group $\mathcal{H}_{j+1,\alpha}(r+1)$ by a locally finite chain lying in this pre-image, and hence also by one of the form $\Sigma(x)$, $x \in \mathcal{H}_{j,\alpha}(r)$. Thus the map Σ is epimorphic.

Our proposition follows now from the first equation of Lemma 7 and from the fact that both Poincaré pairings $\mathcal{H}_{j,\alpha}(r) \otimes \bar{\chi}_{2r-j,\alpha^*}(r) \to \mathbf{C}$ and $\mathcal{H}_{j+1,\alpha}(r+1) \otimes \bar{\chi}_{2r-j+1,\alpha^*}(r+1) \to \mathbf{C}$ are non-degenerate.

Theorem 1 is a direct corollary of this proposition.

Denote by $\partial^- B^{(r+1)}$ the union of the usual boundary $\partial B^{(r+1)}$ and the "left half" of $B^{(r+1)}$ consisting of points z with $z_{n-r} \leq 0$.

Table 1: The groups $E^1_{p,q}$ for the main, anti-invariant and invariant spectral sequences

p	0	1	2	≥ 3
$E^1_{p,q}$	$(H^{lf}_{q-1}(\partial\mathcal{F}_0), L_\alpha)^2$	$(\bar{\mathcal{H}}_{q,\alpha}(r))^2$	$\bar{\mathcal{H}}_{q,\alpha}(r)$	0
$E^{1,-}_{p,q}$	$H^{lf}_{q-1}(\partial\mathcal{F}_0, L_\alpha)$	$\bar{\mathcal{H}}_{q,\alpha}(r)$	0	0
$E^{1,+}_{p,q}$	$H^{lf}_{q-1}(\partial\mathcal{F}_0, L_\alpha)$	$\bar{\mathcal{H}}_{q,\alpha}(r)$	$\bar{\mathcal{H}}_{q,\alpha}(r)$	0

Proposition 4. *For any $r = m, \dots, n-1$ there is short exact sequence*

$$0 \to \mathcal{H}_j(r) \to \bar{\mathcal{H}}_{j+1}(r+1) \xrightarrow{\rho} H^{lf}_{j+1}(B^{(r+1)} \setminus AX, \partial^- B^{(r+1)}; L_\alpha) \to 0, \qquad (18)$$

where the injection $\mathcal{H}_j(r) \to \bar{\mathcal{H}}_{j+1}(r+1)$ is the map $\rightleftharpoons$, and the epimorphism ρ is induced by the reduction modulo $\{z|z_{n-r} \leq 0\}$.

Proof (cf. § 4.2 in [V2]). We filter the disc Ω by the sets $\{\Psi_0 \subset \Psi_1 \subset \Psi_2\}$ where Ψ_0 consists of two points $\pm\sqrt{\delta}$, Ψ_1 consists of two segments $[\sqrt{\delta}, \epsilon]$ and $[-\epsilon, -\sqrt{\delta}]$ (so that the set $\Omega \setminus \Psi_1$ is a 2-cell), and $\Psi_2 \equiv \Omega$. Using the projection (14) we lift this filtration onto the set $B^{(r+1)} \setminus AX$; let $E^r_{p,q}$ be the spectral sequence generated by this filtration and calculating the group $\bar{\mathcal{H}}_{*,\alpha}(r+1)$.

Proposition 5. *For any q, the elements $E^1_{p,q}$ of this spectral sequence are as shown in the second row of Table 1.*

Indeed, the assertion concerning the group $E^1_{0,q}$ follows from the statement b) of Lemma 2. For any element $z \in \bar{\mathcal{H}}_*(r)$, two corresponding elements in two summands $\bar{\mathcal{H}}_*(r)$ of $E^1_{1,*}$ are the homology classes of two cycles (mod $z^{-1}_{n-r}(\Psi_0)$) swept out by the copies of the cycle realizing z transported over two segments as in the definition of the operation $\rightleftharpoons$, see § 3.3, and the corresponding element of the group $E^1_{2,*}$ is swept out by the two-parametric family of cycles obtained from the cycle realizing z by the transportation to all points of the 2-cell $\Omega \setminus \Psi_1$.

Consider an involution acting on the term E^1 of our spectral sequence: it permutes the elements of terms $E_{0,q}$, $E_{1,q}$, corresponding to the same elements of groups $\tilde{H}^{lf}_{q-1}(\partial\mathcal{F}_0, L_\alpha)$, $\bar{\mathcal{H}}_{q,\alpha}(r)$, and does not touch the terms $E_{2,q}$. Denote by $E^{+,r}_{p,q}$ and $E^{-,r}_{p,q}$ the invariant and anti-invariant parts of this involution, respectively.

Lemma 8. *The splitting of the term E^1 into the invariant and anti-invariant parts is compatible with all further differentials and thus defines the splitting of the entire spectral sequence. The groups $E^{+,1}$ and $E^{-,1}$ of the invariant (respectively, anti-invariant) subsequence are as shown in the fourth (respectively, third) row of the Table 1. The unique nontrivial differential $\partial_1 : \bar{\mathcal{H}}_{q,\alpha}(r) \to H^{lf}_{q-1}(\partial\mathcal{F}_0, L_\alpha)$ of the anti-invariant subsequence coincides with the boundary operator in $\mathcal{F}_0$, so that this subsequence is nothing but the exact sequence of the pair $(\mathcal{F}_0, \partial\mathcal{F}_0)$*

calculating the group $H_*^{lf}(\mathcal{F}_0, L_\alpha) \equiv \mathcal{H}_{q,\alpha}(r)$. *The invariant subsequence is isomorphic to the spectral sequence generated by the same filtration on the quotient space* $(B^{(r+1)} \setminus AX)/(\partial^- B^{(r+1)} \setminus AX)$ *and calculating its homology group* $H_{i+1}^{lf}(B^{(r+1)} \setminus AX, \partial^- B^{(r+1)} \setminus AX; L_\alpha)$*; this isomorphism is defined by the reduction modulo* $\{z | z_{n-r} \leq 0\}$.

All this follows immediately from the construction and proves Proposition 4.

By the construction, the induced involution of the limit homology group $\bar{\mathcal{H}}_{i+1}(r+1)$ acts as multiplication by -1 on the subgroup $\rightleftharpoons (\mathcal{H}_i(r))$ and acts trivially on the corresponding quotient group, in particular the invariant subspace of this involution is canonically isomorphic to $H_{i+1}^{lf}(B^{(r+1)} \setminus AX, \partial^- B^{(r+1)} \setminus AX; L_\alpha)$.

Lemma 9. *Let* β *be a sufficiently small closed disc in* $B^{(r+1)}$ *centred at the distinguished point* Δ^+ *(or* Δ^-*) of the fibre* $\{z | z_{n-r} = \sqrt{\delta}\}$ *(respectively,* $-\sqrt{\delta}$*). Then the group* $H_*^{lf}(B^{(r+1)} \setminus AX, \partial^- B^{(r+1)} \setminus AX; L_\alpha)$ *is isomorphic to* $H_*^{lf}(\beta \setminus AX, \partial\beta; L_\alpha)$*; this isomorphism is induced by the reduction modulo the closure of the complement of* β.

This lemma follows from Lemma 2. Together with the Künneth decomposition (15) it proves the statement of Theorem 1′ concerning the group $\bar{\mathcal{H}}_*$, and also the fact that the last summands in (12) are actually generated by the cycles described between Lemmas 4 and 5. Three remaining assertions of Theorems 2–4 (triviality of the action of homomorphisms $J_{\alpha,(m+l)}$ (in Theorem 2) and $\widetilde{Var}_{(m+l)}$ (in Theorem 3) on the last summands in (12) and statement B1 in Theorem 4) follow almost immediately from the construction of all these cycles. Lemma 4 is just the fact Poincaré dual to the surjectivity of the map ρ in (18).

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