TOPOLOGY OF PLANE ARRANGEMENTS AND THEIR COMPLEMENTS

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Abstract. This is a glossary of notions and methods related with the topological theory of collections of affine planes, including braid groups, configuration spaces, order complexes, stratified Morse theory, simplicial resolutions, complexes of graphs, Orlik–Solomon rings, Salvetti complex, matroids, Spanier–Whitehead duality, twisted homology groups, monodromy theory and multidimensional hypergeometric functions.

The emphasis on the most geometrical explanation is done; applications and analogies in the differential topology are outlined.

Some recent results of the theory are presented.

1. Introduction

Finite collections of affine planes in $\mathbb{R}^N$ or in $\mathbb{C}^N$ (shortly, affine plane arrangements) form a remarkable class of algebraic varieties. Indeed,

1) they are a meeting point of topology, combinatorics, linear algebra, representation theory, algebraic geometry, complexity theory, mathematical physics and differential equations;

2) they are a wonderful proving ground for methods and motivations in these fields, having very far generalizations;

3) they provide a successful elementary visualization of abstract algebraic and combinatorial notions and constructions.

Formulas, constructions and theorems once arising in this theory appear then again and again in very distant fields and problems.

One of main problems of the theory asks to which extent the topological properties of the union of several planes (and of the complement of this union) are determined by the formal data, i.e. by the information on the dimensions of all subcollections of planes.

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We shall use this problem to demonstrate such notions and methods as braid groups, configuration spaces, order complexes, stratified Morse theory, simplicial resolutions, complexes of graphs, Orlik-Solomon rings, matroids, Spanier–Whitehead duality, twisted homology groups, monodromy theory, hypergeometric functions, etc.

There are several very good expositions of the theory of arrangements or different its aspects, see e.g [39], [23], [60], [95], [73] and especially [15]. An exhaustive survey of algebraic aspects of the theory of complex hyperplane arrangements is given in S. Yuzvinsky's article [92].

In this short article, I tried to give an elementary introduction to the theory, making emphasis on a) the most geometrical aspects and motivations of the theory, b) the most recent results not reflected yet in introductory texts, c) the subjects that traditionally are treated in more formal and abstract way than it is necessary, d) the results having important applications and generalizations in the fields familiar to me: differential topology, singularity theory, integral geometry, complexity theory...

2. Main definitions, notation and examples

An *affine plane arrangement* is any finite collection of affine planes (of arbitrary, maybe different, dimensions) in $\mathbb{R}^N$.

An arrangement is called *central* if all its planes contain the origin in $\mathbb{R}^N$. In this case one says also that we have a *subspace arrangement.*

One can define also the *complex* plane or subspace arrangements in $\mathbb{C}^N$: they are a special case of usual arrangements in $\mathbb{R}^{2N}$.

In a similar way one defines plane arrangements in $\mathbb{RP}^N$ or in $\mathbb{CP}^N$: they are in the obvious one-to-one correspondence with central arrangements of nontrivial subspaces in $\mathbb{R}^{N+1}$ (respectively, in $\mathbb{C}^{N+1}$).

Any real affine plane in $\mathbb{R}^N$ defines a complex plane of the same dimension in $\mathbb{C}^N$: its *complexification.* Therefore the complexification of any real plane arrangement is well defined.

Suppose that we have a plane arrangement $\mathcal{L}$ consisting of planes $L_1, \dots, L_m$. The union $L_1 \cup \dots \cup L_m$ of these planes is called *the support* of $\mathcal{L}$ and will be denoted by L. For any subset of indices $I \subset \{1, \dots, m\}$ we set

$$L_I \equiv \bigcap_{i \in I} L_i. \tag{1}$$

The first example of a *hyper*plane arrangement is provided by the *coordinate cross* in $\mathbb{R}^N$ given by the equation $x_1 \cdot \ldots \cdot x_N = 0$.

The next famous arrangement, the *diagonal* arrangement $A(N, 2) \subset \mathbb{C}^N$, consists of $\binom{N}{2}$ hyperplanes $V_{ij} \equiv V_{ji}$, $\{i \neq j\} \subset \{1, \dots, N\}$, given by equations $x_i = x_j$. The complement of this arrangement in $\mathbb{C}^N$ can be considered as the *Nth ordered configuration space* of $\mathbb{C}^1$, i.e the space of all ordered collections of N pairwise different points in $\mathbb{C}^1$.

More generally, for any $k \in [2, N]$ the *k-equal arrangement* $A(N, k)$ consists of $\binom{N}{k}$ planes $V(i_1, \dots, i_k)$, $1 \leq i_1 < \dots < i_k \leq N$, given by conditions $x_{i_1} = \dots = x_{i_k}$. We can define such arrangements both in $\mathbb{R}^N$ and in $\mathbb{C}^N$.

Another generalization of the arrangement $A(N, 2)$ is as follows (see [22]). Consider any finite group W of isometries of the Euclidean space $\mathbb{R}^N$ generated by reflections in several hyperplanes (*mirrors*). (Such groups are well-known, see [13]: *irreducible* groups of this type form four infinite series A_m $(m \geq 1)$, C_m $(m \geq 2)$, D_m $(m \geq 4)$, and $I_2(p)$, and seven exceptional cases $G_2, F_4, H_3, H_4, E_6, E_7$ and E_8.) Almost all orbits of the action of W in $\mathbb{R}^N$ have one and the same cardinality. The union of *irregular* orbits of smaller cardinality consists of the mirrors generating the group and their images under its action. All components of the complement of this union are simplicial cones (*Weyl chambers*). The action of W in the space $\mathbb{R}^N$ extends in the obvious way to an action in its complexification $\mathbb{C}^N$. The union D_W of irregular orbits of the latter action consists of complexifications of mirrors and their orbits; it is called the *diagonal* of the group W.

For instance let the mirrors be all the hyperplanes given by equations $x_i = x_j$, $i \neq j$. Then the group W is isomorphic to the permutation group $S(N)$, and the corresponding diagonal arrangement coincides with $A(N, 2)$. This is the case A_{N-1} of the classification of reflection groups.

A hyperplane arrangement in $\mathbb{R}^N$ or $\mathbb{C}^N$ or $\mathbb{RP}^N$ or $\mathbb{CP}^N$ *has normal crossings* if for any subset $I \subset \{1, \dots, m\}$ the plane L_I either is empty or its *codimension* $N - \dim L_I$ is equal to the cardinality of I.

A hyperplane arrangement in $\mathbb{R}^N$ or $\mathbb{C}^N$ is *generic* if, being augmented by the *infinitely distant* hyperplane it becomes an arrangement with normal crossings in $\mathbb{RP}^N$ (respectively, in $\mathbb{CP}^N$).

It is easy to see that for all generic arrangements of m hyperplanes in $\mathbb{C}^N$ the corresponding triples $(\mathbb{CP}^N, \mathbb{C}^N, L)$ are homeomorphic. Generic arrangements form an open dense subset in the space of all ordered collections of m hyperplanes.

3. Basic example: cohomology rings of pure braid groups

Denote the open manifold $\mathbb{C}^N \setminus A(N, k)$ by $M(N, k)$.

Proposition 1 (see [31], [3]). *For any N, the manifold $M(N, 2)$ is a $K(\pi, 1)$-space, i.e. all its higher homotopy groups $\pi_2, \pi_3, \dots$ are trivial.*

Indeed, forgetting the last point of a N-point configuration we obtain a fiber bundle

$$M(N, 2) \to M(N - 1, 2) \tag{2}$$

with fiber equal to $\mathbb{C}^1 \setminus \{N - 1 \text{ points}\}$. Proposition 1 follows by induction from the exact homotopy sequence of this bundle.

The group $\pi_1(M(N, 2))$ is called the *pure braid group* with N strings. The fundamental group of the similar space of non-ordered sets is called simply the *braid group with N strings*, see [8]. For algebraic and homological properties of these groups see, in particular, [51], [4], [40], [66].

A similar statement holds for any finite reflection group.

Theorem 1 (see [22], [28]). *For any finite group W acting by reflections in $\mathbb{R}^N$, the corresponding space $\mathbb{C}^N \setminus D_W$ (i.e. the union of regular orbits of the complexified action in $\mathbb{C}^N$) is a $K(\pi, 1)$-space.*

This theorem was proved by Brieskorn for reflection groups C_m, D_m, G_2, F_4, $I_2(p)$; Deligne has proved it in the general case. The groups $\pi_1(\mathbb{C}^N \setminus D_W)$ for these arrangements are called the *Brieskorn braid groups*, see [22].

Theorem 1 implies that the cohomology rings of these groups are equal to these of spaces $\mathbb{C}^N \setminus D_W$. These rings were calculated in [3] for the case A_{N-1} (i.e. that of the arrangement $D_W \equiv A(N, 2)$) and in [22] for all other reflection groups.

Moreover, the complement $M_{\mathbb{R}}(N, 3)$ of the *real* 3-equal arrangement also is a $K(\pi, 1)$-space for any N: this fact was conjectured by A. Björner and proved by M. Khovanov [47].

The calculation of the cohomology ring in the Arnold's case $M(N, 2)$ is based on the same fiber bundle (2).

Proposition 2 (see [3]). *The group $H^*(M(N, 2))$ is torsion-free and is isomorphic to the tensor product $H^*(\phi_{N-1}) \otimes H^*(\phi_{N-2}) \otimes \cdots \otimes H^*(\phi_1)$ where ϕ_i is the wedge of i circles. In particular the Poincaré polynomial of $H^*(M(N, 2))$ is equal to $(1 + t)(1 + 2t) \cdot \ldots \cdot (1 + (N - 1)t)$.*

Indeed, it is easy to see that the bundle (2) is *homologically simple*, i.e. the fundamental group of the base acts trivially in the homology of its fiber (which is homotopy equivalent to ϕ_{N-1}). The spectral sequence of this bundle stabilizes in the term E_2, therefore we have a ring isomorphism $H^*(M(N, 2)) \simeq H^*(M(N - 1, 2)) \otimes H^*(\phi_{N-1})$. $\square$

The ring structure in this cohomology, also calculated in [3], is as follows. For any plane V_{jk} of our arrangement denote by ω_{jk} the logarithmic form with singularity at this plane, $\omega_{jk} \equiv \frac{1}{2\pi i}\frac{dz_j - dz_k}{z_j - z_k}$; the integral of it along a closed path in $M(N, 2)$ equals the linking number of this path with the plane V_{jk}.

Proposition 3. *For any three different indices $i, j, k \in [1, N]$ the equality*

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0 \tag{3}$$

holds identically in $M(N, 2)$. In particular the similar identity on the corresponding cohomology classes holds in the ring $H^*(M(N, 2))$: if Ω_{jk} is the cohomology class of the form ω_{jk}, then

$$\Omega_{ij} \smile \Omega_{jk} + \Omega_{jk} \smile \Omega_{ki} + \Omega_{ki} \smile \Omega_{ij} = 0. \tag{4}$$

The integer cohomology algebra $H^*(M(N, 2))$ is canonically isomorphic to the quotient algebra of the exterior algebra formally generated by $\binom{N}{2}$ elements Ω_{jk} through the ideal multiplicatively generated by left parts of all possible expressions (4) with arbitrary i, j and k.

4. Orlik–Solomon ring and cohomology of complements of complex hyperplane arrangements

A general statement very similar to Proposition 3 holds for an arbitrary complex central hyperplane arrangement.

Let $L = \{L_1, \dots, L_m\}$ be such an arrangement in $\mathbb{C}^N$, whose planes L_i are given by linear equations $f_i = 0$. A collection of indices $I \subset \{1, \dots, m\}$ is *dependent* if the codimension of L_I is smaller than the expected value $|I|$ (i.e. the corresponding equations f_i are linearly dependent). For any dependent set $I = \{i_1 < \cdots < i_k\}$ denote by $\rho(I)$ the rational differential $(k-1)$-form

$$\sum_{j=1}^{k} (-1)^j \frac{df_{i_1}}{f_{i_1}} \wedge \cdots \wedge \widehat{\frac{df_{i_j}}{f_{i_j}}} \wedge \cdots \wedge \frac{df_{i_k}}{f_{i_k}}. \tag{5}$$

It is easy to see that this form is equal to zero in $\mathbb{C}^N \setminus L$.

Theorem 2 (see [59])**.** *For any central complex hyperplane arrangement $\mathcal{L}$ in $\mathbb{C}^N$, the integral cohomology ring $H^*(\mathbb{C}^N \setminus L)$ is canonically isomorphic to the quotient algebra of the exterior algebra on generators α_j corresponding to hyperplanes of $\mathcal{L}$ through the ideal generated by all elements*

$$\sum_{j=1}^{k} (-1)^j \alpha_{i_1} \smile \cdots \smile \widehat{\alpha_{i_j}} \smile \cdots \smile \alpha_{i_k} \tag{6}$$

corresponding to all dependent collections $I = (i_1, \dots, i_k)$. Moreover, the same ideal is generated by expressions (6) over only minimal *dependent subsets I.*

Example. For the arrangement $A(N, 2)$ the minimal dependent sets are exactly the collections of planes of the form $V_{i_1 i_2}$, $V_{i_2 i_3}$, $\dots V_{i_{q-1} i_q}$, $V_{i_q i_1}$, where $\{i_1, \dots, i_q\} \subset \{1, \dots, N\}$ is any set of indices, $q \geq 3$. It is easy to see that any form (6) defined by such a subset belongs to the ideal generated by similar forms with $q = 3$.

A main step towards Theorem 2 was done in [22].

Corollary 1 (see [22])**.** *The group $H^*(\mathbb{C}^N \setminus L)$ is torsion-free.*

The case of not central hyperplane arrangements in $\mathbb{C}^N$ can be easily reduced to that of central hyperplane arrangements in $\mathbb{C}^{N+1}$.

5. How much the topology of the complement is defined by the dimensional data: a summary

Let $\mathcal{L}$ be an arbitrary affine plane arrangement in $\mathbb{R}^N$, consisting of m planes. Suppose that for any $I \subset \{1, \dots, m\}$ we know the dimension of the plane L_I (and whether this plane is empty or not). What can be then said about the topology of $\mathbb{R}^N \setminus L$? Given a topological invariant of $\mathbb{R}^N \setminus L$, is it determined uniquely by these data ?

For homology and cohomology groups of $\mathbb{R}^N \setminus L$ the answer to the last question is positive, see [41] and §6 below.

For the stable homotopy type the answer also is positive, see [80], [96] and §8 below.

For the multiplicative structure in cohomology the answers are as follows.

A. In the most general situation not, see [94], [95].

B. For complex arrangements of arbitrary dimensions: yes. For hyperplane arrangements it follows from the above Orlik-Solomon theorem. For an arbitrary complex arrangement this was proved in [24] for rational cohomology. Then S. Yuzvinsky [90] proposed an explicit formula for this rational cohomology multiplication, and finally it was proved [25], [29], that the same formula expresses the multiplication in the integral cohomology ring, see §11 below.

C. There is a more general class of real arrangements for which a large part of the multiplicative structure is determined by the dimensional data.

Definition. The arrangement $\mathcal{L}$ in $\mathbb{R}^N$ is called a ≥ 2-*arrangement* if for any two its incident planes $L_I \subsetneqq L_J$ we have $\dim J - \dim I \geq 2$.

For instance any complex arrangement satisfies this condition.

For such arrangements an explicit formula of the multiplication in the cohomology was proved in [25], [29]. In particular it implies the following proposition.

Suppose that two planes L_I, L_J of an arrangement are *transversal*, i.e

$$N - \dim(L_I \cap L_J) = (N - \dim L_I) + (N - \dim L_J). \tag{7}$$

Let us fix also an orientation of $\mathbb{R}^N$. Then any choice of orientations of planes L_I, L_J defines in a standard way an orientation of $L_I \cap L_J \equiv L_{I\cup J}$. If the orientations of all planes L_I of our arrangement are fixed, then for any ordered pair of transversal planes L_I, L_J we get a sign $+$ or $-$ indicating whether the fixed orientation of $L_{I\cup J}$ coincides with the orientation defined by fixed orientations of L_I and L_J.

Proposition 4. *Suppose that we have two plane ≥ 2-arrangements $\mathcal{L}$, $\mathcal{L}'$ in $\mathbb{R}^N$ such that $\dim L_I = \dim L'_I$ for any I, and there are systems of orientations of all these planes such that for any pair of multi-indices I, J satisfying (7) the corresponding signs coincide. Then the cohomology rings of $\mathbb{R}^N \setminus L$, $\mathbb{R}^N \setminus L'$ are isomorphic.*

Certainly, the complex arrangements with equal dimensional data satisfy the conditions of this proposition: if we choose the complex orientations of all planes then all indices will be equal to $+$.

D. On the other hand this condition on orientations cannot be removed: a counterexample see in [94].

E. Still, something good can be said even in the most general case of an arbitrary real affine plane arrangement. The group $H^*(\mathbb{R}^N \setminus L)$ always admits a natural filtration, see §6, 8 below. The corresponding *graded ring* is uniquely determined by the dimensional data and the system of signs as in item C above. Its description also follows from the results of [25], [29], see §11.

The most fragile invariant is the fundamental group of the complement of an arrangement.

Theorem 3 (see [64])**.** *There exist two complex line arrangements in $\mathbb{C}^2$ with equal dimensional data (i.e. sets of lines having a common point) but with nonisomorhic fundamental groups of complements of their supports.*

Finally, if we consider the topology not of the complement of the support L but of this support itself or its one-point compactification $\bar{L}$, then the dimensional information is very strong.

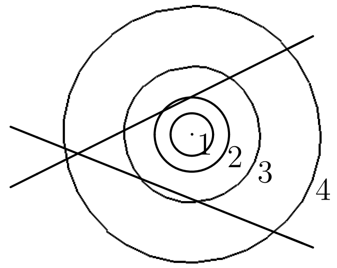

FIGURE 1. Evolution of sets of smaller values

Theorem 4 (see §8)**.** *The homotopy types of topological spaces L and $\bar{L}$ are completely defined by the dimensional data.*

6. Order complex of a poset. Goresky–MacPherson formula

For an arbitrary real affine plane arrangement, a calculation of the cohomology group of its complement was given by M. Goresky and R. MacPherson as a bright application of their *Stratified Morse theory* [41].

In our special case this theory works as follows. Let us fix a generic point $X_0 \in \mathbb{R}^N \setminus L$ and a positive quadratic function $f : \mathbb{R}^N \to \mathbb{R}$ with origin at this point (i.e. equal to $x_1^2 + \cdots + x_N^2$ in some affine coordinates centered at X_0). The *critical* values of f are the critical values of its restrictions on all nonempty planes L_I of our arrangement. If X_0 and f actually are generic then all these values corresponding to different planes L_I are different. Further, for any positive t let us consider the ball $B_t \equiv \{x : f(x) \leq t\}$ in $\mathbb{R}^N$ and the *manifold of lower values* $\Lambda(t) \equiv B_t \setminus L$. If the segment $[a, b]$ contains no critical values of f then $\Lambda(a)$ and $\Lambda(b)$ are homotopy equivalent (and even homeomorphic). If t is sufficiently small then $\Lambda(t)$ is a ball; if t is sufficiently large then $\Lambda(t)$ is a deformation retract of the desired space $\mathbb{R}^N \setminus L$. Thus $\mathbb{R}^N \setminus L$ can be constructed from a topologically trivial space by a finite sequence of local surgeries corresponding to all critical values of f.

For instance let us consider the line arrangement in $\mathbb{R}^2$ shown in Fig. 1 and its complexification $\mathcal{L}$ in $\mathbb{C}^2$. There are four essentially different noncritical values $t_1 < t_2 < t_3 < t_4$: the intersections of the corresponding balls B_{t_i} with $\mathbb{R}^2$ are shown in Fig. 1. Let $\Lambda_i \equiv \Lambda(t_i)$ be the corresponding varieties of our inductive process.

The manifold Λ_1 is topologically trivial. Passages from Λ_1 to Λ_2 and from Λ_2 to Λ_3 are homotopy equivalent to gluing 1-dimensional cells, so that Λ_3 is homotopy equivalent to the bouquet of two circles. Finally,

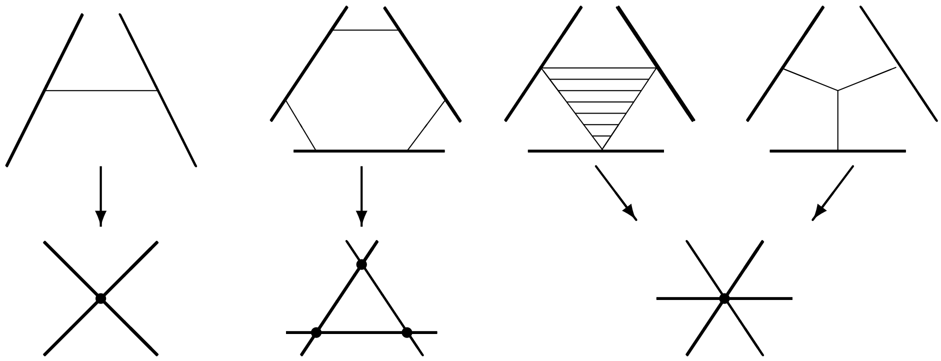

FIGURE 2. Simplicial resolutions of line arrangements

passage from Λ_3 to Λ_4 is equivalent to addition of a 2-dimensional cell. And indeed, it is easy to see that the resulting manifold Λ_4 is homotopy equivalent to the two-dimensional torus.

In general, for an arbitrary arrangement the inductive calculation of homology groups of $\mathbb{R}^N \setminus L$ includes many local problems of the following sort. Suppose that values a, b are non-critical and the segment $[a, b]$ contains exactly one critical value of f; namely, it is the critical value of the restriction of f to some plane L_I. Topological types of manifolds Λ_a, Λ_b differ by a surgery localized in a small neighborhood of the corresponding critical point of f on L_I. How does it relate the cohomology groups of these manifolds?

This problem was solved in [41] in the combinatorial terms of our arrangement. The answer is formulated in terms of the *order complex* of a partially ordered set (= *poset*).

Definition 1. *Given a poset* $(A, <)$*, the corresponding order complex* $\Upsilon(A)$ *is the simplicial complex, whose vertices are the points of the set* A*, and the simplices span all the sequences of such points monotone with respect to the partial order.*

Every plane arrangement $\mathcal{L} = \{L_1, \dots, L_m\}$ defines the poset of all corresponding nonempty sets L_I, $I \subset \{1, \dots, m\}$, and hence the order complex $\Upsilon(\mathcal{L})$.

For instance for three line arrangements shown in the lower row of Fig. 2 the corresponding order complexes are given in Fig. 3. For the diagonal arrangement $A(4, 2)$ (see §2) the order complex is shown in Fig. 4.

To any I with $L_I \neq \emptyset$ the order subcomplex $\Upsilon(I) \subset \Upsilon(\mathcal{L})$ is associated: this is the union of all simplices in $\Upsilon(\mathcal{L})$ all whose vertices correspond to planes L_J containing L_I. Any such subcomplex $\Upsilon(I)$ is

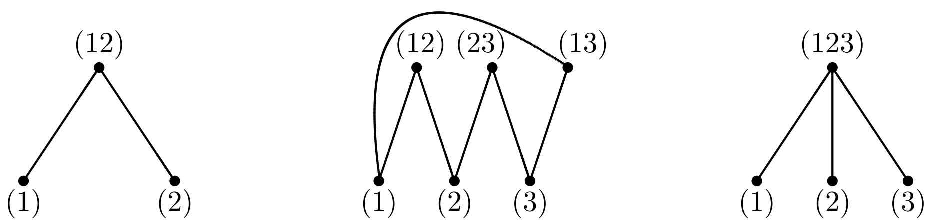

FIGURE 3. Order complexes for some line arrangements

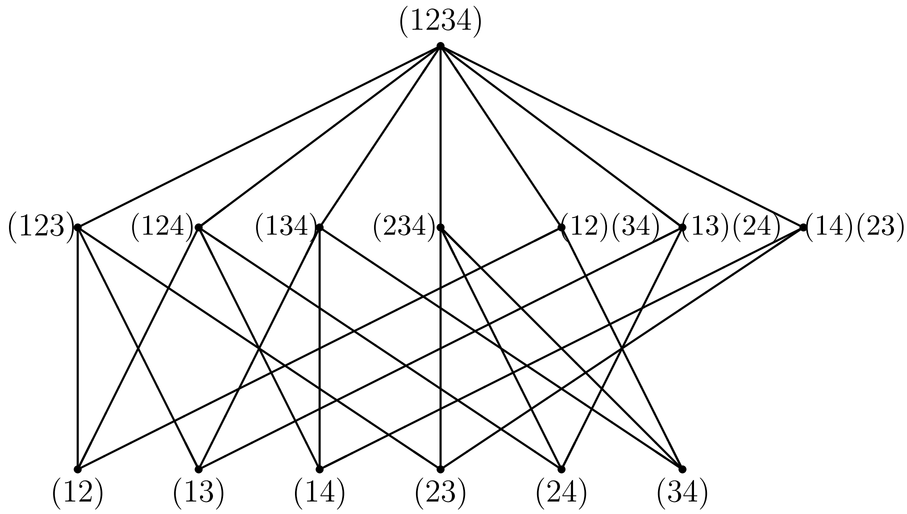

FIGURE 4. Poset and order complex for the diagonal arrangement $A(4, 2)$

contractible: indeed, all its maximal simplices have the common vertex corresponding to the plane $\Upsilon(I)$ itself. Denote by $\partial\Upsilon(I)$ the *link* of this subcomplex, i.e. the union of all its simplices not containing its maximal vertex $\{\Upsilon(I)\}$.

Theorem 5 (see [41])**.** *Suppose that the segment $[a, b]$ contains exactly one critical value of f; let L_I be the corresponding plane. Then*

1) $H^i(\Lambda(b), \Lambda(a)) \simeq H_{N-i-\dim L_I-1}(\Upsilon(I), \partial\Upsilon(I))$;

2) The exact homological sequence of the pair $(\Lambda(b), \Lambda(a))$ splits, i.e. $\tilde{H}^*(\Lambda(b)) \simeq H^*(\Lambda(b), \Lambda(a)) \oplus \tilde{H}^*(\Lambda(a))$; here $\tilde{H}^*$ denotes the cohomology group reduced modulo a point.

Corollary 2 (see [41])**.** *For an arbitrary affine plane arrangement $\mathcal{L}$ in $\mathbb{R}^N$,*

$$\tilde{H}^i(\mathbb{R}^N \setminus L) \simeq \bigoplus_I H_{N-i-\dim L_I - 1}(\Upsilon(I), \partial\Upsilon(I)), \tag{8}$$

summation over all nonempty planes L_I of the arrangement.

In section 8 we shall explain this formula and prove its homotopical version.

Example. Suppose that we have a complex hyperplane arrangement with normal crossings in $\mathbb{C}^N$. Then the group $H^r(\mathbb{C}^N \setminus L)$ is isomorphic to $\mathbb{Z}^{\lambda(r)}$ where $\lambda(r)$ is the number of sets of indices $I = \{i_1, \dots, i_r\}$ of cardinality r such that $L_I \neq \emptyset$. Indeed, for any such set I the corresponding order subcomplex $\Upsilon(I)$ is (the first barycentric subdivision of) an $(r-1)$-dimensional simplex, its link coincides with the boundary of this simplex, the group $H_*(\Upsilon(I), \partial\Upsilon(I))$ has unique non-trivial term $\mathbb{Z}$ in dimension $r-1$, and $\dim_{\mathbb{R}} L_I = 2N-2r$. In particular, for the *generic* arrangement of m hyperplanes in $\mathbb{C}^N$ we have $H^r(\mathbb{C}^N \setminus L) \sim \mathbb{Z}^{\binom{m}{r}}$ for any $r = 0, 1, \dots, N$. In the last case a more strong statement holds: the space $\mathbb{C}^N \setminus L$ is homotopy equivalent to the N-skeleton of (the standard cell decomposition of) the m-dimensional torus. Indeed, let us consider the *universal* hyperplane m-arrangement $\dagger^m$, i.e. the coordinate cross in $\mathbb{C}^m$. There exists an affine embedding $\phi : \mathbb{C}^N \to \mathbb{C}^m$ such that the planes of our generic arrangement in $\mathbb{C}^N$ are preimages of intersections of $\phi(\mathbb{C}^N)$ with coordinate planes in $\mathbb{C}^m$. The cohomology group of the space $\mathbb{C}^m \setminus \dagger^m \sim T^m$ can be calculated as above by means of an arbitrary generic real quadratic function $f : \mathbb{C}^m \to \mathbb{R}$. It has exactly $2^m - 1$ critical values corresponding to all planes $L!_I$, and the surgeries corresponding to the passages through these critical values are homotopy equivalent to adding the cells of the *standard* cell decompositions of the torus. Now, we can choose our function f with center at a point of the embedded plane $\phi(\mathbb{C}^N) \subset \mathbb{C}^m$ and in such a way that it grows very slowly along $\phi(\mathbb{C}^N)$ and very fast in the transversal directions. The "balls" $\{x : f(x) \leq t\}$ will then look like pancakes spread along $\phi(\mathbb{C}^N)$. For some t, such a ball will intersect all planes L_I with $|I| \leq N$ but do not intersect any planes of smaller dimensions. The corresponding manifold $\Lambda(t)$ is homotopy equivalent to $\mathbb{C}^N \setminus L$, on the other hand it is homotopy equivalent to the N-skeleton of the torus $T^m \sim \mathbb{C}^m \setminus L!$.

The isomorphism (8), as it follows from its proof in [41], is not canonical: its realization depends on some choices. However it allows one to define an important increasing filtration in the group $H^*(\mathbb{R}^N \setminus L)$: for any realization of this isomorphism the term F_i of this filtration corresponds to the sum of terms $H_*(\Upsilon(I), \partial\Upsilon(I))$ over all planes L_I of codimensions $\leq i$. This definition already does not depend on the choices, see §§8, 10.

7. Simplicial resolutions and inclusion-exclusion formula. Mayer–Vietoris spectral sequence and its modifications

The simplicial resolutions are a far extension of the combinatorial formula of inclusions and exclusions. They allow us to study effectively the topology of spaces represented as unions of several subspaces, whose (multiple) intersections are much easier than their symmetric differences.

First let us demonstrate this method in the simplest discrete situation. Suppose that a finite set S is represented as the union of finitely many finite sets S_i, $i = 1, \dots, m$, and we need to find the cardinality of S. To do it we construct the simplicial resolution of S. First, we take all sets S_i separately. If some two sets S_i, S_j have a common point, then we draw a segment between the corresponding points in the separated copies of S_i and S_j. If some point belongs to the triple intersection $S_i \cap S_j \cap S_k$ then we get three separated points joined by three segments. On the next step, we add the "interior part" of this triangle, then construct tetrahedra over quadruple intersections, etc. The obtained complex S' is homotopy equivalent to the initial set S: to any point of S there corresponds a simplex in S'. In particular S and S' have equal Euler characteristics. But the Euler characteristic of the finite set S is its cardinality, while that of S' is the number of vertices (i.e. of points of all sets S_i taken separately) minus the number of edges (i.e. the sum of cardinalities of all sets $S_i \cap S_j$ over all pairs $(i \neq j) \subset \{1, \dots, r\}$) plus the number of triangles, etc. The result is nothing else than the exclusion-inclusion formula.

The same method works in the "continuous" case, say if the set $S = \cup S_i$ is a CW-complex, and all sets S_i and all their intersections S_I are its cell subcomplexes. Namely, we consider the $(m-1)$-dimensional simplex Δ whose vertices are in one to one correspondence with the indices $1, \dots, m$. The simplicial resolution of S can be constructed as a subset in $\Delta \times S$. For any set of indices I we take the simplex $\Delta(I)$ whose vertices are the points of the set I. The simplicial resolution S' is defined as the union of all products $\Delta(I) \times S_I$ over all subsets I. The obvious projection $\Delta \times S \to S$ induces the map $S' \to S$. It is easy to see that this map is proper and is a homotopy equivalence. This space S' is often much easier to study than the initial space S.

Example. If there are only two sets S_1, S_2, and $S = S_1 \cup S_2$, then we get the Mayer-Vietoris exact sequence
(9)
$$\cdots \to H_i(S_1 \sqcup S_2) \to H_i(S) \to H_{i-1}(S_1 \cap S_2) \to H_{i-1}(S_1 \sqcup S_2) \to \cdots .$$

Indeed, the corresponding simplex Δ is the segment $[1,2]$. The disjoint union $S_1 \sqcup S_2$ can be realized as the subset $(\{1\} \times S_1) \cup (\{2\} \times S_2) \subset [1,2] \times (S_1 \cup S_2)$, and we can consider the exact sequence of the pair $(S', (\{1\} \times S_1) \cup (\{2\} \times S_2))$.

More generally, for an arbitrary number m of sets S_i the resolved complex S' also has a standard filtration $\phi_1 \subset \cdots \subset \phi_m$: its set ϕ_k is the union over $l = 1, \dots, k$ of all sets $\Delta(I) \times S_I$ with l-element subsets $I \in \{1, \dots, m\}$. The corresponding spectral sequence calculating the homology of $S' \sim S$ is called the *Mayer-Vietoris spectral sequence* of the composite set $S = \cup S_i$. It is useful in some topological problems, however in the study of plane arrangements it is quite useless: in this case a different filtration in the resolution set should be considered, see the next section.

8. Homotopy type of an affine plane arrangement and stable homotopy type of its complement

Let our spaces S_i be affine planes in $\mathbb{R}^N$ forming the arrangement $\mathcal{L}$. In the bottom row of Fig. 2 we give three examples of line arrangements in $\mathbb{R}^N$, in the top row their simplicial resolutions are indicated. Over the right-hand picture we have two different resolutions. The left one of them is constructed exactly as previously: the preimage of the central point is the entire simplex Δ whose vertices correspond to all lines L_i.

In the general situation, let us denote by $\Delta(\mathcal{L})$ the union of all faces $\Delta(I) \subset \Delta$ such that the plane L_I is not empty. The simplicial resolution of the plane arrangement $\mathcal{L}$ constructed previously is a subset of $\Delta(\mathcal{L}) \times L$. Another, more economical resolution is constructed as follows. Instead of the complex $\Delta(\mathcal{L}) \subset \Delta$ we can consider the order complex $\Upsilon(\mathcal{L})$ and define the simplicial resolution as a subspace of $\Upsilon(\mathcal{L}) \times L$: namely as the union of all products $\Upsilon(I) \times L_I$ over all nonempty planes L_I of the arrangement. The order complexes of three arrangements of Fig. 2 are shown in Fig. 3.

For two left arrangements in Fig. 2 both constructions give one and the same space, but for the right-hand one the latter construction gives a different space, see the very right top picture.

These two constructions are homotopy equivalent. Indeed, the order complex $\Upsilon(\mathcal{L})$ can be identified with a subcomplex of the first barycentric subdivision of $\Delta(\mathcal{L})$: the vertex $\{L_I\}$ of $\Upsilon(\mathcal{L})$ goes to the center of the simplex in $\Delta(\mathcal{L})$ spanned by all vertices $\{L_i\}$ with $L_i \supset L_I$. This identification defines an embedding of the latter construction into the former one. Conversely, for any set of indices I with nonempty L_I we can send the center of the simplex $\Delta(I)$ into the vertex $\{L_I\} \in \Upsilon(\mathcal{L}) \subset \Delta(\mathcal{L})$; extending this map by linearity we obtain a map homotopy inverse to our embedding. We shall call two constructions of simplicial resolutions using the complexes $\Delta(\mathcal{L})$ and $\Upsilon(\mathcal{L})$ the *naive* and the *economical* simplicial resolutions, respectively.

Almost all further considerations in this section are equally true for both constructions. In particular we have the following their properties (see Proposition 5). Suppose that $L_I \neq \emptyset$. Some face $\Delta(J)$ of the simplex $\Delta(I)$ is called *marginal* if the corresponding plane L_J is strictly greater than L_I. Denote by $\partial\Delta(I)$ the subcomplex in $\Delta(I)$ formed by all its marginal faces.

Proposition 5. *1. There is a homotopy equivalence $\Delta(\mathcal{L}) \sim \Upsilon(\mathcal{L})$.*
2. For any nonempty plane L_I, the pairs $(\Delta(I), \partial\Delta(I))$ and $(\Upsilon(I), \partial\Upsilon(I))$ are homotopy equivalent. $\Box$

In particular the Goresky–MacPherson formula (8) can be rewritten in the following way

$$\tilde{H}^i(\mathbb{R}^N \setminus L) \simeq \bigoplus_I H_{N-i-\dim L_I - 1}(\Delta(I), \partial\Delta(I)), \tag{10}$$

The space of the resolution of the arrangement with support L will be denoted by L', and its one-point compactification by $\bar{L}'$. The next well-known fact, basic for the entire theory of simplicial resolutions, follows easily from the Borsuk's lemma, see e.g. [77] (unfortunately I do not know the first reference).

Proposition 6. *The obvious projection $\pi : L' \to L$ is a proper map and a homotopy equivalence. Its extension to a map of one-point compactifications, $\bar{\pi} : \bar{L}' \to \bar{L}$, also is a homotopy equivalence.*

Theorem 6. *For any finite affine plane arrangement with support L, there are homotopy equivalences*

$$L' \sim L \sim \Upsilon(\mathcal{L}) \sim \Delta(\mathcal{L}), \tag{11}$$

$$\bar{L}' \sim \bar{L} \sim \bigvee_I \Sigma^{\dim L_I}(\Upsilon(I)/\partial\Upsilon(I)) \sim \bigvee_I \Sigma^{\dim L_I}(\Delta(I)/\partial\Delta(I)), \tag{12}$$

where Σ^k denotes the k-fold suspension, and the bouquets in (12) taken over all nonempty planes L_I of the arrangement.

The middle equivalence in (11) was found in [41], see also [18].

The middle equivalence $\bar{L} \sim \bigvee_I \Sigma^{\dim L_I}(\Upsilon(I)/\partial\Upsilon(I))$ of (12) was proved in [96]; simultaneously the composite equivalence

$$\bar{L}' \sim \vee_I \Sigma^{\dim L_I}(\Delta(I)/\partial\Delta(I))$$

of (12) was proved, see [80]. By Propositions 5, 6 these two equalities involved in the formula (12) are equivalent to one another and to entire this formula.

For other statements of this type see [58], [45].

Corollary 3. *The Goresky–MacPherson formula* (*8*), (*10*).

Indeed, this formula follows from (12) and the Alexander duality

$$\tilde{H}^i(\mathbb{R}^N \setminus L) \simeq \bar{H}_{N-i-1}(L) \tag{13}$$

where $\bar{H}_*$ is the *Borel–Moore homology group*, i.e. the homology group of the one-point compactification reduced modulo the added point; cf. [4]. (An equivalent definition: the Borel–Moore homology group is the homology group of the complex of locally finite singular chains in X.)

The resolution space L' has a very useful filtration: its term $F_i(L)$ is defined as the union of all products $\Delta(I) \times L_I$ (respectively, $\Upsilon(I) \times L_I$) over all nonempty planes L_I of codimensions $\leq i$. In particular $L' = F_N(L)$. This filtration extends to that on $\bar{L}'$: the term $\bar{F}_0$ of the latter filtration is the added point and $\bar{F}_i, i \geq 1$, is the closure of F_i. The filtration mentioned in the end of §6 is Alexander dual to the corresponding filtration in the homology of $\bar{L}' \sim \bar{L}$.

Corollary 4. *The stable homotopy type of the complement of an arbitrary affine plane arrangement $\mathcal{L}$ is determined by the dimensions of all its planes L_I, in particular the same is true for all extraordinary homology and cohomology groups.*

This corollary is based on the following notion.

Definition. Two topological spaces (having homotopy types of CW-complexes) are *Spanier–Whitehead dual* to one another if they are homotopy equivalent to two complementary subsets X and $Y \equiv S^N \setminus X$ of a sphere S^N.

The homology and cohomology groups of such spaces are related by the Alexander duality.

For instance our spaces $\bar{L}$ and $\mathbb{R}^N \setminus L$ are Spanier–Whitehead dual to one another.

The important fact (see e.g. [89]) is that the Spanier–Whitehead duality determines an involution on the set of stable homotopy types: all spaces Spanier–Whitehead dual to stably homotopy equivalent spaces are stably homotopy equivalent to one another. This reduces Corollary 4 to Theorem 6.

The use of simplicial resolutions makes the proof of Theorem 6 especially transparent.

Indeed, let us consider the projection $L' \to \Upsilon(\mathcal{L})$ induced by the standard projection of the space $\Upsilon(\mathcal{L}) \times L \supset L'$. All fibers of this map are planes L_I for certain sets I. The homotopy equivalence $L' \sim \Upsilon(\mathcal{L})$ follows by induction over the consequent contractions of these fibers over different strata of $\Upsilon(\mathcal{L})$, cf. Lemma 1 in §III.3.4 of [77]; at any step of induction the homotopy equivalence follows from the Borsuk's lemma.

To prove (12) we use a version of the induction from §6. For any $t > 0$ we denote by $\bar{L}(t)$ the quotient space $L/(L \cap \{x : f(x) \geq t\})$. Then for sufficiently small t we have $\bar{L}(t) = \{\text{one point}\}$; for sufficiently large t $\bar{L}(t)$ is homotopy equivalent to $\bar{L}$. If the segment $[a, b]$ contains no critical values then $\bar{L}(a)$ is homotopy equivalent to $\bar{L}(b)$, so that all we need is the following lemma.

Lemma 1. *If the segment $[a, b]$ contains only one critical value of f, namely the critical value of its restriction to the plane L_I, then we have a homotopy equivalence*

$$\bar{L}(b) \sim \bar{L}(a) \vee \Sigma^{\dim L_I}(\Upsilon(I)/\partial\Upsilon(I)). \tag{14}$$

Proof. Let us consider also the spaces $\bar{L}'(t) = L'/(L' \cap \pi^{-1}(\{x : f(x) \geq t\}))$. Then the projection π induces homotopy equivalences $\bar{L}'(t) \sim \bar{L}(t)$, and it is enough to prove a version of Lemma 1 with $\bar{L}(b)$ and $\bar{L}(a)$ replaced by $\bar{L}'(b)$ and $\bar{L}'(a)$ respectively. In this proof we use the topological operation of attaching topological spaces by maps. Namely, given two topological spaces X, Y, a subspace $A \subset X$ and continuous map $\phi : A \to Y$, the space $Y \cup_\phi X$ is defined as the quotient space of the disjoint union $X \sqcup Y$ through the relations $a \sim \phi(a)$ for all $a \in A$. In particular if we have $Z = X \cup Y$ then Z can be considered as X attached to Y via the identical embedding $X \cap Y \to Y$. An important property of this operation is its *homotopy invariance*: any homotopy equivalence $f : Y \to Y'$ induces a homotopy equivalence

$$Y \cup_\phi X \sim Y' \cup_{f \circ \phi} X. \tag{15}$$

For any plane L_J of our arrangement, let L'_J be its *proper preimage*, i.e. the set $\Upsilon(J) \times L_J \subset \Upsilon(\mathcal{L}) \times L$. Let $L!a \subset L'$ be the union of proper preimages of all planes L_J such that $L_J \cap \{x : f(x) < a\} \neq \emptyset$. In particular $L!b = L'_I \cup L!a$.

Given a subspace $X \subset L'$, denote by $X_{/t}$ its reduction modulo the set of its points x such that $f \circ \pi(x) \geq t$. Then we have

$$\bar{L}'(b) = (L!b)_{/b} = (L'_I)_{/b} \cup (L!a)_{/b} \equiv (L'_I)_{/b} \cup_{\text{id}} (L!a)_{/b} \tag{16}$$

where id is the identical embedding $(L'_I)_{/b} \cap (L!a)_{/b} \to \cup(L!a)_{/b}$.

Lemma 2. *There is a homotopy equivalence $(L!a)_{/b} \to (L!a)_{/a} \equiv \bar{L}'(a)$ induced by the reduction modulo the layer $\{x : f \circ \pi(x) \in [a, b]\}$.* □

This homotopy equivalence maps the entire set $(L'_I)_{/b} \cap (L!a)_{/b}$ into one point (obtained by the factorization from this layer). Therefore by (15) the space (16) is homotopy equivalent to the wedge of $\bar{L}'(a)$ and the quotient space $(L'_I)_{/b}/((L'_I)_{/b} \cap (L!a)_{/b})$.

The latter space $(L'_I)_{/b}/((L'_I)_{/b} \cap (L!a)_{/b})$ is homotopy equivalent to $(\Upsilon(I) \times L_I)/((\Upsilon(I) \times (L_I))_{/b} \cup (\partial\Upsilon(I) \times L_I)) \sim \Sigma^{\dim L_I}(\Upsilon(I)/\partial\Upsilon(I))$ (cf. [80]); Lemma 1 and Theorem 6 are proved.

9. Examples: resolutions of important arrangements. Complexes of connected graphs and hypergraphs

Let us consider again the diagonal arrangement $A(N, 2)$ in $\mathbb{C}^N$ or $\mathbb{R}^N$, see §2, and its naive simplicial resolution $L' \subset \Delta(\mathcal{L}) \times L$. The smallest plane L_I of this arrangement is the line $\{x_1 = \cdots = x_N\}$. The preimage of any its point is the entire simplex $\Delta \equiv \Delta(\mathcal{L})$ whose $\binom{N}{2}$ vertices correspond to all possible hyperplanes $V_{ij} \equiv \{x_i = x_j\}$.

Let us draw somewhere N points labelled by numbers $1, \ldots, N$. It is convenient to depict any hyperplane V_{ij} by the segment connecting the points i and j. Any face of the simplex Δ defines the graph consisting of segments corresponding to all vertices of this simplex. It is easy to see that the subcomplex of marginal faces $\partial\Delta(\mathcal{L}) \subset \Delta(\mathcal{L})$ consists of all faces corresponding to not connected graphs. Thus the homological study of our arrangement appeals to the homology group of the *complex of connected graphs* which is defined as the quotient complex of the standard (acyclic) triangulation of the simplex Δ through the subcomplex spanned by all faces corresponding to all not connected graphs.

Proposition 7 (see [80]). *The complex of connected graphs with N vertices is acyclic in all dimensions other than $N-2$. Its $(N-2)$-dimensional homology group is isomorphic to $\mathbb{Z}^{(N-1)!}$ and is freely generated by the classes of all* snake-like *(i.e. homeomorphic to a segment) trees, one of whose endpoints is fixed.*

The first assertion of this proposition is a special case of the Folkman's theorem on the homology of geometric lattices [32].

Remark. The number $(N-1)!$ already appeared in this work. Indeed, by Proposition 2 the group $H^{N-1}(M(N,2))$ is $(N-1)!$-dimensional. In the Goresky–MacPherson formula this group corresponds to the summand $H_{N-1}(\Upsilon(I), \partial\Upsilon(I)) \equiv H_{N-1}(\Delta(I), \partial\Delta(I))$ where I is the entire set $\{1, \dots, N\}$.

A nice description of the *co*homology group of the same complex is given in [71].

A natural generalization of this complex is provided by *complexes of i-connected graphs*, see [82], [9], [69], [70]. They also have important applications in the differential and homotopy topology.

In a similar way, if we consider the k-equal arrangement $A(N,k)$ in $\mathbb{R}^N$ or in $\mathbb{C}^N$ (see §2) then the smallest plane L_I is again the line $\{x_1 = \dots = x_N\}$. The corresponding simplex Δ has $\binom{N}{k}$ vertices, its faces are the *k-hypergraphs* with the same N nodes $1, \dots, N$, and the marginal subcomplex $\partial\Delta \subset \Delta$ is the complex of *not connected hypergraphs.* The homology groups of this complex (and of the complement $M(N,k)$ of the arrangement) were studied by A. Björner and V. Welker [19], see also [17], [18].

In particular, the following facts were proved.

Theorem 7 (see [19])**.** *For any $k \geq 2$ the simplex with $\binom{N}{k}$ vertices reduced modulo the union of faces corresponding to non-connected k-hypergraphs is homotopy equivalent to a wedge of spheres, in particular all its homology groups are torsion-free. Moreover, these groups can be nontrivial only in dimensions equal to $N-(k-2)t-2$, $1 \leq t \leq N/k$. The ranks of these groups are multiples of $\binom{N-1}{k-1}$, and in the higher possible dimension $d = N-k$ the rank of H_{N-k} is equal to $\binom{N-1}{k-1}$.*

A general formula for these ranks also is given in [19] (see Theorem 4.5 there), but it is much more complicated.

Remark (see [18]). The topology of the real variety $M(N,k) \subset \mathbb{R}^N$ gives good estimates in the following olympic problem. Suppose we have N coins, some of which are fake, and a weighbridge. It is natural to assume that all regular coins are of the same weight, and the weights of all fake are different from one another and from the weight of regular coins. Given some $k \geq 2$, how many measurements is it enough to do to check that we have at least k regular coins?

Indeed, any measurement separates the space $\mathbb{R}^N$ of all possible collections of weights into three convex parts: two half-spaces and the hyperplane separating them. Any sequence of measurements together with their results specifies some cell (maybe empty) in $\mathbb{R}^N$. Any correct strategy of solving our problem (such strategies are called *decision trees*) should separate our space $\mathbb{R}^N$ into such cells, any of which belongs to either the arrangement $A(N,k)$ or to its complement $M(N,k)$. Thus the number of generators of the total homology group of either of these spaces provides a lower estimate of the number of cells, and hence of the complexity of the decision tree.

Among the origins of the theory of arrangements there is one class of olympic problems more: that on the cut cake, see [93].

10. Combinatorial realization of cohomology classes of complements of arrangements

The Goresky–MacPherson formula (8) has the following direct realization (found essentially in [96], the present form given in [55]). Suppose that an Euclidean metric is fixed in $\mathbb{R}^N$.

Consider a constant vector field V ("power") in $\mathbb{R}^N$. For any r-dimensional simplex of the order subcomplex $\Upsilon(I)/\partial\Upsilon(I)$ (i.e. for a strictly decreasing sequence of $r+1$ planes $L_{I_1} \supset L_{I_2} \supset \ldots \supset L_{I_r} \supset L_I$) and any point $x \in L_I$ consider the sequence of $r+1$ rays in $\mathbb{R}^N$ issuing from x, namely the trajectories of x in the planes $\mathbb{R}^N, L_{I_1}, \ldots, L_{I_r}$ under the action of this power. (We can realize V as the gradient field of a linear function $\theta : \mathbb{R}^N \to \mathbb{R}$, then these rays will be the trajectories of gradients of restrictions of θ to these planes.)

Definition. The constant vector field V is *in general position* with respect to the plane arrangement L if for any I and any simplex in $\Upsilon(I)/\partial\Upsilon(I)$ these rays are linearly independent in $\mathbb{R}^N$.

It is easy to see that such vector fields form an open dense subset in the space $\mathbb{R}^N$ of all constant fields. Let us assume that our field V is generic. Then for any I and simplex as above the convex hull of our $r+1$ rays is linearly homeomorphic to an $(r+1)$-dimensional octant with origin at x. Such octants over all $x \in L_I$ sweep out an $(r+1+\dim L_I)$-dimensional wedge in $\mathbb{R}^N$.

If we have a r-dimensional cycle α of the complex $\Delta(I)/\partial\Delta(I)$, then the sum of (uniformly oriented) corresponding wedges is a relative cycle in $\mathbb{R}^N(\text{mod } L)$, and the relative homology class $\nabla\alpha \in H_{r+1+\dim L_I}(\mathbb{R}^N, L)$ of the latter cycle depends on the class of α in $H_*(\Delta(I), \partial\Delta(I))$ only (up to a sign depending on the choice of the orientation of the plane L_I).

Finally we take the class in $H^*(\mathbb{R}^N \setminus L)$ Poincaré–Lefschetz dual to $\nabla\alpha$ in $\mathbb{R}^N \setminus L$, i.e. defined by intersection indices with the relative cycle $\nabla\alpha$.

This realization of the formula (8) depends on the choice of the direction V, but not very much. Two elements in $\bar{H}_*(\mathbb{R}^N, L)$, corresponding in this way to one and the same class $\alpha \in H_*(\Delta(I), \partial\Delta(I))$ via different generic vector fields can differ by elements of lower filtration only: more precisely, by sums of similar classes coming from the summands $H_*(\Delta(J), \partial\Delta(J))$ corresponding to planes L_J *strictly containing* L_I.

Moreover, if all planes L_I have codimensions ≥ 2 in all greater planes L_J, then the isomorphism (8) is canonical (up to the choice of orientations of all planes L_I): indeed, in this case the space of generic vectors fields V is path-connected.

By the analogy with the knot theory (cf. [63], [85]), such realizations of elements of $H^*(\mathbb{R}^N \setminus L)$ can be called their *combinatorial expressions.*

This construction allows one to investigate the multiplicative structure in $H^*(\mathbb{R}^N \setminus L)$, in particular to determine this structure in the associated graded ring.

11. Multiplication in cohomology.

Let us rewrite the equality (8) as that for associated graded groups:

$$GrH^*(\mathbb{R}^N \setminus L) \cong \oplus H_{k-*-1-\dim\, L_I}(\Upsilon(I), \partial\Upsilon(I)) \tag{17}$$

This isomorphism is canonical (up to the choice of orientations of planes L_I), and the multiplication in the associated graded *ring* is as follows.

Let us consider two planes $L_I, L_J \subset L$ and two cycles A, B of the quotient complexes $\Upsilon(I)/\partial\Upsilon(I)$ and $\Upsilon(J)/\partial\Upsilon(J)$, dim $A = u$, dim $B = v$, represented by chains (=linear combinations of simplices) of subcomplexes $\Upsilon(I), \Upsilon(J)$ with boundaries in $\partial\Upsilon(I)$ and $\partial\Upsilon(J)$ only. The *shuffle product* $A\diamondsuit B$ of these cycles is defined as follows (see [90]).

If L_I and L_J are not transversal (i.e. belong to some proper plane in $\mathbb{R}^N$) or have no intersection points, then $A\diamondsuit B = 0$. Now suppose that L_I and L_J are transversal and $L_K = L_I \cap L_J \neq \emptyset$ (we can take $K = I \cup J$). Let $a \subset A$ and $b \subset B$ be some two simplices with $u+1$ and $v+1$ vertices respectively, i.e. some decreasing sequences of intersection planes of L having $\{L_I\}$ and $\{L_J\}$ as their last elements. Consider all $\binom{u+v+2}{u+1}$ possible *shuffles* of these sequences, i.e. all (non-monotone) sequences of $u+v+2$ planes in which all elements of a and b appear preserving their orders in the sequences a and b. To any such shuffle a monotone sequence corresponds: any element λ of the shuffle coming from the sequence a (respectively, b) should be replaced by the intersection of the corresponding plane with the last plane coming from the sequence b (respectively, a) and staying before λ in the shuffle. The obtained monotone sequence is by definition an $(u+v+1)$-dimensional simplex of the order complex $\Upsilon(K)$. The shuffle product of our simplices a and b is defined as the sum of all such simplices taken with signs equal to parities of the corresponding shuffles (i.e. numbers of transpositions reducing them to the simple concatenation of sequences a and b) multiplied by one sign more, which depends on multi-indices I, J and K only and is defined by the comparison of the fixed coorientation of the plane L_K in $\mathbb{R}^N$ with the ordered pair of coorientations of L_I and L_J. The shuffle product of cycles A and B is defined by linearity. It is a relative cycle defining an element of the summand in the right-hand part of (17) corresponding to the plane L_K; this element depends only on homology classes of A and B in the summands corresponding to L_I and L_J.

Theorem 8 (see [90], [91], [25], [29])**.** *The isomorphism* (*17*) *commutes the shuffle product in its right-hand part and the multiplication in its left part obtained from the usual cohomological multiplication. If all planes L_I have codimensions ≥ 2 in all greater planes $L_{I'}$, then the same is true for the isomorphism* (*8*) *and the multiplication in the ring $H^*(\mathbb{R}^m \setminus L)$ itself, and not in its graded ring only.*

This is a corollary of the explicit construction of relative homology classes described in the previous section. Indeed, the multiplication in the cohomology ring of an oriented manifold M can be realized as follows. Given two classes $\alpha, \beta \in H^*(M)$, we take Borel–Moore cycles $[\alpha], [\beta]$ Poincaré dual to them and meeting transversally, take their intersection $[\alpha] \cap [\beta]$ supplied with natural orientation, and consider the cohomology class Poincaré dual to this intersection.

Given two planes L_I, L_J of our arrangement and classes

$$\alpha \in H_*(\Upsilon(I), \partial\Upsilon(I)), \qquad \beta \in H_*(\Upsilon(J), \partial\Upsilon(J)),$$

we can realize corresponding elements in the left part of (8) with the help of different constant vector fields V_I, V_J in $\mathbb{R}^N$ that are in general position to one another if L_I and L_J have nonempty transversal intersection; if not then these directions should be opposite to one another and transversal to a hyperplane separating or containing these planes.

Proposition 4 follows immediately from Theorem 8.

Exercise: deduce the (Orlik–Solomon) Theorem 2 from this one.

12. Salvetti complex for complexified real hyperplane arrangement

Consider the complexification of a real affine hyperplane arrangement, i.e. the set of complex hyperplanes $L_j \subset \mathbb{C}^N$, $j = 1, \ldots, m$, distinguished by the same linear equations $f_j(x) = 0$, $f_j(\mathbb{R}^N) = \mathbb{R}$. Its complement $\mathbb{C}^N \setminus L$ is an N-dimensional Stein manifold, in particular is homotopy equivalent to a cell complex of dimension $\leq N$. M. Salvetti [65], following some ideas of [28], has constructed explicitly an N-dimensional simplicial complex, embedded into the space $\mathbb{C}^N \setminus S$ as its deformation retract. Here we give an easy description of this construction in the terms of the dual complex.

For any one of our planes L_j, its complement $\mathbb{C}^N \setminus L_j$ can be subdivided into four cells $+_j$ $-_j$, $\uparrow_j$ $\downarrow_j$, given by conditions Re$f_j > 0$, Re$f_j < 0$, $\{\text{Re}f_j = 0, \text{Im}f_j > 0\}$ and $\{\text{Re}f_j = 0, \text{Im}f_j < 0\}$ respectively. To any of 4^m possible sequences of m signs $+, -, \uparrow$ and $\downarrow$ we associate the intersection of corresponding cells (like e.g. $(+_1) \cap (\uparrow_2) \cap (-_3) \cap \ldots \cap (\uparrow_m)$). This intersection of several real affine planes and open half-spaces in $\mathbb{C}^N$ is homeomorphic to a cell. By definition it lies in $\mathbb{C}^N \setminus L$, and any point of $\mathbb{C}^N \setminus L$ belongs to exactly one cell of this sort.

Lemma 3. *The subdivision of the manifold $\mathbb{C}^N \setminus L$ into cells corresponding to all possible sequences of signs $+, -, \uparrow$ and $\downarrow$, augmented with one 0-dimensional cell, defines a cellular structure on the one-point compactification of this manifold.*

The proof is elementary, cf. [33]. $\square$

The *Salvetti complex* (as a topological space) is just the complex dual to this cell decomposition. As a combinatorial object, it is defined as a certain subdivision of this dual complex.

This construction was used in [37] for defining some topological invariants of abstract *oriented matroids* (see §14 and [14]). See also [20].

The above described cell decomposition of $\mathbb{C}^N \setminus L$ can be simplified very much if our arrangement has only normal crossings. In this case to any plane L_I distinguished by several equations $f_i = 0, i \in I$, $f_i(\mathbb{R}^N) \subset \mathbb{R}$, we associate the *imaginary wedge* in $\mathbb{C}^N$ distinguished by the conditions $\{\text{Re } f_i = 0, \text{Im } f_i > 0\}$, $i \in I$. Denote by ∇_I this wedge from which its intersections with L and with all smaller wedges ∇_J, $J \supsetneqq I$, are removed.

Lemma 4 (see [75]). *If L is the complexification of a real hyperplane arrangement with normal crossings, then any nonempty set ∇_I is homeomorphic to a cell of dimension $2N - \#I$, and all these sets form a cellular decomposition of the one-point compactification of $\mathbb{C}^N \setminus L$.*

All the incidence coefficients of the corresponding cell complex are trivial, therefore the Borel–Moore homology group $\bar{H}_*(\mathbb{C}^N \setminus L)$ is free Abelian, with the rank of $\bar{H}_{2N-p}$ equal to the number of nonempty planes L_I with $\#I = p$. Of course, the last statement follows also from the Goresky–MacPherson formula, but the cones ∇_I provide an especially easy its realization.

On combinatorial and topological properties of hyperplane arrangements see also [72], [73], [74], [75], [34], [35], [36], [37], [39], [15]–[20], [38], [43], [57], [58], [59], [60], [93]–[96].

13. Homology of complements of arrangements with twisted coefficients. Resonances

Things become slightly more difficult if we consider homology groups with coefficients in local systems.

Definition. A *linear local system* on a (locally simply connected) topological space X (say, $X = \mathbb{C}^N \setminus L$) is a vector bundle $\pi : M \to X$ with fiber $\mathbb{C}^1$ supplied with a flat connection respecting the $\mathbb{C}$-module structure in the fibers.

In other words, for any point $x \in X$, any sufficiently small neighborhood U of x in X and any point $a \in \pi^{-1}(x)$ we have a distinguished section of our bundle over entire U equal to a at x; this section will be the same if we start from some other point $x' \in U$ and the intersection point a' of the old section with the fiber $\pi^{-1}(x')$. For any two points a_1, a_2 of $\pi^{-1}(x)$ the distinguished section equal to $a_1 + a_2$ at x consists of fiber-vise sums of images of sections equal to a_1 and a_2 at x, and for any $\lambda \in \mathbb{C}$ the section equal to λa at x consists of multiplied by λ images of the section equal to a at x.

An i-dimensional *singular simplex* of the local system Θ is a continuous map of a standard simplex Δ^i into the total space M of the fiber bundle, respecting the flat connection: if a point ξ of the simplex goes to some point $a \in \pi^{-1}(x), x \in X$, then some small neighborhood of ξ goes into the image of the corresponding section over a small neighborhood of x. The group of i-dimensional singular chains with coefficients in the local system Θ is defined as the quotient group of the Abelian group generated by all such locally horizontal maps $\Delta^i \to M$ through the following conditions:

a) if we have two simplices $\phi_1, \phi_2 : \Delta^i \to M$ with the same projection (i.e. $\pi \circ \phi_1 \equiv \pi \circ \phi_2$) then the sum $\phi_1 + \phi_2$ of them is equal to the third simplex ϕ_3 such that $\phi_3(\xi) = \phi_1(\xi) + \phi_2(\xi)$ for any $\xi \in \Delta^i$;

b) for any $\lambda \in \mathbb{C}$ the singular simplex ϕ taken with coefficient λ is equal to the singular simplex mapping any point ξ to $\lambda\phi(\xi)$.

The boundary of such a singular simplex is defined in the obvious way and is a sum of singular simplices (of reduced dimension) of the same local system. We can consider the complex of finite chains (i.e. finite linear combinations of singular simplices) or locally finite chains (i.e. locally finite combinations whose projections to X also are locally finite chains there). The corresponding homology groups will be denoted by $H_i(X, \Theta)$ and $H_i^{lf}(X, \Theta)$, respectively; they are called homology groups *of* (or *with coefficients in*) the local system Θ.

These groups are connected by the canonical homomorphism

$$H_i(X, \Theta) \to H_i^{lf}(X, \Theta). \tag{18}$$

In particular if our bundle M is the direct product $X \times \mathbb{C}^1$ with the obvious flat connection then we get the usual homology groups $H_i(X, \mathbb{C})$ and the Borel–Moore homology groups $\bar{H}_i(X, \mathbb{C})$ respectively. The cohomology groups $H^i(X, \Theta)$, $H^i_{lf}(X, \Theta)$ are defined in the standard way as homology groups of conjugate complexes.

Any linear local system defines in the obvious way the monodromy homomorphism of the fundamental group $\pi_1(X)$ (or, equivalently, of $H_1(X)$) into the group $\mathrm{Aut}(\mathbb{C}^1) \equiv \mathbb{C}^*$: extending our sections over a closed loop in X we multiply the fiber by the *monodromy coefficient* corresponding to this loop.

To any local system Θ there corresponds its *dual system* Θ^*, whose fibers are identified with spaces of $\mathbb{C}$-linear functions on the fibers of the initial system, and this identification respects the flat connections in both. The monodromy coefficients defined by one and the same element of $\pi_1(X)$ in dual local systems are inverse to one another (i.e. their product is equal to 1). If the space X is a d-dimensional oriented manifold then the *Poincaré isomorphisms*

$$H_i^{lf}(X, \Theta) \simeq H^{d-i}(X, \Theta^*), \qquad H^i_{lf}(X, \Theta) \simeq H_{d-i}(X, \Theta^*) \tag{19}$$

relate its homology and cohomology groups with coefficients in dual local systems. (Moreover, a special local system Or, called the *orientation sheaf* makes sense of the Poincaré isomorphisms on non-oriented manifolds: this is true even if Θ and Θ^* are the constant local system. Namely, in any of two equations (19) the right-hand term should be replaced by the similar term in which the coefficient local system is not Θ^* but $\Theta^* \otimes \mathrm{Or}$: its monodromy coefficients coincide with these of

Θ^* up to a sign $+$ or $-$ depending on whether the corresponding loops preserve the orientation of X or not.)

Local systems and their homology groups are an adequate tool for the calculus of ramified differential forms and their integrals, see [26]. Suppose that we have a closed analytic i-form on X, such that the analytic continuation along any closed path c in X multiplies it by a complex number, and this number $\tau(c)$ depends on the class of our path c in the group $H_1(X)$ only. (An important class of such forms will be considered in §15.) Then the integration cycles for this differential form are well defined as elements of the i-dimensional homology group of X with coefficients in a local system, whose monodromy coefficient at any path c is equal to $1/\tau(c)$.

Now suppose that we have a hyperplane arrangement $\mathcal{L} = \{L_1, \dots, L_m\}$ in $\mathbb{C}^N$, $m \geq N$, $X = \mathbb{C}^N \setminus L$, and a linear local system Θ over X; let us denote by $\tau_1, \dots, \tau_m$ the monodromy coefficients of this system corresponding to small circles going around these hyperplanes in the positive direction.

Theorem 9 (see [75], [7], [88]). *Let $\mathcal{L}$ be a* generic *hyperplane arrangement in $\mathbb{C}^N$. Then*

A. If there is at least one coefficient τ_i not equal to 1, then the groups $H_i(X, \Theta)$, $H_i^{lf}(X, \Theta)$ are nontrivial only for $i = N$ and their dimensions are equal to $\binom{m-1}{N}$.

B. The map (18) between these groups is an isomorphism if and only if all numbers τ_i $(i = 1, \dots, m)$, and their product $\tau_0 \equiv \tau_1 \cdot \dots \cdot \tau_m$ are not equal to 1.

C. If all these numbers are different from 1 *and the arrangement $\mathcal{L}$ is the complexification of a real one, then the group $H_N^{lf}(\mathbb{C}^N \setminus L, \Theta)$ is freely generated by classes of all bounded components of* $\mathbb{R}^N \setminus L$.

In the last case of a complexified real generic hyperplane arrangement these facts (the dimensions of both groups, the bijectivity of the map (18), and the assertion of item C) were proved first by K. Aomoto [2] in much stronger assumptions on the coefficients τ_j.

For non-generic arrangements the set of exceptional values of $\tau \equiv (\tau_1, \dots, \tau_m)$ for which the map (18) is not bijective, is more complicated, see in particular Theorem 10 below. Since [75] such values are called *resonances* of our local system.

Example. Let be $N = 1$, and $\mathcal{L}$ the collection of m different points $L_1, \dots, L_m$. If at least one of corresponding coefficients τ_i is different from 1 then the group $H_1^{lf}(\mathbb{C}^1 \setminus L, \Theta)$ is generated by the classes of any $m - 1$ of m parallel rays ∇_j connecting these points with the infinity,

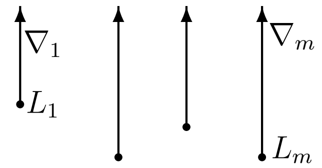

FIGURE 5. Decomposition of the complement of a point arrangement in $\mathbb{C}^1$

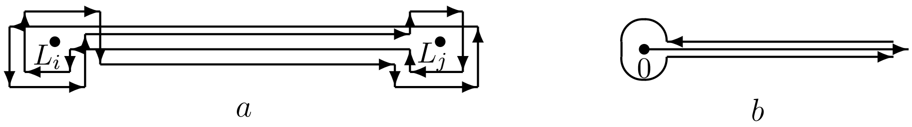

FIGURE 6. The cycle "double loop"

see Fig. 5. Indeed, all these m rays and the complement of the union of them are the cells covering entire $\mathbb{C}^1 \setminus L$. We can calculate the group $H^{lf}_*(\mathbb{C}^1 \setminus L, \Theta)$ with the help of this cellular structure. It is easy to see that for some natural choice of generators of this cellular complex (i.e. of pairs {a cell, a distinguished section of the local system over it}) the incidence coefficients of the unique 2-dimensional cell with all 1-dimensional ones are equal to $\tau_1 - 1, \dots, \tau_m - 1$. This calculates the group $H^{lf}_*(\mathbb{C}^1 \setminus L, \Theta)$; the structure of the "absolute" group $H_*(\mathbb{C}^1 \setminus L, \Theta)$ follows by the Poincaré duality from the similar statement for the group $H_*(\mathbb{C}^1 \setminus L, \Theta^*)$.

If $\tau_j = 1$ then a small circle around the point L_j is a nontrivial element of the group $H_1(\mathbb{C}^1 \setminus L)$ (because its intersection index with the ray ∇_j is not equal to zero). On the other hand it obviously is homological to zero via a locally finite 2-chain bounded by it, therefore the kernel of the map (18) is nontrivial. If $\tau_1 \cdot \ldots \cdot \tau_m = 1$ then the same is true for the big circle embracing all points L_j.

If all numbers $\tau_1, \dots, \tau_m$, and their product $\tau_0 \equiv \tau_1 \cdot \ldots \cdot \tau_m$ are different from 1 then the operator inverse to (18) is provided by *double loops*, i.e. cycles shown in Fig. 6 a). Such a cycle in $\mathbb{C}^1 \setminus L$ goes twice (and in opposite directions) around any of our two singular points L_i, L_j, therefore it can be lifted to a cycle of the local system Θ and defines an element of $H_1(\mathbb{C}^1 \setminus L)$. Since $(1 - \tau_i)(1 - \tau_j) \neq 0$ the image of this cycle in $H^{lf}_1(\mathbb{C}^1 \setminus L)$ is equal to the class of the interval (L_i, L_j) taken with some nonzero coefficient.

Now let us prove Theorem 9 in the case of arbitrary N.

A. Consider again the universal hyperplane m-arrangement, i.e. the coordinate cross $\dagger^m \subset \mathbb{C}^m$, and the local system $\tilde{\Theta}$ on $\mathbb{C}^m \setminus \dagger^m$ with the same monodromy coefficients τ_j. If $\tau_j \neq 1$ for at least one j then the group $H_*(\mathbb{C}^m \setminus \dagger^m, \tilde{\Theta})$ is trivial in all dimensions by the Künneth theorem in $(\mathbb{C}^m \setminus \dagger^m) = (\mathbb{C}^*)^m$. Our generic arrangement $L \subset \mathbb{C}^N$ can be realized as the preimage of $\dagger^m$ under some generic affine embedding $\mathbb{C}^N \to \mathbb{C}^m$; our local system Θ coincides with one induced from $\tilde{\Theta}$ by the same embedding. By the Lefschetz theorem (see [41]), our embedding $\mathbb{C}^N \setminus L \to \mathbb{C}^m \setminus \dagger^m$ is N-connected, in particular induces an isomorphism of homology groups in all dimensions lower than N. Thus $H_i(\mathbb{C}^N \setminus L, \Theta)$ is trivial for $i < N$. By the Poincaré isomorphism (19) the same is true for all groups $H_i^{lf}(\mathbb{C}^N \setminus L, \Theta)$ with $i > N$. The dimension of these groups in the unique remaining dimension N follows from the considerations with the Euler characteristic (which does not depend on the choice of the system Θ).

B. If at least one of numbers τ_j is equal to 1 then, similarly to the one-dimensional example, the small circle around the corresponding plane L_j is a nontrivial element of the kernel of the map (18). If all numbers τ_j and their product τ_0 are different from 1 then the bijectivity of (18) follows from a much more general fact.

Proposition 8. *Let W be a compact N-dimensional complex algebraic manifold, and $\tilde{L}$ a finite set of smooth divisors in it having normal crossings only; let θ be a linear local system on $W \setminus \tilde{L}$ such that all the monodromy coefficients corresponding to small circles going around components of $\tilde{L}$ are different from 1. Then the canonical map*

$$H_*(W \setminus \tilde{L}, \theta) \to H_*^{lf}(W \setminus \tilde{L}, \theta)$$

is an isomorphism in all dimensions.

In particular if $W \setminus \tilde{L}$ is a Stein manifold then both groups can be nontrivial only in the dimension N. This proposition follows easily from the *Leray spectral sequence* for sheaf cohomology, see e.g. [7], [88].

Our assertion on bijectivity of (18) follows from this proposition if we take $W = \mathbb{CP}^N$ and $\tilde{L} =$ the union of L and the improper plane.

C. If our arrangement is the complexification of a real one, then the reversing of the map (18) can be visualized by the "multidimensional double loops" generalizing Fig. 6 a). The construction of them was announced in [75] and described in [88]. Namely, let $\Delta \subset \mathbb{R}^N$ be a bounded connected component of $\mathbb{R}^N \setminus L$. The corresponding "double loop" is an N-dimensional manifold $\kappa(\Delta)$ together with an immersion $K : \kappa(\Delta) \to \mathbb{C}^N \setminus L$ such that

a) this immersion can be lifted to a map into the space M of our local system Θ, locally flat with respect to its connection; in particular it defines (up to a scalar coefficient depending on the choice of this lifting) an element of the group $H_N(\mathbb{C}^N \setminus L, \Theta)$;

b) the map (18) sends this element into the homology class of the component Δ taken with some coefficient, which is different from zero if and only if all monodromy coefficients τ_j corresponding to all walls of our polytope Δ are not equal to 1.

This cycle can be constructed with the help of the embedding of $(\mathbb{C}^N, L)$ in the space $(\mathbb{C}^p, \dagger^p)$ of the universal p-arrangement, where p is the number of walls of Δ. First, we construct an immersion $\mathbb{R}^1 \to \mathbb{C}^1 \setminus \{0\}$ shown in Fig. 6 b): it coincides with the identical map in the segment $[+\epsilon, +\infty)$, with the map $\{x \to -x\}$ in the segment $(-\infty, -\epsilon]$, and maps the segment $[-\epsilon, +\epsilon]$ to a small loop around the origin. The direct product of m copies of such immersions defines an immersion $\Xi_p : \mathbb{R}^p \to \mathbb{C}^p \setminus \dagger^p$ that covers the positive octant in $\mathbb{R}^p \subset \mathbb{C}^p$ with multiplicity 2^p.

Let $\psi_i, i = 1, \dots, p$, be the linear functions $\mathbb{C}^N \to \mathbb{C}$ distinguishing the planes L_j bounding the component Δ; they can be normed so that all of them take positive values in Δ. These functions $(\psi_1, \dots, \psi_p)$ define an embedding $\Psi : \mathbb{C}^N \to \mathbb{C}^p$. The desired cycle $\kappa(\Delta)$ is induced by this map from the universal immersion Ξ_p. Namely, $\kappa(\Delta)$ is defined as the subset in the direct product $\mathbb{R}^p \times (\mathbb{C}^N \setminus L)$ consisting of such pairs (t, x) that $\Xi_p(t) = \Psi(x)$. The immersion K is the restriction of the obvious projection $\mathbb{R}^p \times (\mathbb{C}^N \setminus L) \to \mathbb{C}^N \setminus L$.

Remark. Similar "double loops" reverse the map (18) also in the most general situation described in Proposition 8, but their construction is much more complicated, see §I.10 in [88]. (Direct construction of similar cycles in some interesting particular cases was given in [61].) In Calculus this reversion is called the *regularization of improper integrals*, see also §15 below.

Let us consider a generic linear function $f : \mathbb{R}^N \to \mathbb{R}$ whose values at all 0-dimensional planes L_J are different, and associate with any bounded component Δ its vertex $L(\Delta)$ at which the function $f|_\Delta$ takes its supremum. Let $\nabla(\Delta)$ be the N-dimensional imaginary wedge ∇_J in $\mathbb{C}^N \setminus L$ with the corner at $L(\Delta)$, see the last paragraph of §12. It is not difficult to show that the number of bounded connected components Δ is equal to $\dim H_*(\mathbb{C}^N \setminus L, \Theta)$ (a proof of a more general statement see after Theorem 10 below). Statement C of Theorem 9 is a corollary of the following proposition.

Proposition 9 (see [75]). *If all monodromy coefficients τ_j, $j = 1, \dots, m$, are different from 1, then* *a) the group $H_N(\mathbb{C}^N \setminus L, \Theta)$ is freely generated by the classes of "double loops" $\kappa(\Delta)$ corresponding to all bounded components Δ;*

b) the group $H_N^{lf}(\mathbb{C}^N \setminus L, \Theta)$ is freely generated by the classes of imaginary wedges $\nabla(\Delta)$ corresponding to these components.

We shall prove simultaneously assertion a) for our local system Θ and assertion b) for the dual system Θ^*. Namely, let us lift the wedges $\nabla(\Delta)$ to the total space M^* of the latter system. Then the intersection indices $\langle \kappa(\Delta_1), \nabla(\Delta_2)\rangle$ are well-defined for all pairs of components Δ_1, Δ_2. It is easy to calculate that the matrix consisting of all such intersection indices is triangular with nonzero numbers on the diagonal; in particular it is non-degenerate. $\square$

Many assertions of Theorem 9 and Proposition 9 can be extended to more general situations. However, it this case the set of resonant values $T = (\tau_1, \dots, \tau_m)$ becomes more complicated.

Namely, let us consider the variety $\hat{L} \subset \mathbb{CP}^N$ consisting of the support $L \subset \mathbb{C}^N$ of our arrangement and the improper plane. If this variety is not a divisor with normal crossings then we take some its resolution $(W, \tilde{L}) \to (\mathbb{CP}^N, \hat{L})$ in the sense of [44]. Spaces $W \setminus \tilde{L}$, $\mathbb{CP}^N \setminus \hat{L}$ and $\mathbb{C}^N \setminus L$ are diffeomorphic, hence we can lift the local system Θ to $W \setminus \tilde{L}$. For any irreducible component $\tilde{L}_j$ of $\tilde{L}$, the monodromy coefficient corresponding to a small circle around it is a monomial of the initial coefficients $\tau_1, \dots, \tau_m$. Our collection $T = (\tau_1, \dots, \tau_m)$ is called *resonant* if the value of some of these monomials is equal to 1 for *any* resolution of $\hat{L}$. The trivial monomials $\tau_1, \dots, \tau_m$ and $\tau_0^{-1} \equiv (\tau_1 \cdot \ldots \cdot \tau_m)^{-1}$ always appear among these monomials: they correspond to proper images of initial planes L_j and the improper plane. Proposition 8 implies immediately that if the set T is non-resonant then the map (18) is an isomorphism.

Theorem 10 (see [75]). *Suppose that our hyperplane arrangement $L \subset \mathbb{C}^N$ is the complexification of a real one and has only normal crossings in $\mathbb{C}^N$, and all numbers $\tau_1, \dots, \tau_m$ are not equal to 1. Then*

A. Both groups $H_i(\mathbb{C}^N \setminus L, \Theta)$, $H_i^{lf}(\mathbb{C}^N \setminus L, \Theta)$ are nontrivial only for $i = N$ and their dimensions are equal to the number of bounded components of $\mathbb{R}^N \setminus L$.

B. The group $H_i(\mathbb{C}^N \setminus L, \Theta)$ is freely generated by the classes of double loops corresponding to all these components, and the group $H_i^{lf}(\mathbb{C}^N \setminus L, \Theta)$ is freely generated by the classes of corresponding complex wedges $\nabla(\Delta)$.

C. If moreover the collection of numbers $T = (\tau_1, \dots, \tau_m)$ is non-resonant, then the group $H_i^{lf}(\mathbb{C}^N \setminus L, \Theta)$ is freely generated also by the classes of these bounded components.

A proof of the statement A. follows from the cell decomposition considered in the last part of §12. Indeed, the corresponding cellular complex calculating the homology group $H_i^{lf}(\mathbb{C}^N \setminus L, \Theta)$ becomes, after some norming of its canonical generators, isomorphic to the order complex of our arrangement $\mathcal{L}$ or, equivalently, of its real part $\mathcal{L} \cap \mathbb{R}^N \equiv \{L_j \cap \mathbb{R}^N\}$. By formula (11), the latter complex is homotopy equivalent to the support $L \cap \mathbb{R}^N$ itself, which is obviously homotopy equivalent to the bouquet of spheres corresponding to all these bounded components. The proof of statements B. and C. is the same as for the case of generic arrangements.

Remark. An analog of the deRham theory calculating the homology of complex manifolds (e.g. of spaces $\mathbb{C}^N \setminus L$) with coefficients in local systems was developed in [26], see also [30], [2] and §8 in [92].

14. Matroids and configuration spaces

The simplest example of a configuration space is the space $M(N, 2)$ of collections of N different points in $\mathbb{C}^1$, see §2. In more complicated examples, we can consider collections of subvarieties in some manifolds; a configuration space is the family of all collections such that the pair (the manifold, the union of varieties) has a fixed topological type. Such configuration spaces appear often in integral geometry and theory of special functions, see [62], [34], [75], [88]. An important class of such functions, called *general hypergeometric functions*, appears if all our varieties are hyperplanes in $\mathbb{C}^N$ or $\mathbb{CP}^N$, see [34], [73]. In this case the topology of configuration spaces has especially deep relations with the algebraic geometry. It is convenient to formulate these relations in terms of the theory of *matroids.*

The notion of a matroid is a formalization of dimensional properties of a central hyperplane arrangement, see [50] and [39].

Definition. A *matroid* is a finite set U supplied with a natural-valued function r on the set $2^U \setminus \{\emptyset\}$ of all non-empty subsets of U such that

1) for any such subset $I \in U$ we have $1 \le r(U) \le$ (the cardinality of I);

2) if $I \subset J$ then $r(I) \le r(J)$;

3) for any I, J we have $r(I \cap J) + r(I \cup J) \le r(I) + r(J)$.

Any central hyperplane arrangement $\mathcal{L} = \{L_1, \dots, L_m\}$ defines a matroid, whose elements correspond to the planes L_i and for any collection I of these elements $r(I)$ is the codimension of the intersection L_I of all planes from this collection.

In this case the arrangement $\mathcal{L}$ is called *a realization* of the corresponding matroid.

There exist matroids having *complex* realizations (i.e. realizations by collections of complex hyperplanes in $\mathbb{C}^N$) but no real realizations, and matroids having realizations over finite fields but no complex realizations, etc. (On the other hand, any realization of a matroid over some field defines also its realization over any extension of this field: in particular the complexification of a real realization provides a complex realization).

Specifical properties of real affine hyperplane arrangements (roughly speaking, the fact that two 0-dimensional planes L_I, $L_{I'}$ can lie either to one side of any hyperplane not connecting them or to different sides) are formalized in the notion of an *oriented matroid*, see [14].

The next important question is as follows: given a matroid, what is the set of all its realizations over a given field F?

The study of realizations by central hyperplane arrangements in $\mathbb{C}^3$ (or, equivalently, of arbitrary line arrangements in $\mathbb{CP}^2$) is deeply connected with the theory of integer algebraic varieties (i.e. complex varieties defined by equations with integer coefficients), see [56], [57].

Accordingly to [56], [57], for any integer algebraic subvariety in some $\mathbb{C}^n$ there exists a matroid such that the space of its realizations by plane arrangements in $\mathbb{C}^3$ is homotopy equivalent to our subvariety.

This relation allows one to construct spaces of realizations having very delicious properties: these properties reflect the similar properties of the corresponding algebraic varieties.

For instance, the equation $x^2 = -1$ is related with a matroid having complex realizations but no real realizations. The equation $x^2 + y^2 = 0$ is related with such a matroid that the real dimension of the space of its real realizations is less than the complex dimension of the space of its complex realizations; in particular the latter space is singular, see [74].

The first of these examples (corresponding to the equation $x^2 = -1$) is constructed as follows.

The first four elements $L_1, \dots, L_4$ are *in general position* (i.e., $r(\{I\})$ is equal to 1 for any 1-element set $I \subset \{1, 2, 3, 4\}$, to 2 for any 2-element set and to 3 for 3- and 4- element sets). In the terms of possible realizations by lines in $\mathbb{CP}^2$ or $\mathbb{RP}^2$ this means that these lines meet

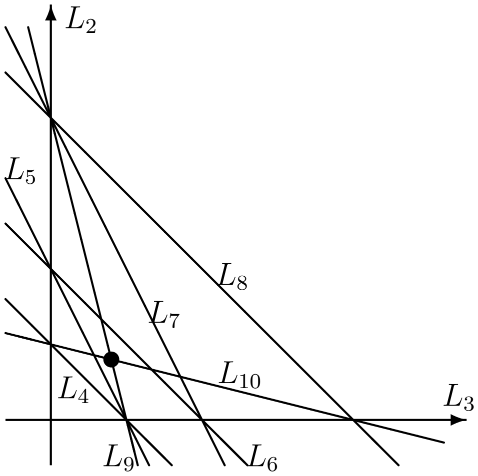

FIGURE 7. Imaginary line arrangement

in general position in the standard sense: none three of them meet at one point. Given a realization of our matroid including these four elements, they fix a coordinate system in $\mathbb{P}^2$: we can take L_1 as the line "at infinity", L_2 as the line $\{x = 0\}$, L_3 as the line $\{y = 0\}$, and choose the scaling of coordinates in such a way that the points $L_2 \cap L_4$ and $L_3 \cap L_4$ have coordinates $(0, 1)$ and $(1, 0)$ respectively.

Further we add the element L_5 with unique non-generic condition $r(L_3, L_4, L_5) = 2$, i.e. the corresponding three lines of any realization in $\mathbb{P}^2$ should intersect at one point. The intersection of this line L_5 with L_2 will be some point with coordinates $(0, \alpha)$. The next line L_6 should pass through this intersection point and be parallel to L_4 (in the terms of the matroid these conditions are expressed as $r(L_2, L_5, L_6) = 2$ and $r(L_1, L_4, L_6) = 2$ respectively). The point $L_6 \cap L_3$ will have coordinates $(\alpha, 0)$. The next line L_7 passes through this point and is parallel to L_5, i.e. we have conditions $r(L_3, L_6, L_7) = 2$ and $r(L_1, L_5, L_7) = 2$. Its intersection point with the line L_2 has coordinates $(0, \alpha^2)$. The next line L_8 passes through the last point and is parallel to L_6 and L_4, i.e. we have $r(L_2, L_7, L_8) = 2$ and $r(L_1, L_4, L_6, L_8) = 2$. Its intersection point with the line L_3 has coordinates $(\alpha^2, 0)$. The next line L_9 should pass through the points $(1, 0)$ and $(0, \alpha^2)$, i.e. $r(L_3, L_4, L_5, L_9) = 2$ and $r(L_2, L_7, L_8, L_9) = 2$. The line L_{10} should pass through the points $(0, 1)$ and $(\alpha^2, 0)$, i.e. $r(L_2, L_4, L_{10}) = 2$ and $r(L_3, L_8, L_{10}) = 2$.

Finally, we claim that the lines L_9 and L_{10} are parallel (i.e. $r(L_1, L_9, L_{10}) = 2$): the intersection point covered by the "black hole" in Fig. 7 should lie at the infinity. This is possible only if $\alpha^4 = 1$. But $\alpha \neq 1$ (because

$L_5 \neq L_4$) and $\alpha^2 \neq 1$ (because $L_8 \neq L_4$). Therefore any complex realization of this matroid corresponds to a number α with $\alpha^2 = -1$.

15. Applications in integral geometry: general hypergeometric functions

The theory of *general* (or *multidimensional*) *hypergeometric functions* was initiated by K. Aomoto [1], [2] and I.M. Gelfand [34]. These functions form an important class of functions given by integral transformations: they represent all these functions to the same extent as the plane configurations represent all algebraic varieties.

The starting point of this theory is the Gauss' hypergeometric integral

$$\Gamma(a; \alpha_1, \alpha_2, \alpha_3) \equiv \int_0^1 z^{\alpha_1}(z-1)^{\alpha_2}(z-a)^{\alpha_3} dz, \tag{20}$$

where α_j and a are complex numbers, $a \notin [0, 1]$[1].

This integral converges absolutely if $\operatorname{Re} \alpha_1 > -1$, $\operatorname{Re} \alpha_2 > -1$. For any fixed a it is a holomorphic function in the domain of the space $\mathbb{C}^3$ of exponents $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ distinguished by these inequalities. The analytical continuation of this function to the space of all $\alpha \in \mathbb{C}^3$ is a meromorphic single-valued function whose poles are the hyperplanes on which either α_1 or α_2 is a negative integer number.

On the other hand, fixing α and moving a we obtain an analytical function on a with a ramification at points 0 and 1, satisfying the famous hypergeometric equations, see [11]. If the exponents α are generic, then the space of solutions of these equations at any point a is two-dimensional and is spanned by different leaves of the analytic continuation of this function.

Much more generally, we can consider the integral

$$\Gamma(\lambda; \alpha) = \int_\Delta f_1^{\alpha_1} \cdot \ldots \cdot f_k^{\alpha_k} dz_1 \wedge \ldots \wedge dz_n, \tag{21}$$

where f_j are some polynomial functions $\mathbb{C}^N \to \mathbb{C}$, depending analytically on parameters $\lambda \in \mathbb{C}^m$, α_j are complex exponents, and integration is taken over some relatively closed (i.e. locally finite) but, generally, not compact (i.e. not finite) n-dimensional cycles in the space of nonzero values of the integration function (or, more precisely, in the space of a covering over this space in which our function becomes single-valued).

[1]Gauss himself wrote it in a different but equivalent form

An important class of such problems is as follows. We fix some matroid and allow the functions f_j be the linear functions $\mathbb{C}^N \to \mathbb{C}$ whose zero sets form all possible complex realizations of this matroid.

The integral (21) also defines a meromorphic function on α (for fixed a), cf. [12], and a branching analytical function on λ for fixed α; the set of its ramification is the set of such values of the parameter λ, that the corresponding functions f_j have the topologically "non-generic" sets of zeros (i.e. the rank function r of the plane arrangement "jumps").

Again, the main problems here are as follows.

1) to describe the polar set of these integrals, considered as meromorphic functions on α;

2) to describe the ramification of these integrals for fixed α and moving parameter λ of the set of functions f_j;

3) to find the number of linearly independent functions on the parameter spaces $\mathbb{C}^m$, given by such integrals.

The solution of these analytical problems is closely related with the following topological ones.

A) the calculation of homology groups related with our collections of functions f_j and containing all possible integration chains Δ;

B) the study of the maps similar to (18) for such groups;

C) the study of the topology (especially of the fundamental group) of the configuration space of all realizations of our matroid;

D) the study of the *homology vector bundle* over this configuration space, whose fiber over an arrangement is the corresponding homology group considered in A); especially the study of the *monodromy representation* of the fundamental group from C) in the fiber of this bundle.

Let us consider all these problems (and their applications) for the classical integral (20). In this case the solution of the problem 1) follows from Fig. 6a): if α_1 and α_2 are not integer, then our improper integral along the interval $(0, 1)$ can be replaced by a similar integral along the "double loop" taken with the coefficient $1/(1-e^{2\pi i\alpha_1})(1-e^{2\pi i\alpha_2})$, which is regular.

Similarly, in the general case of the integral (21) if the collection of exponents $\tau_j \equiv e^{2\pi i\alpha_j}$ is not resonant in the sense of §13 then any integration cycle can be *regularized* by a "multidimensional double loop", see [88].

Remark. The latter assertion is not a formal corollary of Proposition 8, i.e. of the invertibility of the map (18) for non-resonant exponents. Indeed, in the study of integrals as functions of these exponents, the

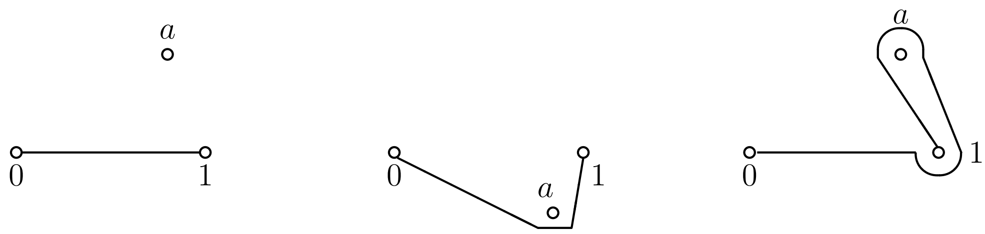

FIGURE 8. Monodromy of the hypergeometric integral

integration cycles cannot be considered as elements of the group

$$H_N^{lf}(\mathbb{C}^N \setminus L, \Theta), \tag{22}$$

because the definition of this group involves some factorization depending on exponents. Still, the geometrical construction of "double loops" works even in this case and allows us to regularize the integrals.

On contrary, if we fix the exponents α_j then we can consider the integration cycles as elements of the group (22). The dimension of the space of different integral functions (considered as functions on parameters λ of functions f_j only) is then no greater than the dimension of this group. This estimate can be not sharp in some exotic examples, however for many important matroids it is sharp. For instance it is so for matroids corresponding to generic arrangements and, moreover, to all arrangements with normal crossings in $\mathbb{C}^N$ but generally not in $\mathbb{CP}^N$, see [75].

The proof (see [88]) is based on the study of the monodromy action in the homology bundle: starting from a single integration cycle and acting on it by all elements of the monodromy group we can obtain a collection of cycles generating the whole group (22).

Example. In the case of the integral (20) the configuration space is the punctured plane $\mathbb{C}^1 \setminus \{0, 1\}$ of all admissible values of the parameter a. Let us move this value along a closed loop in this space going around some singular point, say the point 1. Deforming simultaneously the integration cycle $(0, 1)$ in such a way that at no instant it intersects singular points $0, 1$ and the current value of a, we get the cycle shown in Fig. 8 right. The integral of type (20) along this cycle is equal to the analytical continuation of the initial integral (20) along our loop in the parameter space. The difference between this cycle and the initial one is equal to the interval $(1, a)$ passed in two opposite directions *on two different leafs* of the Riemann surface on which our integration form is single-valued. If α_3 is not integer, then the integral along this cycle is not identically equal to zero, and we get the second integral function independent on the first one. Since the dimension of the group $H_1^{lf}(\mathbb{C}^1 \setminus \{0,1,a\}, \Theta)$ is equal to 2, the lower and upper estimates on the number of linearly independent integral functions coincide and problem 3) is solved in this particular case.

On algebraic properties of analytic functions of this kind see, in particular, [27] and [38].

Similar (although much more complicated) methods allow us to prove analogous results in multidimensional situations, see [75], [88]. In particular if the sets $\{x | f_j(x) = 0\}$ form hyperplane arrangements with normal crossings in $\mathbb{C}^N$ and the set of exponents $\tau_j \equiv e^{2\pi i \alpha_j}$ is non-resonant, then the homological estimate also is sharp, i.e. we have the full number of linearly independent integral functions.

These and similar functions have wonderful applications in mathematical physics, see e.g. [67], [73].

16. How if the collection of planes is infinite ?

There are many important topological subspaces in $\mathbb{R}^N$ that can be represented as continuous families of planes. A great source of such spaces is the *discriminant theory*, see [76], [77]. The above-described strategy of the topological investigation of unions of planes (and complements of such unions), based on simplicial resolutions, works also in these cases (after appropriate modification). In particular we need to consider *continuous order complexes* and construct the *conical resolutions* of such spaces. Here we describe a simple example: that of determinant varieties.

Let $\mathbb{K}$ be any of fields $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The *determinant* variety $Det(\mathbb{K}^n) \subset End(\mathbb{K}^n) \sim \mathbb{K}^{n^2}$ consists of all degenerate operators $\mathbb{K}^n \to \mathbb{K}^n$.

We construct a resolution of this variety that provides a calculation of its Borel–Moore homology groups and, by the Alexander duality, the *most complicated calculation* of cohomology groups of their complementary spaces $GL(\mathbb{K}, n)$.

The *tautological resolution* of $Det(\mathbb{K}^n)$ is defined by elimination of quantifiers (which is an analog of "taking the sets S_j separately" in the justification of the inclusion–exclusion formula). Namely, an operator A belongs to $Det(\mathbb{K}^n)$ if $\exists$ a point $x \in \mathbb{KP}^{n-1}$ such that $\{x\} \subset \ker A$. Define the resolution space $det_1(\mathbb{K}^n)$ as the space of all pairs $(x, A) \in \mathbb{KP}^{n-1} \times End(\mathbb{K}^n)$ such that $\{x\} \in \ker A$. This space admits the (tautological) structure of a $(n^2 - n)$-dimensional $\mathbb{K}$-vector bundle over $\mathbb{KP}^{n-1}$, whose fiber $L(x)$ consists of all A such that $\{x\} \in \ker A$.

The obvious projection $\pi : det_1(\mathbb{K}^n) \to Det(\mathbb{K}^n)$ is regular over operators with one-dimensional kernels, but the pre-image of an operator with $\dim\ker = l$ is isomorphic to $\mathbb{KP}^{l-1}$.

The situation is very similar to the one considered in the arrangement theory: the variety $Det(\mathbb{K}, n)$ is the union of spaces $L(x)$ in the same way as the space L was the union of planes L_i. Keeping the analogy, we construct the order complex of all intersections of these spaces $L(x)$. It is not straightforward because the family of planes $L(x)$ is not discrete, and moreover the set of such planes passing through one and the same point of $Det(\mathbb{K}^n)$ can be continuous. Indeed, the possible intersections of several planes $L(x_j) \subset End(\mathbb{K}^n)$ are just the planes of the form $L(X)$ where X is a subspace of a certain dimension in $\mathbb{K}^n$ (i.e. a point of a certain Grassmannian manifold $G_i(\mathbb{K}^n)$, $i \in [1, n]$), and $L(X)$ consists of all operators whose kernels contain X.

Thus our poset of all planes and their intersections is the *disjoint union of all Grassmann manifolds* $G_1(\mathbb{K}^n), \ldots, G_{n-1}(\mathbb{K}^n), G_n(\mathbb{K}^n)$. The *continuous order complex* of all these Grassmannians is defined as follows. Consider the join $G_1(\mathbb{K}^n) * \ldots * G_n(\mathbb{K}^n)$, i.e., roughly speaking, the naturally topologized union of all simplices whose vertices correspond to points of different Grassmannians. Such a simplex is *coherent* if the planes corresponding to its vertices form a flag. The desired order complex $\Xi(\mathbb{K}^n)$ is the union of all coherent simplices, with topology induced from that of the join. This is a cone with vertex $\{\mathbb{K}^n\} \in G_n(\mathbb{K}^n)$. Its *link* $\partial\Xi(\mathbb{K}^n)$ is the union of coherent simplices not containing the vertex $\{\mathbb{K}^n\}$.

This link $\partial\Xi(\mathbb{K}^n)$ is homeomorphic to the sphere S^M, $M = \frac{1}{2}n(n-1)(\dim_{\mathbb{R}} \mathbb{K}) + n - 2$. (Probably this fact is assumed in Remark 1.4 of [21], see also [79], [77].) Hence $\Xi(\mathbb{K}^n)$ is homeomorphic to a ball.

The *conical resolution* of $Det(\mathbb{K}^n)$ is constructed as a subset of the direct product $\Xi(\mathbb{K}^n) \times Det(\mathbb{K}^n)$. To any plane $X \subset \mathbb{K}^n$ there corresponds a subspace $\Xi(X) \subset \Xi(\mathbb{K}^n)$, namely, the union of all coherent simplices all whose vertices correspond to planes lying in X. This is a cone with vertex $\{X\}$, and is homeomorphic to a closed ball. Define the conical resolution $\delta(\mathbb{K}^n) \subset \Xi(\mathbb{K}^n) \times Det(\mathbb{K}^n)$ as the union of the products $\Xi(X) \times L(X)$ over all planes X of dimensions $1, \ldots, n$. It is easy to see that the obvious projection $\delta(\mathbb{K}^n) \to Det(\mathbb{K}^n)$ induces a homotopy equivalence of one-point compactifications of these spaces (indeed, this projection is proper and semialgebraic, and all its fibers are contractible cones of the form $\Xi(X)$). On the other hand, the space $\delta(\mathbb{K}^n)$ has a nice filtration: its term F_i is the union of products $\Xi(X) \times L(X)$ over planes X of dimensions $\leq i$. The term $F_i \setminus F_{i-1}$ of this filtration is the total space of a fibre bundle over $G_i(\mathbb{K}^n)$. Its fiber over a point $\{X\}$ is the space $(\Xi(X) \setminus \partial\Xi(X)) \times L(X)$, and is homeomorphic to an Euclidean space. Thus the Borel–Moore homology group of this term can be reduced to that of the base. The spectral sequence, generated by this filtration and converging to the Borel–Moore homology group of $Det(\mathbb{K}^n)$ (or, equivalently, to the cohomology group of the complementary space $GL(\mathbb{K}^n)$), degenerates at the first term (i.e. $E_1^{p,q} \equiv E_\infty^{p,q}$) and gives, in particular, the homological *Miller splitting*

$$H_m(GL(\mathbb{C}^n)) = \bigoplus_{k=0}^{n} H_{m-k^2}(G_k(\mathbb{C}^n)) \tag{23}$$

and similar splittings for $\mathbb{K} = \mathbb{R}$ and $\mathbb{H}$.

There is a plenty of other problems in which the technology of conical resolutions works, see [84], [86]. Among them are the theory of knots and generic plane curves (see the next section), topological study of spaces of continuous maps, of smooth functions without complicated singularities, of operators with simple spectra, of nonsingular projective hypersurfaces...

17. Applications and analogies in differential topology

The space $M(N, 2)$ (see §2) can be considered as the space of all embeddings to $\mathbb{C}^1$ of a finite set of cardinality N. In a similar way we can consider the space of all smooth embeddings $S^1 \hookrightarrow \mathbb{R}^n$, i.e. regular knots in $\mathbb{R}^n$; if $n = 3$ then the 0-dimensional cohomology classes of this space are the knot invariants. We can study these and other cohomology classes (in the case of any $n \geq 3$) by essentially the same methods as in §7, 8 (curiously, it was done earlier, see [78]). Consider the space $\mathcal{K}$ of all smooth maps $S^1 \to \mathbb{R}^n$, define the discriminant $\Sigma \subset \mathcal{K}$ as the space of all maps that are not smooth embeddings, and study the group $H^*(\mathcal{K} \setminus \Sigma)$. To do it, we take a conical resolution of the discriminant set Σ. It is possible because this set is swept out by a reasonable family of subspaces in $\mathcal{K}$. These subspaces are parameterized by all unordered pairs of points $(x, y) \subset S^1$. By obvious reasons these pairs are called *chords*, they run over the 2-dimensional *chord space* $\overline{B(S^1, 2)}$.

For any such pair the corresponding subspace $L(x, y)$ consists of all maps $f : S^1 \to \mathbb{R}^n$ such that $f(x) = f(y)$ if $x \neq y$ or $f'(x) = 0$ if $x = y$. Such subspaces form the tautological resolution of Σ. Then we take the order complex of all possible intersections

$$L(x_1, y_1) \cap L(x_2, y_2) \cap ... \tag{24}$$

Theory of arrangements	Knot theory
Space $\mathbb{R}^N$	Space $\mathcal{K} = C^\infty(S^1, \mathbb{R}^n)$
Union of planes $L = \cup L_i \subset \mathbb{R}^N$	Discriminant subset $\Sigma \subset \mathcal{K}$
Set of indices $\{1, \ldots, m\}$	Chord space $\overline{B(S^1, 2)}$
A plane L_i	A subspace $L(x, y)$, $x, y \in S^1$
Disjoint union of planes L_i	Tautological resolution $F_1\sigma$ of Σ
Simplicial resolution L' of L	Conical resolution σ of Σ
Subsets $I \subset \{1, \ldots, m\}$ with $\mathrm{codim} L_I = p$	Combinatorial types of chord configurations J with $\mathrm{codim} L(J) = pn$
A prism L'_I	A J-block in σ
Künneth isomorphism for homology of $\check{L}'_I = \check{\Delta}(I) \times L_I$	Thom isomorphism for the fibration of pure J-blocks by spaces $L(J')$
Shuffle product formulas of §11	Multiplication formulas by Kontsevich (for invariants) and Turchin (for higher cohomology classes)
Integral representations of §§3, 4	Kontsevich integral
Homotopy splitting (12)	Kontsevich stabilization theorem
Combinatorial realization of §10	Combinatorial formulas for invariants [63], [42] and other cohomology classes [85]

TABLE 1. The analogy between the arrangement theory and the knot theory

and limit positions of such intersections (all of them are subspaces in $\mathcal{K}$ whose codimensions are multiples of n), *supply it with a natural topology*, and define the conical resolution in exactly the same way as previously, i.e. as a subset of the direct product of this order complex and the space $\mathcal{K}$. Then we define the filtration on this resolution by the codimensions (divided by n) of these planes and consider the arising spectral sequence.

The homological study of the arising resolution space is known as the theory of *finite-type knot invariants*, see [10], and different its generalizations, including the (equally interesting) calculation of higher dimensional cohomology classes of spaces of knots, see [78], [87], [71]. In Table 1 we give a short list of parallel notions and objects in both theories. Of course, a large part of its right half cannot be explained here; this table is rather a kind of Rosetta stone for those combinatorialists who will study the theory of knot invariants, see [10].

It is necessary also to mention the exceptional value of the Arnold's identity (3) for the construction of the Kontsevich's integral [48], [49].

Of course, the space $\mathcal{K}$ is infinite-dimensional, and formally we cannot use the Alexander duality in it: the usual (i.e. finite-dimensional) cohomology classes of the space of knots $\mathcal{K} \setminus \Sigma$ should correspond to "infinite-dimensional cycles" in Σ, whose definition requires some effort. The strict construction of such cycles corresponding to finite-type cohomology classes uses the techniques of finite-dimensional approximations, see [78].

Similarly, we can consider the space of smooth embeddings of finitely many circles into $\mathbb{R}^n$, it gives us the theory of finite-type cohomology classes of spaces of links.

One more space of this type is that of all plane curves without triple points, see [5], [6], [80], [81], [52]–[55], [68], [46], [83]. It is related very much with the arrangements $A(N, 3)$, see [19] and §2.

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