MONODROMY OF COMPLETE INTERSECTIONS AND SURFACE POTENTIALS

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To Egbert Brieskorn with admiration

Abstract. Following Newton, Ivory and Arnold, we study the Newtonian potentials of algebraic hypersurfaces in $\mathbb{R}^n$. The ramification of (analytic continuations of) these potential depends on a monodromy group, which can be considered as a proper subgroup of the local monodromy group of a complete intersection (acting on a *twisted* vanishing homology group if n is odd). Studying this monodromy group we prove, in particular, that the attraction force of a hyperbolic layer of degree d in $\mathbb{R}^n$ coincides with appropriate *algebraic* vector-functions everywhere outside the attracting surface if $n = 2$ or $d = 2$, and is non-algebraic in all domains other than the hyperbolicity domain if the surface is generic and $(d \geq 3)\&(n \geq 3)\&(n + d \geq 8)$.

Recently W. Ebeling removed the last restriction $d + n \geq 8$, see his Appendix to this article.

1. Introduction

Two famous theorems of Newton assert that

a) a homogeneous spherical layer in Euclidean space does not attract bodies inside the sphere, and

b) exterior bodies are attracted by it to the center of the sphere as by the point-wise particle whose mass is equal to the mass of the entire sphere.

Ivory [I] extended both these theorems to the attraction of ellipsoids, and Arnold [A 82] extended the first of them to the attraction of arbitrary hyperbolic hypersurfaces: such a surface does not attract the particles inside the hyperbolicity domain; see also [G 84].

In any component of the complement of the attracting surface this attraction force coincides with a real analytic vector-function; we investigate the ramification of this function, in particular (following one

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another famous theory of Newton, see [A 87], [AV]) the question if it is algebraic or not.

We describe the monodromy group responsible for the ramification and identify it as a subgroup of the local monodromy group of a complex complete intersection of codimension 2 in $\mathbb{C}^n$. Unlike the usual local monodromy action, this monodromy representation is reducible: e.g. the Newton–Ivory–Arnold theorem depends on the fact that the homology class of the set of real points of a hyperbolic surface defines an invariant element of this action (although this element is not equal to zero: indeed, otherwise even the potential function of the force would be zero, and not only its gradient field, which is wrong already in the Newton's case).

Although we consider mainly the orbit of a very special cycle, formed by all real points of a hyperbolic polynomial, all our calculations can be applied to more general situations, e.g. when the integration cycle is an arbitrary linear combination of real components (maybe non-compact) of an algebraic hypersurface in $\mathbb{R}^n$.

In the case of odd n, this group acts in a vanishing homology group with twisted coefficients (so that the corresponding kernel form $r^{2-n}ds$ of the potential function can be integrated correctly along its elements). In § 2.3 we extend the standard facts concerning vanishing homology of complete intersections to this group, cf. [Ph 65], [G 88].

There is a (non-formal) partition of all classes of isolated singularities of complete intersections into series with varying dimension n of the ambient space $\mathbb{C}^n$ (but with the constant codimension p of the complete intersection), see [E], [AGLV]; e.g., all singularities given by p generic quadrics in the spaces $\mathbb{C}^n$ with different n and fixed p form such a series. To any such series there corresponds a series of reflection groups, also depending on the parameter n; for such n that $n-p$ is even, these groups coincide with the (standard) local monodromy groups of corresponding singularities. The homology groups described in § 2.3 fill in the gap: for $n-p$ odd, the reflection group of the natural series coincides with the monodromy action on such a twisted homology group of the corresponding singularity. (In the marginal case $p=1$, all the reflection groups of the series coincide, see [GZ], [G 88].)

This is a reason why the qualitative behavior of attraction forces in the spaces of any dimension is essentially the same, unlike the usual situation (see e.g. [P], [ABG], [A 87], [AV], [V 94]) when the functions given by similar integral representations behave in very different way in the spaces of dimensions of different parity.

For $n=2$ and arbitrary d, our monodromy group is finite, thus the analytic continuation of the attraction force is finitely-valued, in particular (by the Riemann's existence principle) algebraic, see § 5.1 below. A realistic estimate of the number of values of this continuation is given by Theorem 4. In particular, we get a new series of examples when the attraction force coincides with a *single-valued* (rational) vector-function outside the hyperbolicity domain, see the Corollary to Theorem 4.

For $d = 2$ and arbitrary $n > 2$, the monodromy group is infinite, and the orbit of any integration cycle lies on an ellipsoidal cylinder in the vanishing homology space. Fortunately, the integral of the attracting charge takes zero value on the directing plane of this cylinder, thus the number of its values along the elements of any orbit again is finite, see § 5.2.

In all the other cases (when $d \geq 3$ and $n \geq 3$) it seems likely that the monodromy group defined by the *generic* algebraic surface of degree d in $\mathbb{R}^n$ is large enough to ensure that the Newton's integral (and any other non-zero linear form on the space of vanishing cycles) takes an infinite number of values on the orbit of any non-invariant vector (and the unique invariant vector is presented by the integration cycle corresponding to the hyperbolicity domain of an hyperbolic charge). I can prove this conjecture only if the additional restriction $d + n \geq 8$ is satisfied[1].

Everywhere below all the homology groups $H_*(\cdot)$ are reduced modulo a point.

2. Vanishing homology and local monodromy of complete intersections

Here we recall the basic facts about the local Picard–Lefschetz theory of isolated singularities of complete intersections (see e.g. [H], [E], [AGLV]) and extend them to the case of twisted vanishing homology groups.

2.1. **Classical theory.** Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a holomorphic map, $f = (f_1, \ldots, f_p)$, and suppose that the variety $f^{-1}(0)$ is an *isolated complete intersection singularity* (ICIS) at 0 (i.e. it is a smooth $(n-p)$-dimensional variety in a punctured neighborhood of 0). Suppose that the coordinates in $\mathbb{C}^p$ are chosen generically, then the map $\tilde{f} \equiv (f_1, \ldots, f_{p-1}) : \mathbb{C}^n \to \mathbb{C}^{p-1}$ also defines an ICIS at 0. Let B be a sufficiently small *closed* disc centered at the origin in $\mathbb{C}^n$, and

[1]For a complete proof, removing this restriction, see the Appendix to this article, written by W. Ebeling

$c = (c_1, \dots, c_p)$ a generic point very close to the origin in $\mathbb{C}^p$. The corresponding manifolds $X_f \equiv f^{-1}(c) \cap B$ and $\tilde{X}_f \equiv \tilde{f}^{-1}(c_1, \dots, c_{p-1}) \cap B$ are called the *Milnor fibres* of $f$and $\tilde{f}$. Their homology groups are connected by the exact sequence

$$(1) \qquad \cdots \to H_{n-p+1}(\tilde{X}_f) \to H_{n-p+1}(\tilde{X}_f, X_f) \xrightarrow{\partial} H_{n-p}(X_f) \to \cdots .$$

Proposition 1 (see [M], [H]). *The sequence (1) is trivial outside the fragment presented here. All groups in (1) are free Abelian. Moreover, the spaces X_f and $\tilde{X}_f$ are homotopy equivalent to the wedges of spheres of dimensions $n-p$ and $n-p+1$ respectively.*

The rank of $H_{n-p}(X_f)$ is called the *Milnor number* of the complete intersection f and is denoted by $\mu(f)$. The Milnor numbers of all *quasi-homogeneous* complete intersections are calculated in [GH] (in [MO] for $p=1$); we need the following special case of this calculation.

Proposition 2. *1. The Milnor number of a homogeneous function $f : \mathbb{C}^n \to \mathbb{C}^1$ of degree d with isolated singularity at 0 is equal to $(d-1)^n$.*

2. The Milnor number of a complete intersection $f = (f_1, f_2)$ with isolated singularity at 0, where the functions f_1 and f_2 are homogeneous of degrees a and b respectively, is equal to $((a-1)^n b - (b-1)^n a)/(a-b)$ if $a \neq b$, and to $(a-1)^n(an-a+1)$ if $a=b$.

The rank $\mu(f) + \mu(\tilde{f})$ of the middle group $H_{n-p+1}(\tilde{X}_f, X_f)$ of (1) is equal to the number of (Morse) critical points of the restriction of f_p on $\tilde{X}_f$. The generators of this group are represented by the *Lefschetz thimbles* defined by the (non-intersecting) paths in $\mathbb{C}^1$ connecting the non-critical value c_p of this restriction with all critical values, namely, any of these thimbles is an embedded disc swept out by the one-parametric family of *vanishing spheres* lying in the varieties $f^{-1}(c_1, \dots, c_{p-1}, \tau)$, where τ runs over the corresponding path in $\mathbb{C}^1$: when τ tends to the endpoint (i.e. to a critical value of this restriction) the cycles of this family contract to the corresponding critical point. These vanishing spheres in the variety X_f (which corresponds to the common starting point c_p of these paths) generate the group $H_{n-p}(X_f)$, while the elements of $H_{n-p+1}(\tilde{X}_f)$ define relations among them.

2.2. **Picard–Lefschetz formula for standard homology.** Let $s \subset \mathbb{C}^1$ be the set of all these critical values, then the group $\pi_1(\mathbb{C}^1 \setminus s)$ acts naturally on all groups of (1). This action commutes with all arrows in (1) and is trivial on the left-hand group $H_{n-p+1}(\tilde{X}_f)$. The action on the middle and right-hand groups $H_{n-p+1}(\tilde{X}_f, X_f)$, $H_{n-p}(X_f)$ is determined by the Picard–Lefschetz formula: a class $\delta \in H_{n-p+1}(\tilde{X}_f, X_f)$, being transported along a *simple loop* (see [Ph 67], [V 94]) ω_i, corresponding to the path connecting c_p with the i-th critical value, becomes

$$\delta + (-1)^{(n-p+1)(n-p+2)/2} \langle \partial\delta, \partial\delta_i \rangle \delta_i,$$

where δ_i is the class of the thimble defined by this path, ∂ is the boundary operator in (1), and $\langle \cdot, \cdot \rangle$ is the intersection form in $H_{n-p}(X_f)$. In particular, a similar formula describes the monodromy action of the same loop on $H_{n-p}(X_f)$: it sends an element Δ of this group to

$$\Delta + (-1)^{(n-p+1)(n-p+2)/2} \langle \Delta, \Delta_i \rangle \Delta_i, \tag{2}$$

where $\Delta_i \equiv \partial\delta_i$ is the sphere vanishing along this path.

Proposition 3 (see e.g. [AGV]). *The intersection form $\langle \cdot, \cdot \rangle$ is symmetric if $n-p$ is even and skew-symmetric if $n-p$ is odd. The self-intersection index of any vanishing sphere is equal to* 2 *if $n - p \equiv 0(mod\ 4)$ and to -2 if $n - p \equiv 2(mod\ 4)$.*

In particular, if $n-p$ is even, then any transportation along a simple loop ω_i acts on the group $H_{n-p}(X_f)$ (respectively, $H_{n-p+1}(\tilde{X}_f, X_f)$) as the reflection in the hyperplane orthogonal to the vector Δ_i (respectively, δ_i) with respect to the intersection form in the homology of X_f (respectively, the form induced by the boundary operator from this intersection form). The latter action is a central extension of the former one.

More generally, let F be a k-parametric deformation of f, i.e. a map $F : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^p$ such that $F(\cdot, 0) \equiv f$. For any $\lambda \in \mathbb{C}^k$ lying in a sufficiently small neighborhood D^k of the origin, denote by f_λ the map $F(\cdot, \lambda)$ and by $\tilde{f}_\lambda$ the map $\mathbb{C}^n \to \mathbb{C}^{p-1}$ given by first $p-1$ coordinate functions of f_λ. Set $X_{f,\lambda} = f_\lambda^{-1} \cap B$ and $\tilde{X}_{f,\lambda} = \tilde{f}_\lambda^{-1} \cap B$. If F is "not very degenerate" then for almost all values of λ these varieties are smooth (with boundaries) and have the same topological type; e.g. the varieties $X_f, \tilde{X}_f$ participating in (1) appear in the p-parametric deformation consisting of maps $f_\lambda \equiv (f_1 - \lambda_1, \dots, f_p - \lambda_p)$ and correspond to the particular value $(\lambda_1, \dots, \lambda_p) = (c_1, \dots, c_p)$.

Definition 1. The *discriminant variety* $\Sigma(F)$ of F is the set of such $\lambda \in D^k$ that the topological type of the pair of varieties $(\tilde{X}_{f,\lambda}, X_{f,\lambda})$ does not coincide with that for all neighboring λ, i.e., either the origin in $\mathbb{C}^{p-1}$ is a critical value of $\tilde{f}_\lambda$ or the origin in $\mathbb{C}^1$ is a critical value of $f_p|_{\tilde{X}_{f,\lambda}}$. An exact sequence similar to (1) appears for any $\lambda \in D^k \setminus \Sigma(F)$, as well as the monodromy action of the group $\pi_1(D^k \setminus \Sigma(F))$ on this sequence.

Now suppose that the deformation F keeps $\tilde{f}$ undeformed, i.e., $\tilde{f}_\lambda \equiv \tilde{f}$ for any λ; in particular the action of this group on the left-hand group in (1) is trivial. A standard speculation with the Zariski's theorem (see e.g. [AGV], [V]) allows us to reduce this action to the above-considered action of the group $\pi_1(\mathbb{C}^1 \setminus s)$, and thus to the Picard–Lefschetz operators.

There is a natural map, *Leray tube operation*

$$t : H_{n-p}(X_f) \to H_{n-p+1}(\tilde{X}_f \setminus X_f) \tag{3}$$

described e.g. in [Ph 67], [AGLV], [V 94]: for any cycle γ in X_λ the cycle $t(\gamma)$ is swept out by the small circles in $\tilde{X}_f \setminus X_f$ which are the boundaries of the fibres of the natural fibration of the tubular neighborhood of X_f.

2.3. **Twisted vanishing homology of complete intersections.** Let L_{-1} (respectively, $\pm\mathbb{Z}$) be the local system on $\tilde{X}_f \setminus X_f$ with the fibre $\mathbb{C}^1$ (respectively, $\mathbb{Z}^1$) such that any loop having an odd linking number with X_f acts on this fibre as multiplication by -1. In particular, $L_{-1} \equiv \pm\mathbb{Z} \otimes \mathbb{C}$.

Consider the obvious homomorphism

$$j : H_{n-p+1}(\tilde{X}_f \setminus X_f, L_{-1}) \to H^{lf}_{n-p+1}(\tilde{X}_f \setminus X_f, L_{-1}), \tag{4}$$

where $H^{lf}_*(\cdot)$ denotes the homology of locally finite chains.

The Lefschetz thimbles define elements also in the right-hand group of (4) and in the similar group $H^{lf}_{n-p+1}(\tilde{X} \setminus X_f, \pm\mathbb{Z})$: indeed, they are embedded discs in $\tilde{X}_f \setminus X_f$ with boundary in X_f, and thus their interior parts can be lifted to an arbitrary leaf of the local system L_{-1} or $\pm\mathbb{Z}$. For any such thimble $\delta_i \in H^{lf}_{n-p+1}(\tilde{X} \setminus X_f, \pm\mathbb{Z})$ there is an element $\kappa_i \in H_{n-p+1}(\tilde{X} \setminus X_f, \pm\mathbb{Z})$, the *vanishing cycle* defined by the same path in $\mathbb{C}^1$, such that $j(\kappa_i) = 2\delta_i$, see [Ph 65] and Fig. 1, where such a cycle in one-dimensional $\tilde{X}$ is shown.

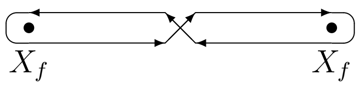

Fig. 1

Theorem 1. *a) The homomorphism (4) is an isomorphism, as well as the similar homomorphism of homology groups reduced mod $\partial\tilde{X}_f$,*

$$H_{n-p+1}(\tilde{X}_f \setminus X_f, \partial\tilde{X}_f \setminus \partial X_f; L_{-1}) \to H^{lf}_{n-p+1}(\tilde{X}_f \setminus X_f, \partial\tilde{X}_f \setminus \partial X_f; L_{-1});$$

b) the dimensions of both groups (4) are equal to $\nu(f) \equiv \mu(f) + \mu(\tilde{f})$, and similar homology groups in all other dimensions are trivial;

c) the right-hand group in (4) is freely generated by the Lefschetz thimbles specified by an arbitrary distinguished *(see e.g. [AGV], [AGLV])* *system of paths connecting the noncritical value c_p of $f_p|_{\tilde{X}}$ with all critical values.*

Corollary. *The left-hand group in (4) is generated by the* vanishing cycles *defined by the same paths.*

Proof of the theorem. The fact that the map (4) (and also its relative version) is isomorphic is a general algebraic fact, which is true for all local systems L_α with monodromy indices $\alpha \neq 1$: this follows from the comparison of the *Leray spectral sequences* (see e.g. [GrH], § III.5) calculating the indicated homology groups and applied to the identical embedding $\tilde{X}_f \setminus X_f \to \tilde{X}_f$.

The assertion of statement b) concerning the right-hand group in (4) follows from the similar assertion concerning the non-twisted vanishing homology group $H_{n-p+1}(\tilde{X}_f, X_f; \mathbb{Z}) \equiv H^{lf}_{n-p+1}(\tilde{X}_f \setminus X_f, \mathbb{Z})$ (see Proposition 1), the fact that $\pm\mathbb{Z} \otimes \mathbb{Z}_2 = \mathbb{Z} \otimes \mathbb{Z}_2 = \mathbb{Z}_2$ (the constant local system with fibre $\mathbb{Z}_2$) and from the formula of universal coefficients. The same reasons prove that the $\mathbb{Z}_2$-torsion of the group $H^{lf}_*(\tilde{X}_f \setminus X_f, \pm\mathbb{Z})$ is trivial in all dimensions.

Statement c) follows now from the fact that the images of thimbles are linearly independent already in the group $H^{lf}_{n-p+1}(\tilde{X}_f \setminus X_f, \pm\mathbb{Z}) \otimes \mathbb{Z}_2$.

Let $\Im$ be the subgroup in $H_{n-p+1}(\tilde{X}_f \setminus X_f, \pm\mathbb{Z})$ generated by vanishing cycles κ_i defined by all possible paths (probably it coincides with entire $H_{n-p+1}(\tilde{X}_f \setminus X_f, \pm\mathbb{Z})$).

Lemma 1. *For any elements $\alpha, \beta \in \Im$, their intersection index is even.*

Indeed, this index is equal to the (well-defined) intersection index of α and $j(\beta)$, and $j(\beta) \in 2H^{lf}_{n-p+1}(\tilde{X}_f \setminus X_f, \pm\mathbb{Z})$.

Define the bilinear form $\langle \cdot, \cdot \rangle$ on $\Im$ equal to half this intersection index.

Proposition 4. *The form $\langle \cdot, \cdot \rangle$ is symmetric if $n-p$ is odd and is skew-symmetric if $n-p$ is even. For any basis vanishing cycle κ_i, $\langle \kappa_i, \kappa_i \rangle$ is equal to 2 if $n-p \equiv 3 (mod\ 4)$ and to -2 if $n-p \equiv 1 (mod\ 4)$.*

In the terms of this form, the monodromy action on the group $\Im$ is defined by the same Picard–Lefschetz formula as before: the monodromy along the simple loop ω_i takes a cycle κ to

$$\kappa + (-1)^{(n-p+1)(n-p+2)/2} \langle \kappa, \kappa_i \rangle \kappa_i. \tag{5}$$

3. Surface potentials and Newton–Ivory–Arnold theorem

3.1. **Potential function of a surface.** Denote by dV the volume differential form in $\mathbb{R}^n$, i.e. the form $dx_1 \wedge \ldots \wedge dx_n$ in the Euclidean positively oriented coordinates $x_1, \ldots, x_n$.

Denote by r the Euclidean norm in $\mathbb{R}^n$, $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, and by C_n the area of the unit sphere in $\mathbb{R}^n$.

Definition 2. The *elementary Newton–Coulomb potential function*, or, which is the same, the *standard fundamental solution of the Laplace operator* in $\mathbb{R}^n$, is the function equal to $\frac{1}{2\pi}\ln r$ if $n = 2$, and to $-r^{2-n}/((n-2)C_n)$ if $n \geq 3$. Denote this function by G.

This function can be interpreted as the potential of the force of attraction by a particle of unit mass placed at the origin, i.e., the attraction force of this particle is equal to $-\text{grad}\ G$.

The attraction force with which a body K with density distribution P attracts a particle of unit mass placed at the point $x \in \mathbb{R}^n$ is equal to minus the gradient of the corresponding *potential function*, whose value at the point x is equal to the integral over K of the differential form $G(x-z)P(z)dV(z)$ (if such an integral exists).

Let F be a smooth function in Euclidean space $\mathbb{R}^n$, and M_F the hypersurface $\{F = 0\}$. Suppose that $\text{grad}\ F \neq 0$ at the points of M_F, so that M_F is smooth.

Definition 3. The *standard charge* ω_F on the surface M_F is the differential form dV/dF, i.e. the $(n-1)$-form such that for any tangent frame $(l_2, \ldots, l_n)$ of M_F and a transversal vector l_1 the product of the values $\omega_F(l_2, \ldots, l_n)$ and (dF, l_1) is equal to the value $dV(l_1, \ldots, l_n)$. The *natural orientation* of the surface M_F is the orientation defined by this differential form.

In particular, the value at a point $x \notin M_F$ of the limit of potential functions of homogeneous (with density $1/\epsilon$) distributions of charges between the surfaces $F = 0$ and $F = \epsilon$ is equal to the integral of the standard charge form

$$G(x-z)\omega_F(z) \tag{6}$$

along the naturally oriented surface M_F.

In a similar way, any function P on the surface M_F defines the charge $P \cdot \omega_F$, which is called the *standard charge with density* P; the potential at the point x of this charge is equal to the integral of the form

$$G(x-z)P(z)\omega_F(z) \tag{7}$$

along the naturally oriented surface M_F. The attraction force of this charge is equal to minus the gradient of this potential function.

In these terms, theorems of Newton and Ivory look as follows.

Theorem. *The potential of the standard charge of the sphere (respectively, an ellipsoid) in $\mathbb{R}^n$ given by the canonical equation (i.e. by a polynomial F of degree 2) is equal to a constant inside the sphere (the ellipsoid), while outside it coincides (up to multiplicative constant) with the potential function defined by any smaller ellipsoid confocal to ours.*

Arnold extended the "interior" part of this theorem to all *hyperbolic* layers.

Definition 4. An algebraic hypersurface M of degree d in $\mathbb{R}P^n$ is *strictly hyperbolic* with respect to a point $x \in \mathbb{R}P^n \setminus M$ if any real line through x intersects M at exactly d different real points. A polynomial $F : \mathbb{R}^n \to \mathbb{R}$ is strictly hyperbolic with respect to the point $x \in \mathbb{R}^n$ if the projective closure $\bar{M}_F$ of the corresponding surface M_F is.

Proposition 5 (see e.g. [ABG]). *If a hypersurface $M \subset \mathbb{R}P^n$ is strictly hyperbolic with respect to a point x, then it is also strictly hyperbolic with respect to any point in the same component of the complement of M. Any strictly hyperbolic hypersurface is smooth.*

Definition 5. The *hyperbolicity domain* of a surface M is the union of points x such that M is hyperbolic with respect to x.

Proposition 6 (see [N]). *The set of all hypersurfaces M of given degree d in $\mathbb{R}P^n$, which are strictly hyperbolic with respect to a given point x, is contractible (or, equivalently, the set of all polynomials of degree d defining them consists of two contractible components).*

In particular, all the strictly hyperbolic surfaces M of a given degree d in $\mathbb{R}P^n$ are situated topologically in the same way: if d is even, then M is ambient (and even rigid) isotopic to the union of $[d/2]$ concentric spheres lying in an affine chart in $\mathbb{R}P^n$; if d is odd, then M is isotopic to the union of $[d/2]$ concentric spheres plus the improper projective hyperplane. The hyperbolicity domain consists of the interior points of the "most interior" spheroid. This spheroid is always convex in $\mathbb{R}P^n$, in particular, the hyperbolicity domain in $\mathbb{R}^n$ may consist of at most two connected components.

The hyperbolic surface M_F separates the space $\mathbb{R}^n$ into *zones*: the k-th zone consists of all points $x \in \mathbb{R}^n \setminus M_F$ such that the minimal number of intersection points of M_F with segments connecting x and points of the hyperbolicity domains is equal to k. In particular, the maximal index k of a zone is equal to $[d/2]+1$ if d is odd and the hyperbolicity domain in $\mathbb{R}^n$ consists of one component, and is equal to $[d/2]$ otherwise.

Given a strictly hyperbolic polynomial F, let us fix some path-component of its hyperbolicity domain in $\mathbb{R}^n$, and number the components of M_F starting from the boundary of this component (which becomes number 1), its neighboring component gets number 2, etc.

Definition 6. The *Arnold cycle* of F is the manifold $\bar{M}_F$, oriented in such a way that in the restriction to its finite part M_F all odd components are taken with the natural orientation (see Definition 3), while all even components are taken with the reversed orientations.

The *hyperbolic potential* (respectively, *hyperbolic potential with density P*) of the surface M_F at a point $x \in \mathbb{R}^n \setminus M_F$ is the integral of the form (6) (respectively, (7)) along the Arnold cycle. As usual, the *attraction forces* defined by these potentials are equal to minus the gradients of the potential functions.

Lemma 2. *This definition of the Arnold cycle is correct, i.e. the orientations of different non-compact components of M_F thus defined are the restrictions of the same orientation of the corresponding components of $\bar{M}_F$.*

The proof is immediate.

Theorem (see [A 82]). *The hyperbolic potential of the surface M_F (and moreover any hyperbolic potential with density P, where P is a polynomial of degree $\leq d-2$) is constant inside the hyperbolicity domain.*

(In other words, the points of the hyperbolicity domain are not attracted by the standard charge on M_F taken with sign 1 or -1 depending on the parity of the number of the component on which this charge is distributed.)

The proof follows Newton's original proof: for any infinitesimally narrow cone centred at the point x, whose direction is not asymptotic for the surface M_F, the forces of attraction to the pieces of M_F cut by the cone annihilate one another. Indeed, let us restrict the polynomial F to the line L in $\mathbb{R}^n$ through x contained in this cone; then this attraction force is equal to the solid angle of our cone multiplied by the sum of the numbers $P(A_i)/F'(A_i)$ over all zeros A_i of the polynomial $F|_L$. The last sum is zero because it is the sum of the residues of a rational function over all its complex poles.

The restriction deg $P \leq d-2$ from the Arnold's theorem ensures that the integration form (7) is "regular at infinity", i.e. extends to a holomorphic form on the projective hypersurface $\bar{M}_F$. Givental [G 84] remarked that a similar statement is true for polynomial potentials of arbitrary degree if the integration cycle M_F is compact in $\mathbb{R}^n$: in this case the potential function in the hyperbolicity domain coincides with a polynomial of degree $\leq$ deg $P-d+2$.

In other domains the potential also coincides with real analytic functions; in the next sections we study the global behavior of these functions, in particular their algebraicity. The ramification of these functions is defined by the action of certain monodromy group on a certain homology group; in the next § 4 we define these objects, and in § 5 we calculate this monodromy group.

4. Monodromy group responsible for the ramification of potentials

4.1. **Homology groups.** For any point $x \in \mathbb{C}^n$, $x=(x_1,\dots,x_n)$, denote by $S(x)$ the cone in $\mathbb{C}^n$ given by the equation

$$(z_1-x_1)^2+\cdots+(z_n-x_n)^2=0. \tag{8}$$

Denote by $@ \equiv @(x)$ a local system over $\mathbb{C}^n \setminus S(x)$ with fibre $\mathbb{Z}$ such that the corresponding representation $\pi_1(\mathbb{C}^n\setminus S(x)) \to \operatorname{Aut}(\mathbb{Z})$ maps the loops whose linking numbers with $S(x)$ are odd to the multiplication by -1.

We specify this local system in such a way that integrals of the form $r(\cdot\ -x)dz_1\wedge\ldots\wedge dz_n$ along the $(n-1)$-dimensional cycles with coefficients in it are well defined. Namely, we consider the two-fold covering over $\mathbb{C}^n\setminus S(x)$, on which this form is single-valued, and the direct image in $\mathbb{C}^n\setminus S(x)$ of this bundle under the obvious projection of this covering. The trivial $\mathbb{Z}$-bundle over $\mathbb{C}^n\setminus S(x)$ is naturally included in this direct image as a subbundle; the desired local system is the quotient bundle of these two local systems. Obviously, integrals of the form $r(\cdot\ -x)dz_1\wedge\ldots\wedge dz_n$ (and of its products by all single-valued functions) along the piecewise smooth n-chains with coefficients in this local system are well-defined, and if these chains are cycles, these integrals depend only on their homology classes.

Let $F:\mathbb{C}^n\to\mathbb{C}$ be a polynomial, $W_F\subset\mathbb{C}^n$ the set of its zeros, and $\bar{W}_F$ the projective closure of W_F.

For any $x\in\mathbb{C}^n$ we denote by $\mathcal{H}(x)$ the group

$$H_{n-1}(W_F\setminus S(x),\mathbb{Z}) \tag{9}$$

in the case of even n, and the group

$$H_{n-1}(W_F \setminus S(x),\ @(x)) \tag{10}$$

if n is odd.

Similarly, denote by $\mathcal{P}H(x)$ the group

$$H_{n-1}(\bar{W}_F \setminus \bar{S}(x), \mathbb{Z}) \tag{11}$$

in the case of even n, and the group

$$H_{n-1}(\bar{W}_F \setminus \bar{S}(x),\ @(x)) \tag{12}$$

in the case of odd n.

Definition 7. If the polynomial F is real (i.e., $F(\mathbb{R}^n) \subset \mathbb{R}$) and strictly hyperbolic, then the Arnold cycle defines correctly an element of the group $\mathcal{P}H(x)$ (and even of the group $\mathcal{H}(x)$ if M_F is compact); these elements are called the *Arnold homology classes* and are denoted by $PA(x)$ and $A(x)$ respectively.

In the case of odd n, integrals of the form (7) along $(n-1)$-chains in $W_F \setminus S(x)$ with coefficients in $@(x)$ are well defined, and the values of these integrals along the cycles depend only on their homology classes in the group (10). Moreover, if $\deg P \leq d-2$, and hence the form (7) is regular at infinity, then it can be integrated along the chains in $\bar{W}_F \setminus \bar{S}(x)$, and the integrals along the cycles depend only on their classes in the group (12).

In the case of even $n > 2$ the form (7) is single-valued, and no problems with the definition of similar integrals along the elements of the group (9) (or even (11) if $\deg P \leq d-2$) arise, and in the exceptional case $n = 2$, when (7) is logarithmic, we remember that we are interested not in the potential, but in its first partial derivatives with respect to the parameter x (i.e. in the components of the attraction force vector). Therefore we integrate not the form (7) but its partial derivatives

$$\frac{x_i - z_i}{(x_1 - z_1)^2 + (x_2 - z_2)^2} P(z)\omega_F, \ i = 1, 2\,;$$

these forms are already single-valued and there is no problem in integrating them along the elements of the group (9) (or (11) if $\deg P \leq d-2$).

4.2. **Homological bundles.** For almost all $x \in \mathbb{C}^n$ the groups $\mathcal{H}(x)$ (respectively, $\mathcal{P}H(x)$) are naturally isomorphic to one another. The set of exceptional x (for which the pair $(\bar{W}_F \setminus \bar{S}(x), W_F \setminus S(x))$ is not homeomorphic to these for all neighboring x') belongs to a proper algebraic subvariety in $\mathbb{C}^n$ consisting of three components:

a) W_F itself,

b) the set of such x that $S(x)$ and W_F are tangent outside x in $\mathbb{C}^n$, and

c) the set of such x that the projective closure of $S(x)$ in $\mathbb{C}P^n$ is "more nontransversal" to the closure of W_F at their infinitely distant points.

For a generic F the last component is empty, and the second is irreducible provided additionally that $n \geq 3$.

Denote this algebraic set of all exceptional $x \in \mathbb{C}^n$ by $\Sigma(F)$.

Consider two fibre bundles over $\mathbb{C}^n \setminus \Sigma(F)$ whose fibres over a point x are the spaces $W_F \setminus S(x)$, $\bar{W}_F \setminus \bar{S}(x)$, and associate with them the homological bundles whose fibres over the same point are the groups $\mathcal{H}(x)$ and $\mathcal{P}H(x)$. As usual, the Gauss–Manin connection in these bundles defines the monodromy representations

$$\pi_1(\mathbb{C}^n \setminus \Sigma(F)) \to \text{Aut } \mathcal{H}(x), \tag{13}$$

$$\pi_1(\mathbb{C}^n \setminus \Sigma(F)) \to \text{Aut } \mathcal{P}H(x). \tag{14}$$

These representations obviously commute with the natural map $\mathcal{H}(x) \to \mathcal{P}H(x)$.

Let u be the potential function of the polynomial charge $P \cdot \omega_F$, i.e. the function defined for any x by the integral of the form (7) along the Arnold cycle. The ramification of (the analytic continuation of) the function u depends on the monodromy action (13) (respectively, (14)) on the Arnold element in $\mathcal{H}(x)$ (respectively, in $\mathcal{P}H(x)$).

Namely, for any multiindex $\nu \in \mathbb{Z}_+^n$ ($\nu \neq 0$ if $n = 2$) consider the linear forms

$$N^{(\nu)} : \mathcal{H}(x) \to \mathbb{C}, \quad PN^{(\nu)} : \mathcal{P}H(x) \to \mathbb{C}, \tag{15}$$

whose values on the cycle γ are equal to the integral along γ of the ν-th partial derivative of the form (7) with respect to the parameter x.

Proposition 7. *For any ν ($\neq 0$ if $n = 2$) and $x \in \mathbb{R}^n \setminus \Sigma(F)$, the ν-th partial derivative of the potential function of the standard charge of the compact hyperbolic surface M_F with density P is finite-valued at x if and only if the linear form $N^{(\nu)}$ takes finitely many values on the orbit of the cycle $A(x)$ under the action of the monodromy group (13). If P is a polynomial of degree $\leq d - 2 - |\nu|$, then the same is true for non-compact hyperbolic surfaces if we replace $A(x)$ by $PA(x)$, $N^{(\nu)}$ by $PN^{(\nu)}$, and the action (13) by (14).*

This is a tautology.

4.3. **The invariant cycle.** In this subsection we show that for any F and $x \in \mathbb{C}^n \setminus \Sigma(F)$ the representation (14) has an invariant vector; if F is a real hyperbolic polynomial and x lies in its hyperbolicity domain, then this cycle coincides with the Arnold homology class.

Denote by $PS \subset \mathbb{C}P^{n-1}$ the common "infinite" part of all cones $\bar{S}(x) \subset \mathbb{C}P^n$ and by @ the local system over $\mathbb{C}P^{n-1} \setminus PS$ such that any system @(x) is induced from it by the obvious projection with center x.

Proposition 8. *The groups*

$$H_{n-1}(\mathbb{C}P^{n-1} \setminus PS) \tag{16}$$

(if n is even) and

$$H_{n-1}(\mathbb{C}P^{n-1} \setminus PS, @) \tag{17}$$

(if n is odd) are one-dimensional. The generators of all these groups are presented by the class of the submanifold $\mathbb{R}P^{n-1} \subset \mathbb{C}P^{n-1} \setminus PS$.

The proof is elementary.

The obvious map $\Pi : \bar{W}_F \setminus \bar{S}(x) \to \mathbb{C}P^{n-1} \setminus PS$ (projection from the center x) is a d-fold ramified covering of complex (and thus oriented) manifolds. The variety $\Pi^{-1}(\mathbb{R}P^{n-1})$ admits thus an orientation (@(x)-orientation if n is odd) induced from the chosen orientation of $\mathbb{R}P^{n-1}$; denote by $\Omega(x)$ the class of this variety in the group $\mathcal{P}H(x)$.

Proposition 9. *1. The classes $\Omega(x)$ for different x constitute a section of the homology bundle over $\mathbb{C}^n \setminus \Sigma(F)$ with fibres $\mathcal{P}H(x)$, which is invariant under the Gauss–Manin connection, in particular these classes are invariant under the representation (14).*

2. If F is a real hyperbolic polynomial and x lies in its hyperbolicity domain, then $\Omega(x)$ coincides with the Arnold homology class $PA(x)$.

This follows immediately from the construction.

4.4. **Reduced Arnold class.** For an arbitrary element γ of the group $\mathcal{P}H(x)$, the corresponding potential function $u_\gamma(x)$ can be defined as the integral of the form (7) along the cycle γ (if this integral exists), in particular the usual potential $u(x)$ coincides with $u_{PA(x)}(x)$.

In this subsection we for any point $x \in \mathbb{R}^n \setminus \Sigma(F)$ replace the corresponding Arnold class $PA(x)$ by another class $P\tilde{A}(x)$, whose potential function $u_{P\tilde{A}(x)}$ ramifies in exactly the same way, but which is more convenient because (as we shall see later)

a) it is represented by a cycle lying in the "finite" part $W_F \setminus S(x)$ of $\bar{W}_F \setminus \bar{S}(x)$ and thus defining an element $\tilde{A}(x)$ of the (much better studied) group $\mathcal{H}(x)$, and

b) if n is even, then this element $\tilde{A}(x)$ can be obtained by the "Leray tube operation" (3) from a certain homology class $\alpha(x) \in H_{n-2}(W_F \cap S(x))$, so that the action (13) on it is reduced to the similar action on this more standard group.

Indeed, it follows from Proposition 9, that if the class $\gamma' \in \mathcal{P}H(x)$ is obtained by the Gauss–Manin connection over some path in $\mathbb{C}^n \setminus \Sigma(F)$ from the Arnold cycle $PA(\mathbf{x})$, where $\mathbf{x}$ is a point in the hyperbolicity domain of a compact hyperbolic surface, then the potential function $u_{\gamma'}(x)$ is a single-valued holomorphic function in $\mathbb{C}^n \setminus \Sigma(F)$. Therefore the ramification of our integrals defined by the class γ coincides with that defined by the class $\gamma - \gamma'$ (if both integrals are well-defined).

For any point $x \in \mathbb{R}^n \setminus M_F$ we choose canonically some class γ' obtained in this way. Namely, we choose an arbitrary point $\mathbf{x} \in \mathbb{R}^n$ in the hyperbolicity domain (if this domain has two components in $\mathbb{R}^n$, then in the component closest to x, i.e. such that the segment connecting x and $\mathbf{x}$ has $\leq [d/2]$ intersections with M_f). Then connect x with $\mathbf{x}$ by a complex line and take the path in this line that goes from $\mathbf{x}$ to x along the real segment and misses any point of W_F along a small arc in the *lower* complex half-line with respect to this direction (i.e. the half-line into which the vector $i \cdot (\mathbf{x} - x)$ is directed). See Fig. 2.

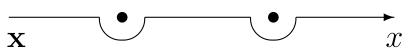

Fig. 2

For any $x \in \mathbb{R}^n \setminus M_F$, denote by $PA_{\text{hyp}}(x)$ the class in $\mathcal{P}H(x)$ obtained from $PA(\mathbf{x})$ by the Gauss–Manin connection over this path. We are interested in the monodromy of the class $PA(x) - PA_{\text{hyp}}(x)$, which will be called the *reduced Arnold class* and denoted by $P\tilde{A}(x)$.

4.5. **Groups $\mathcal{H}(x)$ and the vanishing homology of complete intersections.** We shall consider especially carefully the case when the attracting surface W_F satisfies certain genericity conditions, namely, the following ones.

We say that two holomorphic hypersurfaces in $\mathbb{C}^n$ are *simple tangent* at their common point, if in some local holomorphic coordinates with origin at this point one of them is given by the equality $z_n = 0$, and the second by $z_n = z_1^2 + \cdots + z_{n-1}^2$.

Definition 8. The polynomial F (and the corresponding hypersurface W_F) is *S-generic* if the projective closure $\bar{W}_F$ of W_F is smooth and transversal to the improper hyperplane $\mathbb{C}P^n \setminus \mathbb{C}^n$, its "infinite part" $\bar{W}_F \setminus \mathbb{C}^n$ is transversal in the improper hyperplane $\mathbb{C}P^{n-1}$ to the standard quadric $\{z_1^2 + \cdots + z_n^2 = 0\}$, i.e. to the boundary of any cone $S(x)$, and additionally the set of points at which W_F is *simple* tangent to appropriate cones $S(x)$ is dense in the set of all points of tangency of W_F and these cones at their nonsingular points.

The transversality conditions from this definition can be reformulated as follows: let $\bar{F}$ be the principal (of degree d) homogeneous part of F, and $r^2 \equiv z_1^2+\cdots+z_n^2$, then the function $\bar{F}$ has an isolated singularity at 0, and also the pair of functions $(\bar{F}, r^2)$ defines a (homogeneous) complete intersection with an isolated singularity at 0.

Theorem 2. *Suppose that the algebraic surface $W_F = \{F = 0\}$ in $\mathbb{C}^n$ is S-generic,* $\deg F = d$. *Then for a generic x the ranks of both groups (9), (10) (in particular, of the group $\mathcal{H}(x)$) are equal to $(d-1)^n + (2(d-1)^n - d)/(d-2)$ if $d > 2$, and to $2n$ if $d = 2$.*

Indeed, the pair of functions $(F, r^2(\cdot \ - x))$ defining the manifolds $W_F, S(x)$ is a perturbation of the complete intersection $(\bar{F}, r^2)$, changing only terms of lower degree of these polynomials. Thus the pair $(W_F, W_F \cap S(x))$ for smooth W_F and nondiscriminant x is homeomorphic to the pair $(\tilde{X}_f, X_f)$ from (1), and the local system @(x) is isomorphic to the system $\pm\mathbb{Z}$ on $\tilde{X}_f \setminus X_f$, see § 2.3. For the group (10) the assertion of the theorem follows now from Theorem 1 and Propositions 1 and 2.

Denote by ∂W_F the "infinite part" $\bar{W}_F \setminus \mathbb{C}^n$ of $\bar{W}_F$. Then the group (9) is Poincaré–Lefschetz dual to the group $H_{n-1}(\bar{W}_F, \partial W_F \cup (\bar{W}_F \cap \bar{S}(x)))$. Consider the homological exact sequence of the triple $(\bar{W}_F, \partial W_F \cup (\bar{W}_F \cap \bar{S}(x)), \partial W_F)$. By Proposition 1 and Poincaré duality in the manifolds W_F, $W_F \cap S(x)$, the only nontrivial fragment in this sequence is

$$\begin{aligned} &0 \to H_{n-1}(\bar{W}_F, \partial W_F) \to H_{n-1}(\bar{W}_F, \partial W_F \cup (\bar{W}_F \cap \bar{S}(x))) \to \\ &\qquad \to H_{n-2}(\bar{W}_F \cap \bar{S}(x), \partial W_F \cap \bar{S}(x)) \to 0, \end{aligned} \tag{18}$$

and the assertion of our theorem about the group (9) follows from Proposition 2.

Remark. It is easy to see that the map

$$H_{n-2}(W_F \cap S(x)) \to H_{n-1}(W_F \setminus S(x)), \tag{19}$$

conjugate with respect to Poincaré dualities to the third arrow in (18), coincides with the Leray tube operation (3), in particular in this case this operation is monomorphic.

So we have identified the pair $(W_F, W_F \cap S(x))$ with a standard object of the theory of singularities of complete intersections. The pair of functions $(F, r^2(\cdot - x))$ defining this complete intersection participates in three important families, which depend on n, 1 and $n+1$ parameters respectively. Since all of them keep the first function F unmoved, we describe only the corresponding families of second components. The first family consists of all functions $r^2(\cdot - \tilde{x})$, $\tilde{x} \in \mathbb{C}^n$; the second of all functions $r^2(\cdot - x) - \tau$, $\tau \in \mathbb{C}$, and the third of all functions

$$\rho_\lambda \equiv z_1^2 + \cdots + z_n^2 + \lambda_1 z_1 + \cdots + \lambda_n z_n + \lambda_0. \tag{20}$$

Denote the parameter space of the third deformation by T; the parameter spaces $\mathbb{C}^n$ and $\mathbb{C}^1$ of the first and second families are obviously included in it.

Define the set Σ_T as the set of all such points $\lambda \in T$ that the variety $W_F \cap \{\rho_\lambda = 0\}$ is not smooth; the intersection of Σ_T with the parameter space of the first (respectively, the second) subfamily coincides with $\Sigma(F)$ (respectively, the set s of critical values of the restriction of $r^2(\cdot - x)$ on W_F, see § 2.2).

By the Zariski theorem, the obvious homomorphism $\pi_1(\mathbb{C}^1 \setminus s) \to \pi_1(T \setminus \Sigma_T)$ is monomorphic, in particular the monodromy group generated by the action of the latter group in $\mathcal{H}(x)$ coincides with the standard monodromy group of the complete intersection $(\bar{F}, r^2)$ considered in § 2.2, 2.3.

Definition 9. The monodromy group defined by the Gauss–Manin representation $\pi_1(\mathbb{C}^1 \setminus s) \to Aut(\mathcal{H}(x))$ (or, equivalently, $\pi_1(T \setminus \Sigma_T) \to Aut(\mathcal{H}(x))$) is called the *big* monodromy group, while the similar monodromy group defined by the natural action (13) is the *small* one.

Below we shall see that the small monodromy group actually is a proper subgroup of the big one. To describe it we need several more reductions and notions.

The subgroup $\mathcal{J}(x) \subset \mathcal{H}(x)$ for any n is defined as that generated by all vanishing cycles in $W_F \setminus S(x)$ defined by all paths in $\mathbb{C}^1 \setminus s$ connecting 0 with all the points of s, see § 2: for even n it coincides with the image of the Leray tube map (19), for odd n it is just the group $\Im$ described in the end of § 2.3.

On this subgroup there is a symmetric bilinear form $\langle \cdot, \cdot \rangle$: in the case of odd n it was defined before Proposition 4 (as half the intersection index), and in the case of even n it is induced by the tube *mono*morphism (19) from the intersection index on the group $H_{n-2}(W_F \cap S(x))$. By Propositions 3 and 4, for any vanishing cycle $\alpha \in \mathcal{H}(x)$ $\langle \alpha, \alpha \rangle$ is equal to 2 if $[\frac{n+1}{2}]$ is odd and to -2 if $[\frac{n+1}{2}]$ is even.

Lemma 3. *For any n, the action of the big monodromy group on $\mathcal{H}(x)$ preserves the subgroup $\mathcal{J}(x)$ and the bilinear form $\langle \cdot, \cdot \rangle$ on it.*

This follows immediately from the Picard–Lefschetz formulae (2), (5).

Now suppose that the polynomial F is real and hyperbolic.

Theorem 3. *For any point x from the k-th zone of $\mathbb{R}^n \setminus \Sigma(F)$, $k \leq [d/2]$, the reduced Arnold class $P\tilde{A}(x) = PA(x) - PA_{\rm hyp}(x)$ can be represented by a cycle with support in $W_F \setminus S(x)$ which is homological in $\mathcal{H}(x)$ to the sum of k pairwise orthogonal vanishing cycles. In particular, its homology class $\tilde{A}(x)$ belongs to the subgroup $\mathcal{J}(x)$, and its self-intersection index $\langle \tilde{A}(x), \tilde{A}(x) \rangle$ is equal to $2k$ if $[\frac{n+1}{2}]$ is odd, and to $-2k$ if $[\frac{n+1}{2}]$ is even.*

Indeed, these vanishing cycles are constructed as follows. If the point $y \in \mathbb{R}^n \setminus M_F$ is sufficiently close to a component of M_F, then in a small disc $B \subset \mathbb{C}^n$ centered at y the pair $(W_F, S(y))$ is diffeomorphic to the pair consisting of the plane $\{x_1 = 1\}$ and the cone $S(0)$; it is easy to see that both groups $H_{n-1}(B \cap W_F \setminus S(y))$ and $H_{n-1}(B \cap W_F \setminus S(y), @(y))$ are isomorphic to $\mathbb{Z}$ and generated by vanishing cycles defined by the one-parametric family of maps $(F, r^2(\cdot - y) - \tau)$, $\tau \in \mathbb{C}^n$ (in the first case this cycle is equal to the tube around the vanishing cycle in $W_F \cap S(y)$).

Lemma 4 (see [V 94], Lemma 2 in § III.3.4). *If we go from the hyperbolicity domain along a line in $\mathbb{R}^n$ and traverse a component of M_F, then the Arnold class corresponding to the point after the traversing is equal to the sum of this vanishing cycle and of the similar Arnold cycle for the point before it transported by the Gauss–Manin connection over the arc of the path from Fig. 2 connecting them.*

In particular, the difference $PA(x) - PA_{hyp}(x)$ for x from the k-th zone is homologous to the sum of k vanishing cycles; by construction all these cycles lie in the finite domain $W_F \setminus S(x)$. The homology class of this sum in $\mathcal{H}(x)$ is exactly the promised reduced Arnold class $\tilde{A}(x)$, see § 4.4. It remains only to prove that these cycles are pairwise orthogonal. To do it, consider a model hyperbolic surface: the union of $[d/2]$ concentric close spheres of radii $1, 1+\varepsilon, \dots, 1+([d/2]-1)\varepsilon$ (which do not intersect one another even in the complex domain) and, if d is odd, one plane distant from these spheres.

Although this surface is not S-generic, the above-described construction of the cycle $\tilde{A}(x)$ can be accomplished for any point x in the k-th zone where $k \leq [d/2]$ and, if d is odd and $k = [d/2]$, then x lies much closer to the exterior ovaloid than to the additional plane. Then any of our k vanishing cycles lies on the complexification of its own sphere, in particular they do not intersect one another, and our assertion is proved for the (very degenerate) model hyperbolic surface. We can change this surface arbitrarily weakly so that its closure $\bar{W}_F$ becomes S-generic and transversal to $S(x)$, but the topological shape of the pair $(W_F, S(x))$ does not change in a large ball in $\mathbb{C}^n$ containing all our k vanishing cycles. Therefore they have zero intersection indices also for a certain generic hyperbolic polynomial. Finally, the set of nongeneric real hyperbolic polynomials, all whose "nongenericity" lies in the complex domain, has codimension at least 2 in the space of all strictly hyperbolic polynomials, and, by Proposition 6, the space of pairs of the form {a strictly hyperbolic polynomial F of degree d in $\mathbb{R}^n$; a point x of its k-th zone with $k \leq [d/2]$} is open and path-connected; this gives our assertion also for arbitrary generic F.

5. Description of the small monodromy group and finiteness theorems in the cases $n = 2$ and $d = 2$

5.1. **The two-dimensional case.** Let $n = 2$. Denote by $\eta(F)$ the number of factors $x_1^2 + x_2^2$ in the decomposition of the principal part $\bar{F}$ of the polynomial F into the simplest real factors. (Of course, if $\eta(F) > 0$ then F is not S-generic.)

Theorem 4. *The attraction force of the standard charge, distributed on a hyperbolic curve $\{F = 0\}$ of degree d in $\mathbb{R}^2$ coincides in the k-th zone with the sum of two algebraic vector-functions, any of which is $\leq \binom{d-\eta(F)}{k}$-valued. The same is true for the standard charge with polynomial density P of degree $\leq d-2$.*

If the hyperbolic curve $\{F = 0\}$ is compact and the density function P is holomorphic, then the corresponding attraction force coincides in the k-th zone with the sum of two analytic finite-valued (and even algebraic if P is a polynomial) vector-functions, any of which also is $\leq \binom{d-\eta(F)}{k}$-valued.

Corollary. *If d is even and $\bar{F} \equiv (x_1^2 + x_2^2)^{d/2}$, then the attraction force coincides with a rational vector-function in the "most nonhyperbolic" $(d/2)$-th zone.*

Example. If $d = 2$, then $\eta(F) \neq 0$ only in the Newtonian case (when M_F is a circle). In this case the attraction force is single-valued, in all the other irreducible cases it is 4-valued in the 1-st zone.

Proof of Theorem 4. If $n = 2$, then the surface $S(x)$ consists of two complex lines through x, collinear to the lines $\{x_1 = \pm i \cdot x_2\}$. The reduced Arnold class $\tilde{A}(x)$ corresponding to a point x from the k-th zone is represented by $2k$ small circles in $W_F \setminus S(x)$ around the intersection points of these two lines with W_F: k circles around the points of any line. It follows from the construction of Arnold cycles that all these circles close to one line are oriented in accordance with the complex structure of the normal bundle of this line, while close to all points of the other they are oriented clockwise. The total number of such intersection points in the finite domain for any line is equal to $d - \eta(F)$. Moving the point x in $\mathbb{C}^2 \setminus \Sigma(F)$ we can only permute these $d - \eta(F)$ circles (and, if W_F is smooth, all permutations can be realized). Therefore the orbit of the monodromy group consists of $\binom{d-\eta(F)}{k}^2$ elements; this implies Theorem 4.

Remark. Already in this case we see that the small monodromy group actually is smaller than the big one. Indeed, the standard ("big") monodromy group of the complete intersection $(\bar{F}, r^2)$ in $\mathbb{R}^2$ is just the permutation group of all $2d$ points of the Milnor fibre. In particular, the orbit of the reduced Arnold class from the k-th zone under this action consists of $\binom{2d}{k,k,2d-2k}$ points, which is much more than $\binom{d}{k}^2$ provided by Theorem 4 in the case $\eta(F) = 0$.

Remark about Ivory's second theorem. Given a hyperbolic surface, do there exist other surfaces defining the same attraction force in some exterior zone? If yes, these surfaces define the same ramification locus of the analytic continuations of these forces. In the case of irreducible plane curves this locus consists of $d(d-1)$ lines tangent to W_F and parallel to the line $x_1 = i \cdot x_2$ plus $d(d-1)$ lines parallel to the line $x_1 = -i \cdot x_2$. If $d = 2$, the set of curves for which these ramification loci coincide consists of all conics inscribed in a given rectangle whose sides are parallel to these two directions. It is easy to see that this set is one-parametric and coincides with the family of confocal conics. For larger d, such copotential families do not exist or at least are exceptional, because the number $2d(d-1)$ of conditions that the curves of such a family should satisfy becomes much greater than the dimension of the space of curves.

5.2. **Reduction of the kernel of the form $\langle\cdot,\cdot\rangle$ and the case of conical sections.** Denote by $\mathrm{Ker}\mathcal{J}(x)$ the kernel of the bilinear form $\langle\cdot,\cdot\rangle$ on the group $\mathcal{J}(x)$, i.e. the set of all $\gamma \in \mathcal{J}(x)$ such that $\langle\gamma,\alpha\rangle = 0$ for any α. By the Picard–Lefschetz formula, this subspace is invariant under the monodromy action, and hence this action on the quotient lattice $\tilde{\mathcal{J}}(x) \equiv \mathcal{J}(x)/\mathrm{Ker}\mathcal{J}(x)$ is well defined.

Theorem 5. *If F is S-generic, $x \in \mathbb{C}^n \setminus \Sigma(F)$, and P a polynomial of degree p, then any form $N^{(\nu)}$ (see (15)) with $|\nu| \geq p+2-d$ takes zero value on* $\mathrm{Ker}\mathcal{J}(x)$.

Proof. Let n be even, so that $\mathcal{J}(x) = t(H_{n-2}(W_F \cap S(x_0)))$, see (19). By Poincaré duality in $W_F \cap S(x_0)$, the condition $\gamma \in \mathrm{Ker}\ \mathcal{J}(x_0)$ implies that the cycle $t^{-1}(\gamma) \in H_{n-2}(W_F \cap S(x_0))$ is homologous in the projective closure $\bar{W}_F \cap \bar{S}(x_0) \subset \mathbb{C}P^n$ of $W_F \cap S(x_0)$ to a cycle which lies in the improper subspace $\bar{W}_F \cap \bar{S}(x_0) \cap (\mathbb{C}P^n \setminus \mathbb{C}^n)$. The tube around this homology provides the homology of γ to some cycle belonging to $\partial W_F \setminus \bar{S}(x_0) \equiv (\bar{W}_F \setminus \bar{S}(x_0)) \cap (\mathbb{C}P^n \setminus \mathbb{C}^n)$. The last space is an $(n-2)$-dimensional Stein manifold, thus γ is homologous to zero in $\bar{W}_F \setminus \bar{S}(x_0)$. On the other hand, the forms $D_x^{(\nu)}|_{x=x_0} G(x-y)P(y)\omega_F(y)$ with $|\nu| \geq 2+p-d$ can be extended to holomorphic forms on $\bar{W}_F \setminus \bar{S}(x_0)$, thus their integrals along γ are equal to zero.

In the case of odd n, the condition $\gamma \in Ker\ \mathcal{J}(x_0)$ also implies that γ is homologous in $\bar{W}_F \setminus \bar{S}(x_0)$ (as a cycle with coefficients in $@(x_0) \otimes \mathbb{C}$) to a cycle in the improper subspace: indeed, by Poincaré duality this condition implies that γ defines a trivial element of the group $H^{lf}_{n-1}(\bar{W}_F \setminus S(x_0), \partial W_F \setminus S(x_0); @(x_0))$, and hence, by the relative part of Theorem 1a), also of the group $H_{n-1}(W_F \setminus S(x_0), \partial W_F \setminus S(x_0); @(x_0) \otimes \mathbb{C}))$. The rest of the proof is the same as for even n.

Corollary. *In the conditions of Theorem 5, the linear form $N^{(\nu)}$ induces a form on the quotient lattice $\tilde{\mathcal{J}}(x)$, and the number of different values of this form on any orbit of the monodromy action on $\mathcal{J}(x)$ coincides with similar number for the induced form and induced monodromy action on $\tilde{\mathcal{J}}(x)$.*

Theorem 6. *For any $n \geq 3$ the potential of the standard charge (6) distributed on a strictly hyperbolic surface $\{F = 0\}$ of degree 2 in $\mathbb{R}^n$ coincides in the 1-st zone with an algebraic function.*

Proposition 10. *If n is even, $n > 2$, and F is a generic quadric in $\mathbb{C}^n$, then the pair consisting of the corresponding lattice $\mathcal{J}(x)$ and the* *bilinear form $\langle\cdot,\cdot\rangle$ on it coincides with that defined by the extended root system $\tilde{D}_{n+1}$. For odd n this pair is a direct sum of the lattice $\tilde{D}_{n+1}$ and the $(n-1)$-dimensional lattice with zero form on it.*

This fact in the case of even n and non-twisted homology is proved in [E], and the calculation for odd n is essentially the same.

Proof of Theorem 6. If F is a *generic* quadric, then by the Proposition 10 the lowered form $\langle\cdot,\cdot\rangle$ on the quotient lattice $\tilde{\mathcal{J}}(x)$ is isomorphic to the canonical form on the lattice D_{n+1}, in particular is elliptic. Hence the orbit of any class in this lattice (in particular of the coset of the reduced Arnold class) under the reduced monodromy action is finite, and any linear form takes finitely many values on it.

Finally, the *non-generic* quadric F can be approximated by a one-parameter family F_τ, $\tau \in (0, \epsilon]$, of generic quadrics. The analytic continuation of the potential function $u = u(F)$ is equal to the limit of similar continuations of potentials $u(F_\tau)$. Hence the number of leaves of $u(F)$ is majorized by the (common) number of leaves of any of the $u(F_\tau)$.

This proof estimates the number of leaves of potential functions of quadrics by the numbers of elements of length $\sqrt{-2}$ in the lattice D_{n+1}. As we shall see in the next subsection, this majorization is not sharp: a more precise upper bound is the number of integer points in the intersection of the sphere of radius $\sqrt{-2}$ with a certain affine sublattice of corank 1 that does not pass through the origin.

5.3. **Principal theorem on the small monodromy group.** The obvious map $\Pi : \bar{W}_F \setminus \bar{S}(x) \to \mathbb{C}P^{n-1} \setminus PS$ (see § 4.3) induces a homomorphism Π_* of the group $\mathcal{J}(x)$ to the group (16) (if n is even) or (17) (if n is odd). Denote by $\mathcal{M}(x)$ the kernel of this homomorphism.

Theorem 7. *Suppose that the polynomial $F : \mathbb{C}^n \to \mathbb{C}$ is S-generic. Then for any $x \in \mathbb{C}^n \setminus \Sigma(F)$*

a) the map Π_* is epimorphic, in particular $\mathcal{M}(x)$ is a sublattice of corank 1 in $\mathcal{J}(x)$;

b) $\mathcal{M}(x)$ is spanned by all vectors but one of some basis of vanishing cycles in $\mathcal{J}(x)$;

c) the small monodromy group in $\mathcal{J}(x)$ is generated by reflections (with respect to the form $\langle\cdot,\cdot\rangle$) in all the basis vanishing cycles generating $\mathcal{M}(x)$;

d) the set of these basis vanishing cycles in $\mathcal{M}(x)$ is transitive under the action of this small monodromy group;

e) the subgroup Ker $\mathcal{J}(x) \subset \mathcal{J}(x)$ *belongs to $\mathcal{M}(x)$.*

If F is a real hyperbolic polynomial of degree d, x a point from the k-th zone, $1 \leq k \leq [d/2]$, and $\tilde{A}(x)$ the corresponding reduced Arnold cycle, then additionally

f) $\tilde{A}(x)$ belongs to k times the generator of the quotient group $\mathcal{J}(x)/\mathcal{M}(x) \sim \mathbb{Z}$, in particular does not belong to $\mathcal{M}(x)$;

g) the linear form $\langle \tilde{A}(x), \cdot \rangle$ on $\mathcal{M}(x)$ is not trivial.

For the proof of this theorem and next Theorem 8 see § 6.

Corollary. *The orbit of any element of $\mathcal{J}(x)$ under the small monodromy group lies in some affine hyperplane parallel to $\mathcal{M}(x)$.*

Indeed, this follows from Theorem 7c) and Picard–Lefschetz formula.

Definition 10. A polynomial $P : \mathbb{C}^n \to \mathbb{C}$ is *very degenerate* with respect to W_F if it is equal to 0 at all points $y \in W_F$ at which appropriate surfaces of the form $S(x)$ are tangent to W_F at their smooth points.

Theorem 8. *Suppose that W_F is S-generic, and the polynomial P is not very degenerate with respect to W_F. Then there exist multiindices $\nu \in \mathbb{Z}_+^n$ with arbitrarily large $|\nu|$ such that for a generic $x \in \mathbb{C}^n \setminus \Sigma(F)$, the restriction on $\mathcal{M}(x)$ of the linear form $N^{(\nu)}$ (see (15)) is not trivial.*

5.4. **Main conjectures.** **Conjecture 1.** *If the hyperbolic polynomial F of degree $d \geq 3$ in $\mathbb{C}^n$, $n \geq 3$, is S-generic, then the potential function of the standard charge (7) with not very degenerate P does not coincide with algebraic functions in the components of $\mathbb{R}^n \setminus \Sigma(F)$ other than the hyperbolicity domain; moreover, the same is true for some arbitrarily high partial derivatives of this potential function.*

Theorem 7 reduces this conjecture to the following Conjecture 2 (proved recently by W. Ebeling, see the Appendix).

Definition 11. A triple $(A; \langle \cdot, \cdot \rangle; g)$ consisting of an integer lattice A, an *even* integer-valued symmetric bilinear form on it and a group $g \subset \mathrm{Aut}(A)$ generated by the reflections in hyperplanes orthogonal to several elements a_i of length $\sqrt{-2}$ in A, is called *completely infinite* if for any element $a \in A$ such that not all numbers $\langle a, a_i \rangle$ are equal to 0, any nonzero linear form $A \otimes \mathbb{C} \to \mathbb{C}$ takes infinitely many values on the orbit of a under the action of the group g.

Conjecture 2. *For any S-generic polynomial F of degree $d \geq 3$ in $\mathbb{C}^n$, $n \geq 3$, the triple consisting of the group $\mathcal{M}(x)$, the bilinear form equal (up to sign if $[\frac{n+1}{2}]$ is odd) to the form $\langle \cdot, \cdot \rangle$ defined before Lemma*

3, and the "small" monodromy group on $\mathcal{M}(x)$, is completely infinite.

In [V 94] this conjecture was proved if additionally $n + d \geq 8$.

Proposition 11. *Conjecture 2 implies Conjecture 1.*

Proof. Let x be a nondiscriminant point in the k-th zone, $1 \leq k \leq [d/2]$, for which the assertion of Theorem 8 with a certain ν is satisfied. By Theorem 7b), g) there is a vanishing cycle $\Gamma \in \mathcal{M}(x)$ such that $\langle \tilde{A}, \Gamma \rangle \neq 0$. By the Picard–Lefschetz formula, the monodromy along the corresponding simple loop takes $\tilde{A}$ to $\tilde{A} + \lambda\Gamma$, $\lambda \neq 0$. By Conjecture 2, for generic x the form $N^{(\nu)}$ takes infinitely many values on the orbit of the added term $\lambda\Gamma$ under the action of the small monodromy group. On the other hand, this infinite number is estimated from above by the number $q(q-1)$, where q is the number of values of the form N^ν on the orbit of $\tilde{A}(x)$, in particular this number q is also infinite.

Finally, for the points x from the $([d/2]+1)$-th zone (if it exists) the assertion of the Conjecture 1 follows from the fact that the potential function defined by the charge (7) obviously extends to an analytic function on $\mathbb{R}P^n \setminus M_F$, hence its algebraicity in the $([d/2]+1)$-th zone is equivalent to that in the zone separated from it by a piece of the improper subspace in $\mathbb{R}P^n$; the number of the latter zone is surely less than $[d/2]+1$.

6. Proof of Theorems 7, 8

All the main characters of statements a)–e) of Theorem 7 corresponding to all S-generic F of the same degree in $\mathbb{C}^n$ and all $x \notin \Sigma(F)$ are isomorphic to one another, therefore we can assume that F is a real hyperbolic polynomial and x a real point.

The proof of statement e) follows immediately from that of Theorem 5.

Any induction step from the proof of Theorem 3 obviously increases the image of $\tilde{A}(x)$ under the map Π_* by a generator of the target homology group; all such k steps are locally topologically equivalent, and hence add a fixed generator of this target group with the same sign. This proves statement f) of Theorem 7, and statement a) is a direct corollary of it.

For any $k = 1, 2, \dots, [d/2]$, and any point x in the k-th zone, consider the difference of the projective Arnold class $PA(x)$ and the element in $\mathcal{P}H(x)$ obtained as in the definition of the reduced Arnold cycles (i.e. by transportation along an arc in the lower complex half-line) from a similar class $PA(x')$, x' in the $(k-1)$-st zone. By Lemma 4, if x and

x' are sufficiently close to one another and to the k-th component of M_F separating them, then this class can be realized by a cycle lying in a small disc B containing both these points x, x'. Denote by $a(x)$ the class of this cycle in the group $\mathcal{H}(x)$; by continuity this class $a(x)$ is well defined also for arbitrary x from the same zone (not necessarily close to M_F). By Lemma 4, for all x not in the hyperbolicity domain the corresponding maps Π_* send the elements $a(x)$ into the same element of the group (16) or (17).

Theorem 9. *If $n > 2$, then*

a) all classes $a(x)$, corresponding to all points $x \in \mathbb{R}^n \setminus \Sigma(F)$, x not in the hyperbolicity domain or in the $([d/2]+1)$-th zone, can be obtained from one another by the Gauss–Manin connection in the homology bundle $\{\mathcal{H}(x) \to x\}$ over some path in $\mathbb{C}^n \setminus \Sigma(F)$. These classes $a(x)$ do not belong to $\mathcal{M}(x)$, and any of them, being added to the set of $\dim \mathcal{J}(x) - 1$ basis elements of $\mathcal{M}(x)$, mentioned in statements b), c) of Theorem 7, completes this set to a basis in $\mathcal{J}(x)$;

b) for arbitrary x in the k-th zone, $1 \leq k \leq [d/2]$, the linear form $\langle a(x), \cdot \rangle$ on the group $\mathcal{M}(x)$, defined by our bilinear form, is nontrivial.

6.1. **Comparison of big and small monodromy groups.** Now we compare the fundamental groups of $\mathbb{C}^n \setminus \Sigma(F)$ and of the complement of the discriminant variety Σ_T of the deformation (20) of the complete intersection $(\bar{F}, r^2)$. Since F is S-generic, the set $\Sigma(F)$ consists of only two components, W_F and the set of $x \notin W_F$ such that $S(x)$ is tangent to W_F; if $n > 2$, then the latter component is irreducible.

Let us choose the distinguished point $\mathbf{x}$ of the space $T \setminus \Sigma_T$ in the hyperbolicity domain of the subspace $\mathbb{R}^n \setminus \Sigma(F)$. The group $\pi_1(T \setminus \Sigma_T)$ acts in the usual way on the group $\mathcal{H}(\mathbf{x})$ and generates the "big" monodromy group, see § 4.5.

Let Λ be a generic 2-plane in T, and $L = \Lambda \cap \mathbb{C}^n$; $\bar{U}$ a small neighbourhood of L in the projective compactification of T, and $U = \bar{U} \cap T$ the affine part of $\bar{U}$. Let L' be a generic line in Λ through $\mathbf{x}$ sufficiently close to L, so that $L' \subset U$ and L' intersects Σ_T transversally.

Lemma 5. *The obvious maps $\pi_1(L \setminus \Sigma(F)) \to \pi_1(\mathbb{C}^n \setminus \Sigma(F))$ and $\pi_1(L' \setminus \Sigma_T) \to \pi_1(U \setminus \Sigma_T) \to \pi_1(T \setminus \Sigma_T)$ are epimorphic.*

The proof follows directly from the generalized Lefschetz theorem (see [GM]).

Thus the small and big monodromy groups are generated by simple loops lying in $L \setminus \Sigma_T$ and $L' \setminus \Sigma_T$, respectively. Let us compare these collections of loops.

Lemma 6. *The group $\mathcal{J}(\mathbf{x})$ is generated by the cycles vanishing along the paths of an arbitrary distinguished system in L' connecting the distinguished point $\mathbf{x}$ with all points of $L' \cap \Sigma_T$.*

Indeed, the group $\pi_1(L' \setminus \Sigma_T)$ acts on the group $\mathcal{J}(\mathbf{x})$; this monodromy action is described by the Picard–Lefschetz formulae, see § 2. Lemma 6 follows from these formulae, from Lemma 5, and from the fact that the group $\mathcal{J}(\mathbf{x})$ coincides with the linear hull of the orbit of any vanishing cycle under the action of the big monodromy group, see [Gab], [E].

The set $L \cap \Sigma(F)$ consists of several points of two kinds: the points of transversal intersection of L and W_F and points $x \notin W_F$ such that $S(x)$ is tangent to W_F.

Lemma 7. *a) Close to a generic point y of the submanifold $W_F \subset \mathbb{C}^n \subset T$ (i.e. to a point at which the generating lines of the cone $S(y)$ are transversal to W_F) the variety Σ_T is smooth and has simple tangency with $\mathbb{C}^n$ along W_F. In particular, the intersection of Σ_T with any 2-plane Λ transversal to W_F coincides close to the points of $\Lambda \cap W_F$ with a smooth curve having simple tangency with the line $\Lambda \cap \mathbb{C}^n \equiv L$;*

b) if F is S-generic, then close to a generic point of the variety $(\Sigma(F) \setminus W_F) \subset \mathbb{C}^n \subset T$ the variety Σ_T is smooth and intersects $\mathbb{C}^n$ transversally along $(\Sigma(F) \setminus W_F)$.

The proof is immediate.

Thus the cardinality of $L' \cap \Sigma_T$ is equal to the cardinality of $L \cap \Sigma(F)$ plus $\deg F$: to any point of $L \cap (\Sigma(F) \setminus W_F)$ there corresponds one close point of $L' \cap \Sigma_T$, while to any point of $L \cap W_F$ there correspond two such points; see Fig. 3a.

Since the point $\mathbf{x}$ lies in the hyperbolicity domain, all points of $L \cap W_F$ are real. For any such point y belonging to the k-th component of M_F, let $y_+ \in \mathbb{R}^n \setminus \Sigma(F)$ be a close point in the k-th zone. For such a point y_+, the class $a(y_+)$ was defined before Theorem 9.

Let us agree to choose the distinguished system of paths in L' in such a way that the paths connecting $\mathbf{x}$ with any two points of $L' \cap \Sigma_T$ arising from the same point y of $L \cap W_F$ go together up to a small common neighborhood of these two points and are close to the real segment in L connecting $\mathbf{x}$ and y, while the paths in L' connecting $\mathbf{x}$ with any other points of $L' \cap \Sigma_T$ do not touch this small neighborhood; see Fig. 3b.

Definition 12. A point of $L' \cap \Sigma_T$ is of the first kind (respectively, of the second kind) if it arises from a close point of W_F (respectively, of

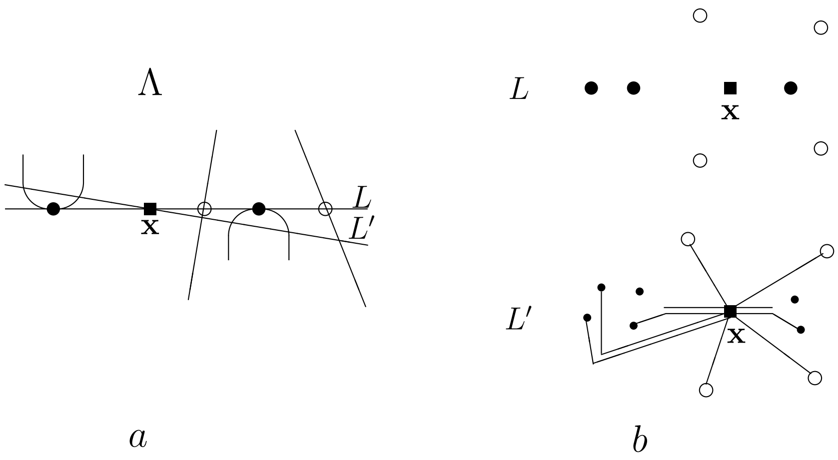

Fig. 3. Lines L and L' and discriminant points in them

$\Sigma(F)\setminus W_F$) in L after the move $L \to L'$. A cycle in $\mathcal{J}(\mathbf{x})$ vanishing over a path of our distinguished system in L' that connects $\mathbf{x}$ with a point $y \in \Sigma_T$ is called a cycle of the first kind (respectively, of the second kind) if this point y is of the first (respectively, the second) kind.

In Fig. 3b the points of $L \cap W_F$ and the points of the first kind in L' are shown by small black circles, while the points of $L \cap (\Sigma(F) \setminus W_F)$ and the points of the second kind in L' are shown by white circles.

Lemma 8. *a) Two cycles of the first kind in $\mathcal{H}(\mathbf{x})$, vanishing over two distinguished paths connecting $\mathbf{x}$ with two points of $L' \cap \Sigma_T$ arising from the same close point y of $L \cap W_F$, coincide (maybe up to sign);*

b) this cycle coincides (maybe up to sign) with the cycle $a(y_+)$ transported from the point y_+ to $\mathbf{x}$ along the path described in the definition of the reduced Arnold class. In particular, the map Π_* sends the homology class of any such cycle into a generator of the corresponding group (16) or (17);

c) the monodromy action in the group $\mathcal{H}(\mathbf{x})$, defined by any simple loop in $L \setminus \Sigma(F)$ going around some point of $L \cap W_F$, is trivial;

d) any cycle in $\mathcal{H}(\mathbf{x})$ vanishing over a path in $L \setminus \Sigma(F)$ connecting $\mathbf{x}$ with a point of $\Sigma(F) \setminus W_F$ belongs to the subspace $\mathcal{M}(\mathbf{x})$. In particular, the same is true for any cycle of the second kind defined by a path of our distinguished system in $L' \setminus \Sigma_T$ connecting $\mathbf{x}$ with a point (of the second kind) of Σ_T.

Proof. Consider the space of complex lines through $\mathbf{x}$ transversal to Σ_T in the plane Λ. Obviously this space is a projective line with several points removed, one of which is the point $\{L\}$. Consider a small loop in this space, which starts and finishes at the point $\{L'\}$ and goes once around the point $\{L\}$. This loop takes one of the two distinguished paths from statement a) of the lemma into the other, thus this statement follows.

Statement c) is a direct consequence of a). Indeed, the loop considered there is homotopic in $\Lambda \setminus \Sigma_T$ to a loop in $L' \setminus \Sigma_T$ which turns around two discriminant points defining the same vanishing cycle, thus its monodromy action is equal to the square of the reflection in the hyperplane orthogonal to this vanishing cycle.

Statement b) follows from Lemma 4 and the local shape of the pair $(W_F, \mathcal{S}(\lambda))$ where $\mathcal{S}(\lambda)$ is the variety of zeros of the polynomial (20) defined by the discriminant point λ of the first kind. The way in which the pairs of distinguished paths connecting x with different pairs of points of the first kind miss one another is not important, because by the proof of Theorem 3 all the cycles of the first kind that vanish over the paths going from $\mathbf{x}$ to the points arising from different points of $L \cap W_F$ on the same side of $\mathbf{x}$ in $\operatorname{Re} L$ are pairwise orthogonal.

Statement d) of the lemma follows immediately from the constructions.

Thus, the vanishing cycles of the first (respectively, second) kind are exactly those that are sent by the map Π_* into a generator of the group (16) or (17) (respectively, into a zero class).

Lemma 9. *Any vanishing cycle of the first kind in $\mathcal{J}(\mathbf{x})$ can be transformed into any other by a sequence of reflections in the hyperplanes orthogonal to cycles of the second kind and to this cycle itself.*

By the Picard–Lefschetz formula, this lemma follows from the next one.

Lemma 9′. *There exists a distinguished system of paths in $L' \setminus \Sigma_T$ connecting $\mathbf{x}$ with all points of $L' \cap \Sigma_T$, such that all vanishing cycles of the first kind defined by this system are equal to each other.*

Proof. (This proof simulates that of the well-known fact that the fundamental group of the complement of a smooth irreducible algebraic hypersurface in $\mathbb{C}^n$, $n \geq 2$ is isomorphic to $\mathbb{Z}$.)

Let y_1 be any point of $L \cap W_F$. Let us fix an arbitrary path γ_1 in $L \setminus \Sigma_T$ connecting $\mathbf{x}$ with y_1. Denote by A^n the space of complex lines in $\mathbb{C}^n$, and by $\operatorname{Reg}(\Sigma(F))$ the subset of A^n consisting of lines transversal to $\Sigma(F)$. Consider a path $\chi_1 : [0,1] \to A^n$ such that $\chi_1(0) = L$, $\chi_1([0,1)) \subset \operatorname{Reg}(\Sigma(F))$, the last point $\chi_1(1)$ is a line transversal to $\Sigma(F)$ everywhere except for one point of simple tangency with W_F, and one of the two points of $\chi_1(\tau) \cap W_F,\ \tau = 1-\varepsilon$ that coalesce at this tangency point is obtained from the point y_1 of the similar set corresponding to the value $\tau = 0$ during the deformation of the set $\chi_1(\tau) \cap W_F,\ \tau \in [0, 1-\varepsilon]$.

Consider the continuous deformation $\gamma_1[\tau]$, $\tau \in [0,1]$, of the path γ_1 such that $\gamma_1[0] = \gamma_1$, $\gamma_1[\tau] \subset \chi_1(\tau)$, and for any τ the path $\gamma_1[\tau]$ connects in $\chi_1(\tau) \setminus \Sigma(F)$ a point of $\chi_1(\tau) \cap W_F$ with some distinguished point $\mathbf{x}(\tau) \in \gamma_1[\tau] \setminus \Sigma(F)$, $\mathbf{x}(0) = \mathbf{x}$. At almost the final instant $\tau = 1-\varepsilon$, the endpoint $\gamma_1[1-\varepsilon](1)$ of the path $\gamma_1[1-\varepsilon]$ lies very close to some other point of $\chi_1(1-\varepsilon) \cap W_F$ (with which it coalesces at the instant $\tau = 1$). Connect this new point with $\mathbf{x}(1-\varepsilon)$ by a path $\gamma_2[1-\varepsilon]$ in $\chi_1(1-\varepsilon)\setminus\Sigma(F)$ that goes very close to $\gamma_1[1-\varepsilon]$ but does not intersect it except for the initial point. Then construct a continuous family of paths $\gamma_2[\tau] \subset \chi_1(\tau)$, $\tau \in [0, 1-\varepsilon]$, such that for any τ the corresponding path $\gamma_2[\tau]$ connects a point of $\chi_1[\tau] \cap W_F$ with $\mathbf{x}(\tau)$ and does not intersect other points of $\chi_1(\tau) \cap \Sigma(F)$ or of the path $\gamma_1[\tau]$. At the instant $\tau = 0$ we get a path $\gamma_2 \equiv \gamma_2[0] \subset L$ connecting $\mathbf{x}$ with some point y_2 of W_F.

Then consider a new path $\chi_2 : [0,1] \to A^n$, $\chi_2([0,1)) \subset \mathrm{Reg}\,(\Sigma(F))$, connecting L with some new simple tangent line to W_F and having no extra nontransversalities with $\Sigma(F)$, in such a way that at the last instant $\tau = 1$ one of the two points of $\chi_2(\tau) \cap W_F$ that coalesce at the tangency point is obtained by deformation along our path χ_2 from one of the points y_1 or y_2, and the other two points of these two pairs do not coincide. Arguing as before, we construct a third path in $L \setminus \Sigma(F)$, connecting $\mathbf{x}$ with some third point of $L \cap W_F$, and so on.

After the $(d-1)$-th step we get a system of d nonintersecting paths in $L \setminus \Sigma(F)$, connecting $\mathbf{x}$ with all points of $L \cap W_F$. Complete this family to any distinguished collection of paths connecting $\mathbf{x}$ with all points of $L \cap \Sigma(F)$. For the close perturbation $L' \subset T$ of L, take a close distinguished system of paths in L', connecting the point $\mathbf{x}$ with all points of $L' \cap \Sigma_T$ in such a way that to any path in L connecting $\mathbf{x}$ with W_F there correspond two paths connecting $\mathbf{x}$ with two close points of the first kind. This system of paths is the desired one. For instance, the cycles vanishing along the (perturbed) paths γ_1 and γ_2 define the same vanishing homology class in $\mathcal{J}(\mathbf{x})$: indeed, a similar assertion for the cycles in the group $\mathcal{H}(\mathbf{x}(1-\varepsilon)) \equiv H_{n-1}(W_F \setminus S(\mathbf{x}(1-\varepsilon)), \mathbb{Z})$ or $H_{n-1}(W_F \setminus S(\mathbf{x}(1-\varepsilon)), @(\mathbf{x}(1-\varepsilon)))$ is proved just as the statement a) of Lemma 8, and for other values of $\tau \in [0, 1-\varepsilon]$ it follows by continuity. Lemmas 9$'$ and 9 are thus proved.

Now we are ready to prove statement b) of Theorem 7. Indeed, by Lemma 6 the group $\mathcal{J}(\mathbf{x})$ is generated by the vanishing cycles of the first and second kind. By Lemma 9 and the Picard–Lefschetz formula, all vanishing cycles of the first kind lie in the linear span of an arbitrary one of them (for which we can take the class obtained by the Gauss–Manin connection from $a(x)$, x from the **k**-th zone, $1 \leq k \leq [d/2]$, see statement b) of Lemma 8) and the vanishing cycles of the second kind (which lie in $\mathcal{M}(\mathbf{x})$, see statement d) of Lemma 8).

Statement c) of Theorem 7 follows immediately from statement c) of Lemma 8, and statement d) follows from the fact that the variety $\Sigma(F) \setminus W_F$ is irreducible.

Proof of Theorem 9a). We can assume that the points y_1 and y_2, whose classes $a(y_1)$ and $a(y_2)$ we want to transfer to each other, lie very close to the "interior" (i.e. closest to the hyperbolicity domain) components of M_F bounding corresponding zones. For such y_i the class $a(y_i)$ is realized by a cycle generating the group $H_{n-1}(W_F \cap B \setminus S(y_i))$ or $H_{n-1}(W_F \cap B \setminus S(y_i), @(y_i))$, where B is a small neighbourhood of y_i; see Lemma 4. Thus, for the desired path connecting y_1 and y_2 we can take the path that goes very close to the set of generic points of W_F (i.e. of such points y close to which all the generating lines of the cones $S(y)$ are transversal to W_F and hence the pairs $(W_F, S(y))$ have locally the same topological structure).

Statement b) of Theorem 9 follows from Theorem 3 and the connectedness of Dynkin diagrams of isolated singularities of complete intersections.

Proof of the statement g) of Theorem 7.

First of all, this statement is true in the case when M_F is an ellipsoid with different eigenvalues. Indeed, by Theorem 3 in this case $\tilde{A}(x)$ is a vanishing cycle, and the assertion follows from the connectedness of the Dynkin diagram and the fact that the group $\mathcal{M}(x)$ is nontrivial for such F, see e.g. [E].

For arbitrary d, consider the model (not S-generic) hyperbolic surface M_F consisting of $[d/2]$ ellipsoids $\alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 = j, \quad j = 1, 1+\varepsilon, \dots, 1+([d/2]-1)\varepsilon$, where all α_i are positive and distinct, plus, if d is odd, a distant hyperplane. The class $\tilde{A}(x)$ for x from the k-th zone, $1 \leq k \leq [d/2]$, is then equal to the sum of k vanishing cycles, each of which lies in the complexification of its own ellipsoid; see the proof of Theorem 3. By the previous special case of a single ellipsoid, in each of these k complexified ellipsoids $\mathcal{E}_i$ there is a compact cycle Γ defining an element of the group $H_{n-1}(\mathcal{E}_i \setminus S(x))$ if n is even, or in

$H_{n-1}(\mathcal{E}_i \setminus S(x), @(x))$ if n is odd, such that $\langle \tilde{A}(x), \Gamma \rangle \neq 0$ and the map Π_* sends the homology class of Γ into the zero homology class.

Consider a perturbation of our model hyperbolic polynomial F which replaces it by a S-generic one and is so weak that it does not change the topology of the variety $W_F \cup S(x)$ inside a sufficiently large disc, in which the cycles Γ and $\tilde{A}(x)$ lie. The cycle $\tilde{\Gamma}$ close to Γ in the moved manifold W_F satisfies all the above conditions, and statement g) of Theorem 7 is proved for *some* S-generic hyperbolic polynomial. For an arbitrary such polynomial this statement follows from the fact that all the generic surgeries separating different path-components of the space of all strictly hyperbolic S-generic surfaces of given degree in $\mathbb{R}^n$ (these surgeries correspond to the smooth hyperbolic surfaces in $\mathbb{R}P^n$ simple tangent to the non-proper plane) preserve the homology classes $\tilde{A}(x)$ (provided that the corresponding point x and the distinguished point in the hyperbolicity domain do not change in this surgery). (In formal terms, this preservation means that these homology classes corresponding to the polynomials before and after the surgery are transposed into one another by the natural connection over any *short* connecting them path in the space of all *complex* S-generic polynomials.)

6.2. **Proof of Theorem 8.** Let c be a point of simple tangency of a cone $S(x_0)$ and W_F such that $P(c) \neq 0$. Let Υ be an affine complex line through x_0 in $\mathbb{C}^n$, transversal to the common tangent hyperplane of $S(x_0)$ and W_F at c; let ξ be an affine coordinate on it with the origin at x_0. Consider the one-parametric family of surfaces $S(x(\xi))$, $x(\xi) \in \Upsilon$. The elements $S(x(\xi))$ of this family with ξ from a small *punctured* neighborhood of the origin are transversal to W_F in a small disc B centred at c, and the vanishing element $\gamma(\xi)$ of the group $H_{n-1}(B \cap W_F \setminus S(x(\xi)))$ (if n is even) or $H_{n-1}(B \cap W_F \setminus S(x(\xi)), @(x(\xi)))$ (if n is odd) is well defined (up to sign) by this family. By the Picard–Lefschetz formulae of § 2, in both cases the rotation of ξ around 0 sends $\gamma(\xi)$ to $-\gamma(\xi)$.

Define the function $\Xi(\xi)$, $\xi \in \mathbb{C}$, as the integral of the form (7) with $x = x(\xi) \in \Upsilon$ along the cycle $\gamma(\xi)$. It is sufficient to prove that there are arbitrarily high derivatives of this function not equal identically to 0. This follows from the next lemma.

Lemma 11. *The function $\Xi(\xi)$ is represented by a power series of the variable $\sqrt{\xi}$, whose leading (of smallest degree) term with non-zero coefficient has degree* 1.

(Of course, all even powers of this series vanish.)

Proof. Using the Leray residue theorem, we can replace the integral (7) along the cycle $\gamma(\xi)$ by the integral of the form $G(x(\xi) - z)P(z)/F(z)dz_1 \wedge \ldots \wedge dz_n$ along the Leray tube $t\gamma(\xi) \in H_n(B \setminus (W_F \cup S(x(\xi))))$ or $\in H_n(B \setminus (W_F \cup S(x(\xi))), @(x(\xi)))$. Close to c the holomorphic function $\mathbb{C}^n \to \mathbb{C}$ is defined, which assigns to any point the coordinate ξ of the origin $x(\xi)$ of the cone $S(x(\xi))$ containing it. Choose this function for the last local coordinate w_n at c; by the Morse lemma we can choose the remaining coordinates $w_1, \ldots, w_{n-1}$ in such a way that W_F is locally given by $w_n = w_1^2 + \cdots + w_{n-1}^2$. In these coordinates our differential form becomes

(21)

$$(w_n - \xi)^{-(n-2)/2}(w_n - w_1^2 - \cdots - w_{n-1}^2)^{-1} I(w_1, \ldots, w_n) dw_1 \wedge \ldots \wedge dw_n,$$

where the function I does not vanish at c. Let $I = I_0 + I_1 + \cdots$ be the expansion of I into the sum of quasihomogeneous polynomials of degrees $0, 1, \ldots$ respectively with respect to the weights $\deg w_1 = \cdots = \deg w_{n-1} = 1, \deg w_n = 2$. Using the corresponding group of quasihomogeneous dilations $(w_1, \ldots, w_{n-1}, w_n) \to (\tau w_1, \ldots, \tau w_{n-1}, \tau^2 w_n)$ we see, that the integral along $t\gamma(\xi)$ of the form similar to (21), in which I_m is substituted instead of I, is a homogeneous function in ξ of degree $(m+1)/2$. It is easy to calculate that this function corresponding to the constant polynomial $I_0 \neq 0$ is not the identical zero function; this proves our Lemma.

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