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Cohomology of Knot Spaces

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In this article a system of numerical knot invariants is described. The construction of these invariants is based on the study of the discriminant, i.e., the set of maps $S^1 \to S^3$ possessing singularities or self-intersections. The discriminant is a singular hypersurface in the space of maps; its nonsingular points correspond to maps with a single transversal self-intersection point, while its singular points are maps with vanishing derivative, nontransversal or multiple self-intersections.

Any numerical invariant of the regular isotopy type of knots may be described in terms of the discriminant: to each nonsingular piece of the discriminant (i.e., to each connected component of its nonsingular points) we can assign its index — the difference between the values of the invariant for nearby knots separated by this piece of the discriminant.

Conversely, suppose that some numerical indices are assigned to all the nonsingular components of the discriminant. In order to determine an invariant correctly, this family of indices must satisfy the following homology condition: the sum of these components, taken with the corresponding coefficients (i.e., their indices) must have no boundary in the space of maps $S^1 \to S^3$. The listing of such admissible families of indices is a problem in homology theory that can be solved by standard methods of this theory.

We construct a spectral sequence that yields such families of indices. This spectral sequence $E_r^{p,q}$ is generated by the natural stratification of the discriminant into degeneracy levels of the maps $S^1 \to S^3$. For $r \geqslant 1$ the sequence is contained in the domain $p < 0, p + q \geqslant 0$. See Figure 1. The knot invariants correspond to elements of the groups $E_\infty^{-i,i}$, $i \geqslant 1$.

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In general, to any element of the group $E_\infty^{p,q}$ there corresponds a $p+q$-dimensional cohomology class of the knot space. We explicitly describe the E_1 term of the spectral sequence and present algorithms (ready for computer implementation) computing all the terms $E_\infty^{-i,i}$, the corresponding invariants, and the values of these invariants on arbitrary (tame) knots.

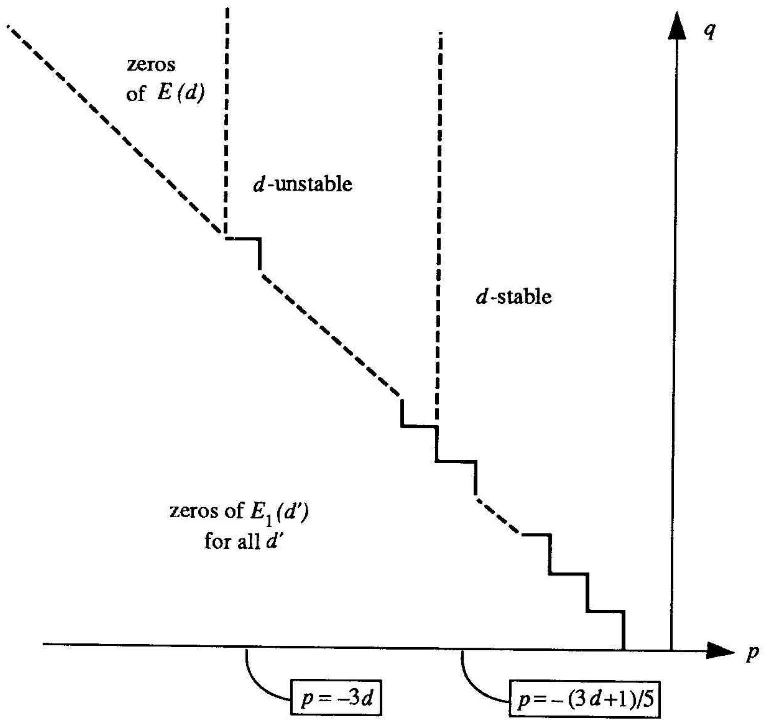

FIGURE 1

The natural question about our invariants (do they constitute a complete system of invariants?) is in fact that of the convergence of the spectral sequence (at least on the diagonal $p+q=0$). I hope that the answer is positive.

Our first nontrivial invariant comes from the term $E^{-2,2}$ and coincides with the coefficient of the monomial in x^2 of the Conway polynomial, but the second (coming from $E^{-3,3}$) cannot be reduced to Conway polynomials: it distinguishes the trefoil knot and its mirror image. Three invariants come from $E^{-4,4}$. The values of these five invariants for all simple knots having diagrams with $\leqslant 7$ overlaps or less are shown in Figure 2 in

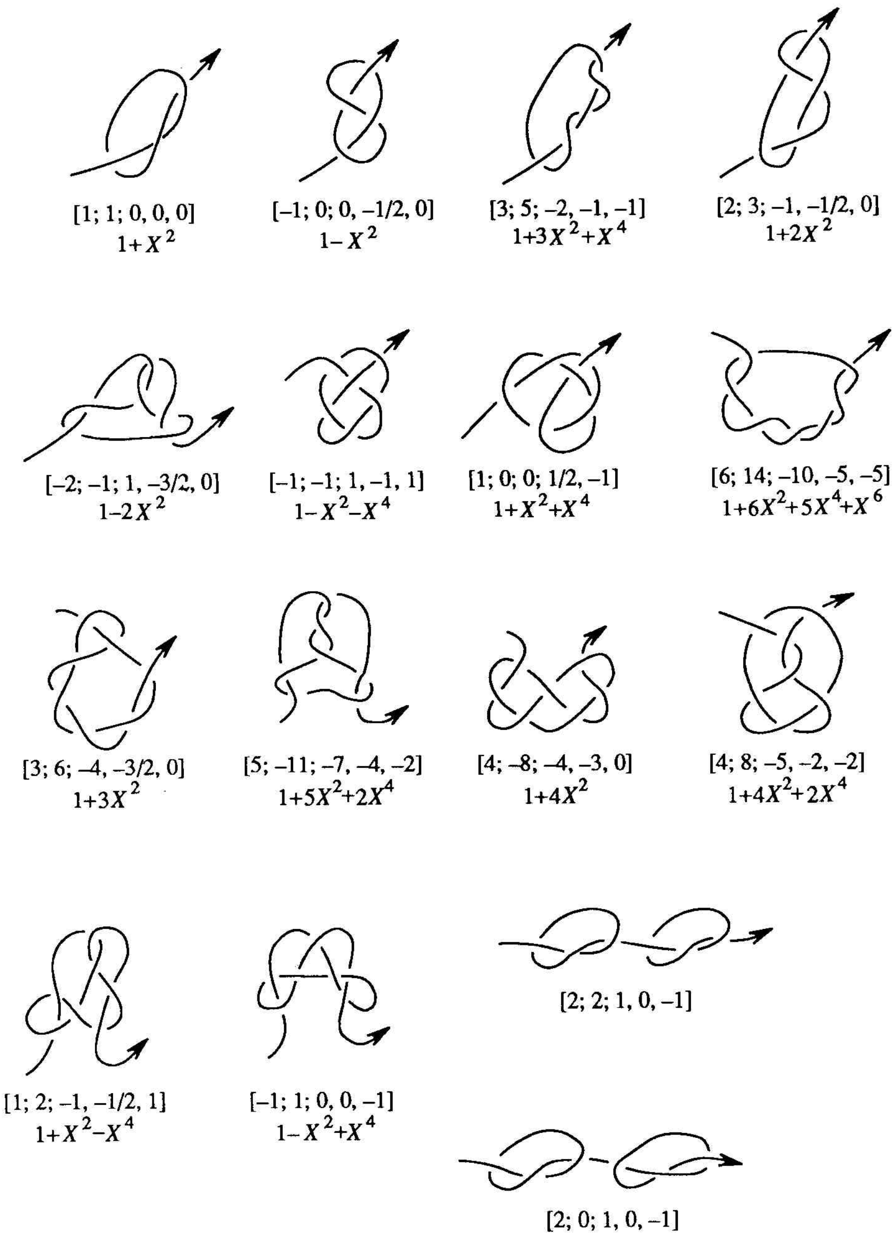

FIGURE 2

the square brackets; the Conway polynomials are written below them for comparison. For the missing knots, which are all mirror images of knots actually shown on the figure, the values can be obtained by the following rule: all our invariants coming from terms $E_\infty^{-i,i}$ with even (odd) i have the same values (resp., values of opposite sign).

Our spectral sequence can be used in a much more general situation. For example, its analogs converge to the cohomology of the embedding spaces $M^m \to \mathbb{R}^n$, $n \geqslant 2m+2$, or give invariants of embeddings $M^m \to \mathbb{R}^{2m+1}$; for such problems, see Subsection 6.6. Other versions of this spectral sequence were used in [V1, V2, V3].

The idea of constructing knot invariants as cycles of the discriminants is not new. As I was told by V. I. Arnold, O. Ya. Viro expressed this idea around 1985.

0.1. Noncompact knots. For the convenience of computations, rather than ordinary knots (i.e., smooth embeddings $S^1 \to S^3$), we consider so-called noncompact knots, i.e., nonsingular embeddings $\mathbb{R}^1 \to \mathbb{R}^3$ that at infinity asymptotically approach a fixed straight line in $\mathbb{R}^3$ (see §1). The space of all (including singular) smooth embeddings $\mathbb{R}^1 \to \mathbb{R}^3$ with such asymptotic behavior will be denoted by K; this space is homotopically trivial. By Σ we denote the discriminant of this space, i.e., the set of maps belonging to K and possessing singularities or self-intersections.

It is easy to see that the connected components of the space of noncompact knots $K-\Sigma$ are in one-to-one correspondence with isotopy classes of ordinary knots $S^1 \to S^3$. In particular, our numerical invariants may be viewed as elements of the group $H^0(K-\Sigma)$. We study this group, together with the other groups $H^i(K-\Sigma)$, $i \geqslant 0$.

0.2. Geometrical interpretation of the spectral sequence. Our spectral sequence generalizes the one from another paper [V3] in this volume; the sequence from [V3] converges to the cohomology of iterated loop spaces. Perhaps the reader will benefit by first reading §2 in [V3], which does not depend on the rest of [V3].

The invariants obtained from our spectral sequence form a group with a natural increasing filtration. Sums of invariants have degree $\leqslant k$ when they come from terms $E_\infty^{-i,i}$, where $i \leqslant k$. The geometric interpretation of this filtration is the following.

Any nonsingular piece of the discriminant Σ consists of maps having exactly one point of transversal self-intersection. This piece has a well-defined transversal orientation, i.e., there is a method for calling one of the two adjacent components of the knot space positive and the other negative. Any numerical knot invariant α assigns an index to our piece of the discriminant, the index being defined as the jump that occurs in the value of α when we pass through the piece in the positive direction.

Consider a nonsingular self-intersection point of Σ, i.e., an immersion $\mathbb{R}^1 \to \mathbb{R}^3$ with two points of transversal self-intersection. Near such a point the discriminant is represented by two irreducible components, each of which contains two nonsingular pieces (see Figure 3).

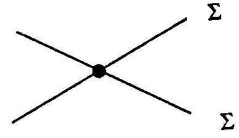

FIGURE 3

To such a point (and to the self-intersection piece of the discriminant that contains this point) we assign an index of the second order, defined as the difference between the indices assigned above to the two smooth pieces of any of the irreducible components of Σ near that point. (This number does not depend on the choice of component, because the sum of jumps of the invariant α equals 0 when we go all the way around the circle). Similar higher order indices are assigned to nonsingular pieces of multiple self-intersections of the discriminant.

The invariant α has filtration k iff all its corresponding indices of order $> k$ vanish.

EXAMPLE. There are no nontrivial invariants of order 1. Indeed, any first order invariant defines the same index on all the nonsingular pieces of the discriminant. But for the piece of the discriminant containing the map shown on Figure 4, such an index always equals zero: two nearby knots, separated by this piece, are equivalent.

0.3. Coding and computation of invariants. Let us call two nonsingular points of k-fold self-intersection of the discriminant *related* if we can move from one to the other without leaving the set of nonsingular k-fold self-intersection, only intersecting (transversally) the set of $(k+1)$-fold self-intersection a finite number of times. It is easy to see that for any k the number of classes of such related nonsingular points (*families*) is equal to the number of nonordered partitions into pairs of a set of cardinality $2k$: these classes are well-defined by the order of the pairs of points on the line identified by the corresponding maps $\mathbb{R}^1 \to \mathbb{R}^3$. For example, for $k = 2$ such partitions are shown on Figure 5, and the corresponding maps of the line appear on Figure 6.

Any invariant of order k (= filtration k) is determined by the following data. For any $i = 1, \dots, k$ and any class of related nonsingular i-fold self-intersections of the discriminant, one of the maps $\mathbb{R}^1 \to \mathbb{R}^3$ from this class is fixed as well as the i-th order index of this map.

The algorithm for computing our knot invariants is the following. We connect the given knot with the trivial one by a homotopy in general position in K. During this homotopy we intersect the discriminant at its nonsingular points several times. The value of the invariant for the given knot is equal to the sum of indices of all these intersection points taken with the sign + or –, depending on whether the direction in which Σ is intersected is positive or negative. The indices themselves, in their turn, are computed as follows. Each of the intersection points is connected by a path in general position in Σ with the distinguished map in the same

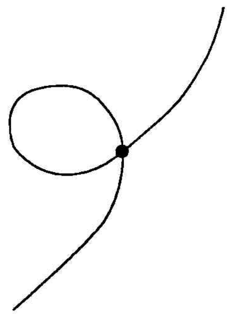

FIGURE 4

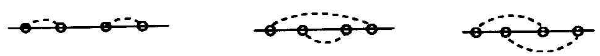

FIGURE 5

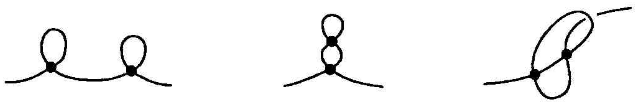

FIGURE 6

(unique) family (= class of related maps). The required index is equal to the sum of the index of the distinguished map (if it is the map on Figure 4, this index equals 0) and the (second order) indices of all the pieces of the self-intersection set of the discriminant that are met along the way (= path), taken with weight $+1$ or -1 depending on the directions of intersection. The computation of these indices is similarly reduced to third order indices, etc.; the whole procedure has k branching steps, where k is the filtration order of the invariant. Here at the last step (i.e., the calculation of the indices of the k-fold self-intersection points of Σ) it is no longer necessary to construct any path: the kth order index is entirely determined by the family to which each of these points belongs.

EXAMPLE. Let us describe the simplest invariant and find its value for the trefoil knot. This invariant has filtration 2. The corresponding index for the map shown on Figure 4 equals 0, while the second order indices of the three maps on Figure 6 are respectively equal to 0, 0, 1. Let us compute the value of this invariant for the trefoil knot (Figure 7, *a*). This knot can be joined to the trivial one, intersecting the discriminant only once (at the point shown on Figure 7, *b*), and it remains to compute the index at that point. In its turn, this point can be joined with the simplest self-intersecting curve shown on Figure 7, *e* (or Figure 4) by a path which will intersect the set of 2-fold self-intersections of the discriminant only once (Figure 7, *d*) at a point of index 1. Finally, the value of our invariant is the integer

$$0 + (-1)[0 + (-1) \cdot 1] = 1.$$

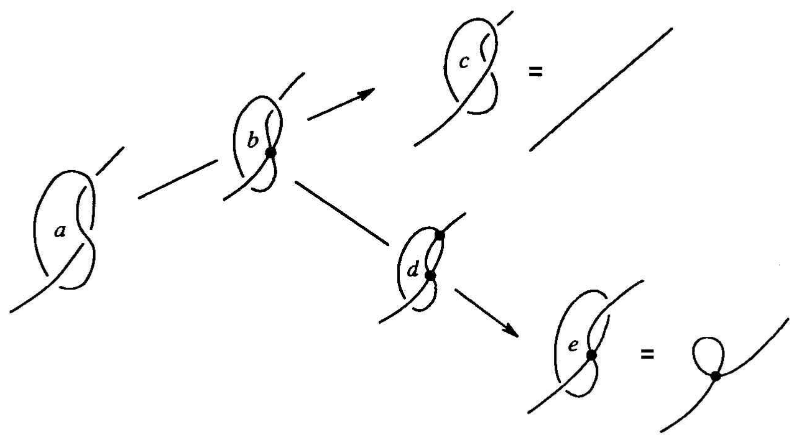

FIGURE 7

Here the first zero is the value of the invariant on the trivial knot, the first coefficient (-1) is due to the direction of intersection of Σ at the point in Figure 7, *b*, the second zero is the index of the map from Figure 7, *e*, the second coefficient (-1) is the toll for the crossing corresponding to the arrow between Figures 7, *b* and 7, *e* and, finally, the last 1 is the index of second order for the curve on Figure 7, *d*.

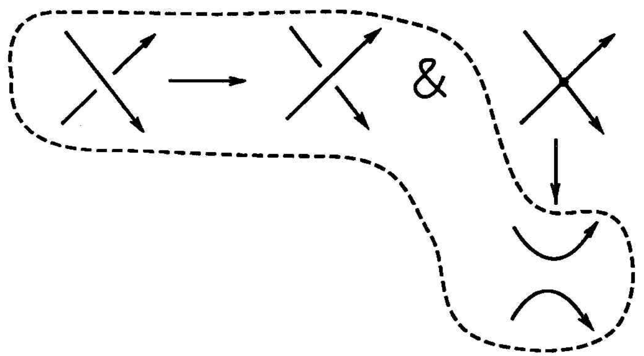

FIGURE 8

REMARK. The sequence of local surgeries carried out by our algorithm corresponds to the one from the skein relation algorithm for computing the Conway-Jones polynomials, if after each surgery we additionally "unlink" the self-intersecting curve obtained: the local reduction of the Conway algorithm is the part of Figure 8 encircled by the dotted line. In this unlinking, part of the information is lost. When we are computing the simplest invariant (of filtration 2) then the lost information is unessential, because the value of the corresponding index of the curve before unlinking can be recovered from the unlinked curve.

0.4. The groups $E_1^{-i,i}$. For any i and any coefficient group G, the group $E_1^{-i,i}$ is naturally isomorphic to the kernel of the homomorphism $h_i\colon X_i \to Y_i$ defined below.

DEFINITION. A collection of $2i$ pairwise distinct points on the oriented line $\mathbb{R}^1$, arbitrarily partitioned into i pairs, will be called an $[i]$-*configuration*. An $\langle i\rangle$-*configuration* is a collection of $2i-1$ points partitioned into $i-2$ pairs and one triple. Two $[i]$- or $\langle i\rangle$-configurations are called *equivalent*, if they can be sent into each other by an orientation-preserving diffeomorphism of $\mathbb{R}^1$.

The group X_i is the direct sum of the groups X_i^0 and X_i^1, defined as free G-modules whose generators correspond bijectively to equivalence

classes of $[i]$-configurations (respectively, $\langle i\rangle$-configurations).

EXAMPLES. $X_1^1 = 0$. The group X_1^0 is isomorphic to G and is generated by the two-point configuration in $\mathbb{R}^1$. Further, $X_2 = X_2^0 \oplus X_2^1 = G^3 \oplus G$: the group X_2^1 is generated by any triple of points in $\mathbb{R}^1$, the generators of the group X_2^0 are shown on Figure 5.

To each $[i]$-configuration and $\langle i\rangle$-configuration corresponds an affine subspace of codimension $3i$ in K: this subspace consists of all maps $\mathbb{R}^1 \to \mathbb{R}^3$ which identify all the points within each pair and within the triple.

Now let us describe the group Y_i. It is defined as the direct sum of two groups: Y_i^1 and Y_i^*. The group Y_i^1 is isomorphic to the direct sum of three copies of the group X_i^1, the generator of Y_i^1 is an $\langle i\rangle$-configuration in which one of the points in the triple has been distinguished. Such an $\langle i\rangle$-configuration with distinguished point will be called an $\widetilde{\langle i\rangle}$-*configuration*. The group Y_i^* for $i \geqslant 2$ is isomorphic to the direct sum of $2i-1$ copies of the group X_{i-1}^0: the standard generator of the group Y_i^* is an $[i-1]$-configuration (considered up to equivalence) to which a "singular" point $*$ (not coinciding with any of the $2i-2$ points of the $[i-1]$-configuration) has been added. Such an augmented $[i-1]$-configuration will be called an i^*-*configuration*. The group Y_1^* is defined as the one-dimensional G-module with generator corresponding to the solitary point $* \in \mathbb{R}^1$.

EXAMPLES. $Y_2^* = G^3$. See Figure 9. $Y_2^1 = G^3$, all the generators of this group are shown in the top row of Figure 10.

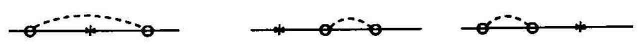

FIGURE 9

To each i^*-configuration corresponds a subspace of codimension $3i$ in K consisting of all maps $\mathbb{R}^1 \to \mathbb{R}^3$ identifying points within each pair (from the partitioning of the configuration) and possessing a singularity (vanishing derivative) at the point $*$.

The homomorphism $h_i\colon X_i \to Y_i$ is illustrated by Figure 10. On this figure (and subsequent ones) the triples (appearing in the definition of $\langle i\rangle$-configurations) are shown by three dotted curves joining the three points of the triple. If in such a triple a point has been distinguished (after which the $\langle i\rangle$-configuration becomes an $\widetilde{\langle i\rangle}$-configuration), the dotted curve joining the two undistinguished points must be deleted.

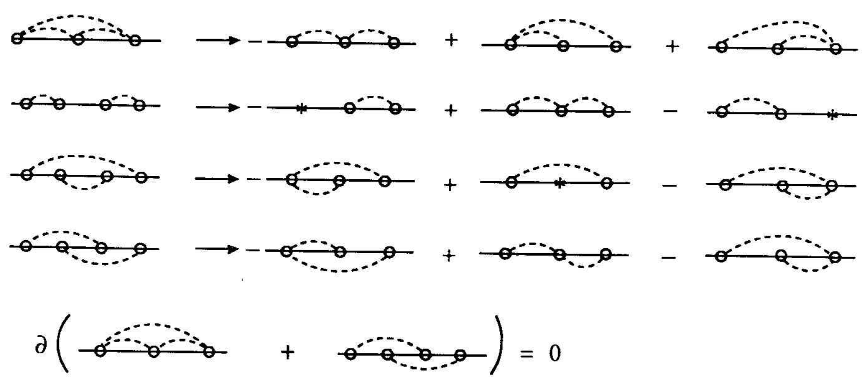

FIGURE 10

Let us give a formal definition of the homomorphism h_i. Suppose α is a generator of X_i^1, i.e., an $\langle i\rangle$-configuration. Its image $h_i(\alpha)$ is the linear combination of the three generators in Y_i^1 corresponding to the three $\widetilde{\langle i\rangle}$-configurations that geometrically coincide with α, where their coefficients (in the linear combination) are -1 (for the $\widetilde{\langle i\rangle}$-configuration in which the middle point of the triple is distinguished) and $+1$ for the other two.

Now suppose α is a generator in X_0^i, i.e. an $[i]$-configuration. Then $h_i(\alpha)$ is the linear combination of the $2i-1$ generators of the group Y_i corresponding to all finite segments of the line $\mathbb{R}^1$ bounded by points of the configuration α. Each such generator is represented by an i^*- or an $\widetilde{\langle i\rangle}$-configuration, consisting of $2i-1$ points and coinciding with the given configuration α outside the corresponding segment; the endpoints of this segment are replaced by a single point — the midpoint of the segment. If these endpoints belonged to the same pair in the configuration α, then the midpoint becomes the point $*$ and we get an i^*-configuration, i.e., a generator of the module Y_i^*. If the endpoints belong to different pairs, then these two pairs are replaced by one triple in which the new point (obtained by identifying the endpoints) is distinguished; this yields an $\widetilde{\langle i\rangle}$-configuration, i.e., a generator of Y_i^1. Each of these $2i-1$ generators appears in the linear combination $h_i(\alpha)$ with coefficient $(-1)^{i+m-1}$, where m is the number (in increasing order on $\mathbb{R}^1$) assigned to the "largest" (in that order) of the two identified points. Thus, the homomorphisms h_i: $X_i \to Y_i$ are entirely defined. For any $i \geqslant 1$ and any coefficient group, the group $E_1^{-i,i}$ from our spectral sequence is naturally isomorphic to the kernel of the homomorphism h_i.

EXAMPLES. Let $i = 1$. Then $X_1 \cong Y_1 \cong G$ and h_1 is an isomorphism — compare with the example at the end of §0.2.

Let $i = 2$. Then $X_2 = X_2^0 \oplus X_2^1 \cong G^3 \oplus G$; $Y_2 = Y_2^1 \oplus Y_2^* \cong G^3 \oplus G^3$. The action of the homomorphism h_2 is shown on Figure 10. It turns out in particular that the sum of the first and fourth generator of the group X_2 in this figure belongs to the kernel of h_2 (and so a knot invariant corresponds to this sum). This is precisely the second order invariant (considered above) that distinguishes the trefoil knot from the trivial one.

0.5. Values of basis invariants for mirror-image knots. The system of invariants corresponding to the basis elements of the graded group

$$\bigoplus_i E_\infty^{-i,i}$$

possesses a standard group of substitutions. For example, to any invariant of ith order we can add any invariants of lesser orders. By an appropriate use of such substitutions, we can insure the following amphichieral property of the invariants.

THEOREM. *Suppose the coefficient group of the spectral sequence possesses division by 2. Then all the invariants corresponding to all the terms $E_\infty^{-i,i}$ of our sequence may be chosen so as to assume identical values on mirror image knots for i even and opposite values for odd i.*

0.6. Plan of the paper. In §2 we construct the principal spectral sequence that computes the cohomology of the complements to the discriminant in the space $\Gamma^d \in K$ of finite-dimensional (polynomial) approximation of knot space. This spectral sequence stabilizes as $d \to \infty$. In §3 we prove that its terms $E_1^{p,q}$ (even nonstable ones) vanish when $p + q < 0$ and compute the term E_1 of the stable spectral sequence. In §4 we describe the algorithm for computing $E_r^{-i,i}$, $r = 2, 3, \dots, \infty$ of the stable sequence and for calculating the corresponding knot invariants, as well as the algorithm that computes the value of such an invariant (given by the output file of the first algorithm) for any tame knot. In §5 we present the first results of our computations: the five simplest invariants (of orders 2, 3 and 4) and their values on all prime knots possessing diagrams of $\leqslant 7$ overlaps.

One notation — the double colon (::) denotes the absence or end of the proof.

I am grateful to V. I. Arnold, A. B. Sossinsky, and A. G. Khovanski for useful discussions and interest in this work.

§1. Definitions and notations

1.1. Noncompact knots. A *knot* is an oriented C^1-submanifold of S^3 diffeomorphic to the circle. Two knots are called *equivalent* if they can be sent into each other by an orientation-preserving isotopy of S^3. The equivalence of knots in $\mathbb{R}^3$ is defined in a similar way.

1.1.1. PROPOSITION. *The obvious inclusion $\mathbb{R}^3 \to S^3$ induces a bijection between classes of equivalent knots in $\mathbb{R}^3$ and in S^3. Two knots in S^3 are equivalent if and only if the embeddings $S^1 \to S^3$ that determine them are homotopic in the class of embeddings (i.e., in the class of maps whose restrictions to S^1 are C^1-diffeomorphisms of S^1 onto its image).* ::

At the distinguished point $* \in S^3$, let us fix some tangent direction in S^3.

DEFINITION. A smooth map $S^1 \to S^3$ is said to be *normed* if it sends the distinguished point $* \in S^1$ (and only this point) to the distinguished point $* \in S^3$ and the derivative of this map at the distinguished point has the same direction as the fixed direction (in particular, it is nonzero). A knot is called *normed* if it is the image of a normed map.

The classification of normed knots is equivalent to the classification of all knots, and this follows from the following elementary statement.

1.1.2. LEMMA. *Any class of equivalent knots in S^3 contains a normed knot. Any two normed knots are equivalent if and only if the normed embeddings $S^1 \to S^3$ that determine them are homotopic in the class of normed embeddings.* ::

1.1.3. LEMMA. *The set of all normed maps $S^1 \to S^3$ is homotopically trivial (in particular, it can be contracted onto the identical embedding of the large circle corresponding to the distinguished direction at the distinguished point in S^3).* ::

The space of normed maps $S^1 \to S^3$ is denoted by K. The set of maps of the class K that are not embeddings is called the *discriminant* and is denoted by Σ.

Normed maps are conveniently represented as maps $\mathbb{R}^1 \to \mathbb{R}^3$ that have a fixed asymptotic direction at infinity. (Without loss of generality we can assume that this direction is given by the vector $(1, 1, 1)$ in the standard linear coordinates x, y, z in $\mathbb{R}^3$.) This representation motivates the following definition.

DEFINITION. By a *noncompact knot* we mean a regular submanifold in $\mathbb{R}^3$ diffeomorphic to $\mathbb{R}^1$ whose closure in the one-point compactification S^3 of the space $\mathbb{R}^3$ is a normed knot. Two noncompact knots are equivalent if the corresponding normed knots in S^3 are given by embeddings homotopic in the space of (normed) embeddings.

1.2. Approximation of knots. Here we will fix a system of finite-dimensional subspaces of the space K of normed maps such that any (noncompact) knot is equivalent to some knot determined by a point of one of these subspaces. Namely, we consider the space of all polynomials of the form

$$t^{d+1} + a_1 t^d + \cdots + a_d t \tag{1}$$

and suppose that $\tilde{\Gamma}^d$ is the space of maps $\mathbb{R}^1 \to \mathbb{R}^3$ given by three polynomials

$$(x, y, z)(t) = (P_1(t), P_2(t), P_3(t)) \tag{2}$$

where all the P_i are of the form (1).

Suppose d is even, then[1] $\tilde{\Gamma}^d$ is an affine subspace of the space K of normed maps. This subspace is not in general position with the discriminant. For example, it contains maps that have an uncountable number of self-intersections: such are all the maps of the form (2) for which $P_1 = P_2 = P_3$ is any nonmonotone polynomial of the form (1).

Let us move the space $\tilde{\Gamma}^d$ slightly so that it is in general position with respect to the discriminant. To do this, note that for any odd $w > 1$ the space $\tilde{\Gamma}^d$ can be viewed as a subspace of $\tilde{\Gamma}^{w(d+1)-1}$. The standard embedding

$$I_w : \tilde{\Gamma}^d \to \tilde{\Gamma}^{w(d+1)-1}$$

is given by the change of parameter $t = s^w + s$.

Set, for example, $w = 3$. By moving the affine subspace $I_3(\tilde{\Gamma}^d) \subset \tilde{\Gamma}^{3d+2}$ as little as we wish, we can put it into general position with respect to the discriminant $\Sigma \cap \tilde{\Gamma}^{3d+2}$. The subspace obtained by this move will be denoted by Γ^d. By the Weierstrass approximation theorem, any noncompact knot in $\mathbb{R}^3$ is equivalent to a knot given by a map of the class Γ^d for a sufficiently large d. We will consider further the cohomologies of the spaces $\Gamma^d \setminus \Sigma$ and their behavior as $d \to \infty$.

[1] For an appropriate choice of the identification $\mathbb{R}^1 \to S^1 \setminus \{*\}$.

Here is the first reduction. By the Alexander duality theorem

$$\tilde{H}^i(\Gamma^d \setminus \Sigma) \cong \bar{H}_{3d-i-1}(\Sigma \cap \Gamma^d), \qquad (3)$$

where $\tilde{H}^*$ is cohomology taken modulo the fundamental cocycle, $\bar{H}_*$ is closed homology, i.e., homology of the one-point compactification taken modulo the compactifying point.

Thus, we are led to the study of closed homology of the discriminant $\Sigma \cap \Gamma^d$.

§2. Construction of the spectral sequence

2.1. Configurations of singularities and self-intersections. For any even d, the discriminant $\Sigma \cap \Gamma^d$ is a stratified semialgebraic variety, consisting of different affine planes, which we will now describe.

2.1.1. Definitions. Suppose A is an arbitrary finite sequence of integers of the form $\{a_1 \geqslant a_2 \geqslant \cdots \geqslant a_g\}$, $a_g \geqslant 2$. Denote by $|A|$ the number Σa_i, by $\#A$ the number of terms a_j of this sequence.

By an *A-configuration* we will mean any family of $|A|$ pairwise distinct points on the line $\mathbb{R}^1$ partitioned into $\#A$ groups whose cardinalities are respectively $a_1, \dots, a_{\#A}$.

Now suppose b is a non-negative integer.

By a *(A, b)-configuration* we mean a pair consisting of an A-configuration and an additional family of b pairwise distinct points on $\mathbb{R}^1$; here we make no supplementary assumptions on the eventual coincidence of some of these b points with the points constituting the A-configuration.

Example. The $[i]$-configurations, the $\langle i \rangle$-configurations and the i^*-configurations defined in Subsection 0.4 are (in our present notations) (A, b)-configurations, where respectively

$$A = (\underbrace{2, \dots, 2}_{i}), \quad b = 0; \qquad A = (3, \underbrace{2, \dots, 2}_{i-2}), \quad b = 0;$$

$$A = (\underbrace{2, \dots, 2}_{i-1}), \quad b = 1$$

and all the $2i - 1$ points determining the configuration are distinct.

Two (A, b)-configurations are said to be *equivalent* (or belonging to the same class) if they can be sent into each other by an orientation-preserving diffeomorphism of the line $\mathbb{R}^1$.

The dimension of the space of configurations equivalent to the given one is equal to the number of geometrically distinct points involved in its definition, and in particular is no greater than the number $|A| + b$.

A smooth map $\theta: \mathbb{R}^1 \to \mathbb{R}^3$ is said to *respect* the configuration if it has singularities at all b distinguished points of this configuration and if all the points of each of the groups of cardinality $a_1, \dots, a_{\#A}$ are mapped into one point in $\mathbb{R}^3$. (In general, this point is a different one for each of the $\#A$ groups.)

The set of maps $\mathbb{R}^1 \to \mathbb{R}^3$ that respect the given configuration is an affine subspace in the space K (or in the space of all smooth maps); its codimension is equal to $3(|A| - \#A + b)$. As a rule, the intersection of this subspace with the space Γ^d defined in §1 has the same codimension in Γ^d.

For any (A, b)-configuration $\mathcal{J}$, denote by $\chi(\Gamma^d, \mathcal{J})$ the set of maps of the class Γ^d that respect $\mathcal{J}$.

2.1.2. Lemma. *For almost any choice of the space Γ^d, for any (A, b)-configuration $\mathcal{J}$, we have the following statements.*

(A) *For almost any (A, b)-configuration $\mathcal{J}'$ equivalent to $\mathcal{J}$, the set $\chi(\Gamma^d, \mathcal{J}')$ is an affine subspace of codimension $3(|A| - \#A + b)$ in Γ^d (in particular it is empty if $|A| - \#A + b > d$).*

(B) *Suppose $|A| - \#A + b = k \leqslant d$. Then in the set of all configurations equivalent to $\mathcal{J}$ the subset of those configurations $\mathcal{J}'$ for which $\chi(\Gamma^d, \mathcal{J}') = \varnothing$ is of codimension $\geqslant 3d - 3k + 1$, while the set of $\mathcal{J}'$ such that the codimension of $\chi(\Gamma^d, \mathcal{J}')$ in Γ^d equals $3k - i$, $i \geqslant 1$ is a subset of codimension $\geqslant i(3d - 3k + i + 1)$. In particular, when $k \leqslant (3d + 1)/5$, the codimension of any set $\chi(\Gamma^d, \mathcal{J})$ defined by any (A, b)-configuration $\mathcal{J}$ satisfying $|A| - \#A + b = k$ is exactly equal to $3k$.*

(C) *Suppose $|A| - \#A + b = k > d$. Then in the set of configurations equivalent to $\mathcal{J}$ the set of all $\mathcal{J}'$ such that $\dim \chi(\Gamma^d, \mathcal{J}') = l \geqslant 0$ is of codimension no less than $(l + 1)(3k - 3d + l)$. In particular, the set of all $\mathcal{J}'$ such that $\chi(\Gamma^d, \mathcal{J}') \neq 0$ is of codimension $\geqslant 3(k - d)$ and is empty when $k > 3d$.*

All this immediately follows from the (weak) transversality theorem. See, for example, [AVG].

Everywhere in the sequel we will assume that all the statements of this lemma hold for our space Γ^d.

2.2. Generating families.

2.2.1. Suppose we are given an arbitrary finite set $L \subset \mathbb{R}^1$. A *generating family* of the set L is by definition a nonordered family of pairwise distinct nonordered pairs of points

$$(t_1 \neq t_1'), \dots, (t_l \neq t_l'), \quad t_j \in \mathbb{R}^1, \quad t_j' \in \mathbb{R}^1$$

such that for all the maps $\theta: \mathbb{R}^1 \to \mathbb{R}^3$ the following statements are equivalent:

(a) the map θ identifies all the points of the set L,

(b) for any $j = 1, \dots, l$ the map θ identifies the points t_j and t_j'.

Obviously, a family of pairs $T = \{(t_1, t_1'), \dots, (t_l, t_l')\}$ is a generating family for L iff all the t_j, t_j' belong to L and for any two points of L there is a chain of pairs from the family T that joins them.

2.2.2. Definition. The set of pairs $T = \{(t_1, t_1'), \dots, (t_l, t_l')\}$ is said to be a *generating family* for the A-configuration $\mathcal{J}$, if it is the union of certain generating families for all the $\#A$ sets constituting the A-configuration.

Obviously the number of pairs constituting the generating families of a given A-configuration may vary from $\sum(a_i - 1) = |A| - \#A$ to $\sum a_i(a_i - 1)/2$; these two numbers coincide iff all the a_i are equal to 2.

2.2.3. A family (T, V), where $T = \{(t_1, t_1'), \dots, (t_l, t_l')\}$, while V is a certain set of points $v_1, \dots, v_b \in \mathbb{R}^1$, is said to be a *generating family* for the (A, b)-configuration $\mathcal{J}$, if the family T is a generating family for the corresponding A-configuration, while V is exactly the set of the b points that transform the A-configuration into a (A, b)-configuration. Two families (T, V), (T_1, V_1) are said to be *equivalent* if they may be sent into each other by a certain orientation-preserving diffeomorphism $\mathbb{R}^1 \to \mathbb{R}^1$. Obviously, equivalent families generate equivalent configurations; the converse is not true in general.

2.3. The resolvent of the discriminant. Here we will construct an auxiliary space σ whose closed homology coincides with that of $\Gamma^d \cap \Sigma$.

2.3.1. Denote by Ψ the set of nonordered pairs of points in $\mathbb{R}^1$ (which may coincide): $\Psi = \mathbb{R}^2/\delta$, where δ is the involution $(t, t') \to (t', t)$. If we remove the diagonal $t = t'$ from Ψ, we obtain a configuration space of subsets of cardinality 2 in $\mathbb{R}^1$, which we will denote by $\overset{\circ}{\Psi}$. We will call the map $\Psi \to \mathbb{R}^N$ *polynomial* if it is defined by a polynomial map $R^2 \to \mathbb{R}^N$, invariant with respect to the reflection in the diagonal $\{t = t'\}$.

2.3.2. Lemma. *There exists a polynomial embedding $\Psi \to \mathbb{R}^N$ (where N is sufficiently large) such that there are no sets of $2\binom{3d}{2} \equiv 3d(3d-1)$ pairwise distinct points belonging to Ψ that are mapped into one $(2\binom{3d}{2}-2)$-dimensional affine plane in $\mathbb{R}^N$.* ::

2.3.3. Fix such an embedding λ. For any (A, b)-configuration $\mathcal{J}$, consider the set of pairs $(\theta, (T, V))$, where θ is an arbitrary map of the class Γ^d respecting the configuration $\mathcal{J}$, while

$$(T, V) = \{(t_1, t_1'), \dots, (t_l, t_l'), v_1, \dots, v_b\}$$

is any generating family for this configuration. To any such pair assign the convex hull of the $l + b$ points in the space $\Gamma^d \times \mathbb{R}^N$ whose projection on Γ^d is the same point θ, while the projections on $\mathbb{R}^N$ are the points

$$\lambda(t_1, t_1'), \dots, \lambda(t_l, t_l'), \lambda(v_1, v_1), \dots, \lambda(v_b, v_b).$$

It follows from the last statement of Lemma 2.1.2 that the number $l + b$ of these points is no greater than $\binom{3d}{2}$, and therefore, this convex hull is a $l + b - 1$ dimensional simplex. Moreover, by the choice of the map λ, projections in $\mathbb{R}^N$ of two such simplices in $\Gamma^d \times \mathbb{R}^N$ corresponding to distinct pairs $(\theta, (T, V))$ have no common inner points.

Define the set $\sigma \subset \Gamma^d \times \mathbb{R}^N$ as the union of all such simplices over all pairs $(\theta, (T, V))$, all (A, b)-configurations $\mathcal{J}$, and all families (A, b).

These simplices will be called *standard simplices* of the space σ, and their projections into $\mathbb{R}^N$ standard simplices in $\mathbb{R}^N$.

Note that of course we obtain an equivalent definition of σ by considering only maximal standard simplices corresponding to the $(\sum\binom{a_i}{2} + b)$-element families (T, V).

2.3.4. Lemma. *The set σ is semialgebraic in $\Gamma^d \times \mathbb{R}^N$.*

This follows immediately from the constructions. ::

2.3.5. Theorem. *The obvious map $\sigma \to \Sigma \cap \Gamma^d$ induced by the projection $\Gamma^d \times \mathbb{R}^N \to \Gamma^d$ is proper and induces an isomorphism*

$$\bar{H}_*(\sigma) \to \bar{H}_*(\Sigma \cap \Gamma^d). \tag{4}$$

The proof is standard, compare [V1].

2.4. The filtration in the resolvent of the discriminant. The set σ can be fibered into affine planes in a natural way. Indeed, restrict the natural projection $\pi: \Gamma^d \times \mathbb{R}^N \to \mathbb{R}^N$ to σ. The inverse image of any point of the space $\mathbb{R}^N$ under this map is an affine space (maybe empty) whose projection into Γ^d is the subspace determined by certain conditions of the form "the map θ has a singularity at the point v" or "θ identifies the points t, t'." Any family of conditions of this form determines a subspace $\chi(\Gamma^d, \mathcal{J})$ in Γ^d for a certain (A, b)-configuration $\mathcal{J}$ (see 2.1).

Choose the increasing filtration $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_{3d} = \sigma$ on the set σ by defining each σ_i as the union of all subspaces in $\sigma \subset \Gamma^d \times \mathbb{R}^N$ whose projections on Γ^d are subspaces $\chi(\Gamma^d, J)$ for some (A, b)-configurations J with $|A| - \#A + b \leqslant i$.

Consider the homology spectral sequence $E^r_{p,q}(d)$ converging to the group $\overline{H}_*(\sigma)$ and generated by this filtration. Transform it into a cohomological sequence by renaming the term $E^r_{p,q}(d)$ as $E_r^{-p,\,3d-1-q}(d)$. This is the spectral sequence that was promised in the introduction; to distinguish it from other spectral sequences, we shall call it the *principal spectral sequence*. By Alexander duality in Γ^d, it converges to the group $H_*(\Gamma^d \setminus \Sigma)$.

By construction, this spectral sequence is contained in the region $\{p, q \mid p < 0\}$ and any of its terms $E_\infty^{p,q}$ can be nonzero only if $p+q \geqslant 0$.

2.5. Main properties of the principal spectral sequence.

THEOREM. *For any choice of space Γ^d satisfying the conditions of Lemma 2.1.2, we have:*

(A) *the term $E_1^{p,q}(d)$ of the principal spectral sequence that computes the cohomology of $\Gamma^d \setminus \Sigma$ with arbitrary coefficients may be nonzero only for the following conditions on p and q:*

$$p + q \geqslant 0, \tag{i}$$

$$p \geqslant -3d, \tag{ii}$$

$$p \leqslant -2; \tag{iii}$$

(B) *for $i \leqslant (3d+1)/5$ and arbitrary coefficient group, the group $E_1^{-i,i}(d)$ is isomorphic to the group* Kerh_i *described in Subsection 0.4; in particular, it does not depend on d;*

(C) *for any choice of an even number $d' > d$ and a space $\Gamma^{d'}$, which also satisfies the assumptions of Lemma 2.1.2, the corresponding principal spectral sequence $E_r^{p,q}(d')$ coincides with the sequence $E_r^{p,q}(d)$ for the following p, q, r:*

(i) *for $r = 1$ and p, q belonging to the "d-stable" region $\{p, q | p+q \geqslant 0, -p \leqslant (3d+1)/5\}$, see Figure 1;*

(ii) *for any $r > 1$ and p, q such that no differential from the nonstable domain acts into $E_{r'}^{p,q}$ for some $r' < r$.*

COROLLARY 1. *If we choose $\Gamma^d, \Gamma^{d'}$ in general position, we have*

$E_r^{-i,i}(d) = E_r^{-i,i}(d')$ *for all $r = 1, 2, \ldots, \infty$ and all even $d' > d \geqslant (5i-1)/3$.*

COROLLARY 2. *For all i, the homomorphism $\overline{H}_{3d-1}(\sigma_i) \to \overline{H}_{3d-1}(\sigma)$ induced by inclusion is injective.*

Condition (ii) of statement (A) of the theorem follows from Lemma 2.1.2, condition (iii) from the explicit form of the column $p = -1$ of the term $E_0^{p,q}$. Condition (i) and statement (B) of the theorem will be proved in §3. The statement (C) for $r = 1$ follows from the cellular structure of the spaces $\sigma_i - \sigma_{i-1}$, which will be exhibited in Subsection 3.2, while for the other r it can be proved similarly to Theorem 4.5 from [V1]. In the most important case $p + q = 0$, all the terms $E_r^{-q,q}$ can be computed by means of an explicit algorithm, which will be presented in §4 (and does not depend on d, as long as d is sufficiently large).

REMARK. In the statement of the theorem, we could have weakened the restrictions on $i \equiv -p$ and q. Namely, statement (B) of the theorem is valid for all $i \leqslant d-1$, while the d-stable region from statement (C) may be defined as the intersection of the region $p + q \geqslant 0$ with the union of the regions $p \geqslant -(3d+1)/5$ and $q < 3d + 4p$. In a similar way, Corollary 1 is valid for all $i \leqslant d - 1$.

2.6. The stable spectral sequence and stable invariants. Statement (C) of Theorem 2.5 allows us to define the stable spectral sequence $\{\underleftarrow{E}_r^{p,q}\}$: its term $\underleftarrow{E}_r^{p,q}$ is equal to the common term $E_r^{p,q}(d)$ of all spectral sequences corresponding to finite but sufficiently large d.

CONJECTURE. *For all p, q there exists a d_0 such that for all (even) $d > d_0$ and all r we have $E_r^{p,q}(d) = \underleftarrow{E}_r^{p,q}$.*

For example, Corollary 1 to the previous theorem says that this conjecture is true for $p + q = 0$.

The isomorphism $E_\infty^{-i,i}(d) \cong E_\infty^{-i,i}(d')$ from this corollary agrees with the homology map: for $d' > d \geqslant (3i+1)/5$ there is a canonical isomorphism

$$\overline{H}_{3d-1}(\sigma_i(d)) \cong \overline{H}_{3d'-1}(\sigma_i(d')). \tag{5}$$

This isomorphism agrees with Alexander duality. Suppose θ, θ' are equivalent noncompact knots in K and $\theta \in \Gamma^d \setminus \Sigma$, $\theta' \in \Gamma^{d'} \setminus \Sigma$. Then the elements of the groups (5) corresponding to each other by the isomorphism (5) assume the same values on these knots.

All this (including the construction of the isomorphism (5)) directly follows from the computational algorithm for the terms $E_r^{-i,i}$ of our spectral sequence. See §§3 and 4.

DEFINITION. By a *stable invariant of the i-th order* we mean any element of the group $\overline{H}_{3d-1}(\sigma_i(d))$, $d \geqslant (5i-1)/3$ whose image under the natural map of this group into $\overline{H}_{3d-1}(\sigma_i(d)-\sigma_{i-1}(d))$ is nontrivial. Two elements of such groups corresponding to different d determine the same invariant of the i-th order if they correspond to each other under the isomorphism (5).

§3. The term E_1 of the principal spectral sequence

By definition, the group $E_1^{-i,q}$ of the principal spectral sequence is isomorphic to the closed $(3d-1+i-q)$-dimensional homology group of the space $\sigma_i - \sigma_{i-1}$. Here we will begin the computation of these groups and, in particular, we will prove that $E_1^{-i,q} = 0$ when $q - i < 0$. For stable values of i (i.e., if $i \leqslant (3d+1)/5$), we introduce a CW-complex structure on the one-point compactification of the space $\sigma_i - \sigma_{i-1}$ and explicitly describe all the differentials of the corresponding chain complex.

3.1. Partition of the space $\sigma_i - \sigma_{i-1}$ according to configuration types. Suppose $\mathcal{J}$ is an arbitrary (A, b)-configuration. By a *$\mathcal{J}$-block* in σ we mean the union of the interior points of all possible standard simplices in $\Gamma^d \times \mathbb{R}^N$ such that the points $(t_1, t_1'), \dots, (t_l, t_l'), v_1, \dots, v_b$ involved in the definition of these simplices constitute a generating family of some (A, b)-configuration equivalent to $\mathcal{J}$. Obviously the space $\sigma_i - \sigma_{i-1}$ is the union of $\mathcal{J}$-blocks corresponding to all possible (A, b)-configura tions $\mathcal{J}$ such that $|A|-\#A+b=i$ (and blocks corresponding to nonequivalent configurations do not intersect).

3.1.1. DEFINITION. We shall say that an (A, b)-configuration is *non-complicated* if either $A = 2, \dots, 2, b = 0$ or $A = 3, 2, \dots, 2, b = 0$ or $A = 2, \dots, 2, b = 1$ and the last point v_1 of this configuration does not coincide with any of the $2\#A$ points constituting the corresponding A-configuration.

The union of all $\mathcal{J}$-blocks over all possible complicated (A, b)-configurations satisfying $|A| - \#A + b = i$ is a closed subset in $\sigma_i - \sigma_{i-1}$. Let us denote this union by S_i.

3.1.2. THEOREM. *For any i, we have $\overline{H}_k(S_i) = 0$ for $k \geqslant 3d-2$. In particular, if $k \geqslant 3d-2$, the quotient map*

$$\overline{H}_k(\sigma_i - \sigma_{i-1}) \to \overline{H}_k((\sigma_i - \sigma_{i-1}) - S_i) \qquad (6)$$

is surjective, while if $k \geqslant 3d-1$, it is an isomorphism.

For the proof see subsections 3.4–3.7 below.

It follows from this theorem that in order to compute the zero-dimensional cohomology of the space $\Gamma^d \setminus \Sigma$ (i.e., the $(3d-1)$-dimensional closed homology of σ), it suffices to consider only $\mathcal{J}$-blocks with non-complicated $\mathcal{J}$. By Lemma 2.1.2, the dimension of these blocks is no greater than $3d-1$; this immediately implies part (i) of statement (A) of Theorem 2.5. Statement (B) of this theorem directly follows from the form of these blocks; for the appropriate argument, see subsections 3.2, 3.3.

3.2. The cellular decomposition of the space $\sigma_i - \sigma_{i-1}$ for stable values of i.

3.2.1. Suppose $i \leqslant (3d+1)/5$. In this case, by Lemma 2.1.2, any $\mathcal{J}$-block in $\sigma_i - \sigma_{i-1}$ is the total space of a locally trivial fibration whose base is the set of all (A, b)-configurations equivalent to $\mathcal{J}$ and whose fiber is the product of the $3(d-i)$-dimensional affine space by the union of all the interior points of all possible standard simplices in $\mathbb{R}^N$ defined by all the generating families of the configuration $\mathcal{J}$. The base of this fibration is obviously homeomorphic to an open cell, and therefore its total space is homeomorphic to the Cartesian product of the three spaces indicated. Let us decompose the last factor into open cells corresponding to all possible generating families of $\mathcal{J}$ consisting of inner points of the corresponding standard simplices in $\mathbb{R}^N$. Such a decomposition of the factor induces a decomposition of the entire product, i.e., of our $\mathcal{J}$-block. This decomposition will be called the *canonical decomposition* of the $\mathcal{J}$-block.

By the *canonical decomposition* of the space $\sigma_i - \sigma_{i-1}$, we will mean its decomposition into all possible cells of the canonical decompositions of all the $\mathcal{J}$-blocks belonging to $\sigma_i - \sigma_{i-1}$.

3.2.2. PROPOSITION. *For $i \leqslant (3d+1)/5$, the decomposition of the one-point compactification of the space $\sigma_i - \sigma_{i-1}$ into the cells of the canonical decomposition and the added point induces the structure of a CW-complex in this compactification.* ::

3.2.3. Let us describe the corresponding chain complex with coefficients in $\mathbb{Z}_2$. Its generator, i.e., an arbitrary pair of the form "(A, b)-configuration $\mathcal{J}$ with $|A| - \#A + b = i$, considered up to equivalence; the generating family of this configuration" will be depicted by means of a family of $|a| + b$ points on the line (see Figure 11 and compare Figures

5, 9, and 10). The points belonging to the A-configuration are shown by little circles $\circ$, while the b singular points are shown by little stars $*$. Circles belonging to the same pair from the generating family are joined by a dotted line; some of the little stars can geometrically coincide with the little circles. The dimension of this generator is equal to the sum of four numbers: $3(d-i)-1$, b, the number of dotted arcs and the number $\rho(\mathscr{J})$ of geometrically distinct points belonging to the configuration (this same number is the dimension of the space of configurations equivalent to $\mathscr{J}$). This dimension may be greater than $3d-1$. For example, this is the case for the (A, b)-configuration with $A = 4$, $b = 0$ and generating family consisting of all 6 possible pairs of points taken from the four points that generate this configuration.

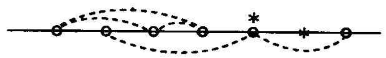

FIGURE 11

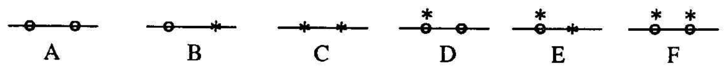

FIGURE 12

FIGURE 13

The differential of such a generator is the sum of similar generators obtained from the given one in the following two ways.

(1) The generators are contained in the same $\mathscr{J}$-block. Remove from the picture of our generator any dotted arc. If its endpoints are still joined by a chain of dotted arcs, then the generator depicted by the figure thus obtained is part of the boundary of the given generator with coefficient $1 \pmod 2$. For example, in Figure 11, we can remove any of the three arcs joining the first, third, and fourth points (counting from the left).

(2) The generators are obtained by a degeneracy of the given configuration. Consider any of the $\rho(\mathscr{J})-1$ line segments whose endpoints are neighboring points of the configuration $\mathscr{J}$. To each such segment (except the so-called *forbidden segments*, see below) assign the generator whose picture coincides with the given one outside of the segment; the segment itself must be contracted to a point that inherits all the arcs that joined the endpoints of the segment with other points of the configuration. The type of the point obtained (and the admissibility of the contraction) depends on the type of the segment's endpoints. There are six possible types, see Figure 12. Let's consider them.

A. The type A can be divided into five subtypes. A1: if both points are joined by arcs to some third point, then contraction is forbidden. A2: if the two points are joined together and neither of them is joined to any third point, then we obtain a point of type $*$. A3: if the points are joined together and at least one of them is joined to another point (but we do not have A1), then we obtain a point of type $\overset{*}{\circ}$. A4: if these two points are joined only by chains of $\geqslant 3$ arcs, then we obtain a point of type $\circ$. A5: if the points are not joined by any chains, then we also obtain $\circ$.

B. The point obtained is of type $\overset{*}{\circ}$.

C, E, F: contraction is forbidden.

D. If these two points are joined directly or by a chain of two arcs, then contraction is forbidden; if they are joined by a chain of $\geqslant 3$ arcs or are not joined at all, then the point obtained is of the form $\overset{*}{\circ}$.

All the generators of our complex obtained by the methods (1) or (2) appear in the boundary of the given cell with coefficient $1 \pmod 2$, all the other generators, with coefficient 0. Thus, we have entirely described the boundary operator of the $\mathbb{Z}_2$-chain complex corresponding to the canonical CW–complex structure on the one-point compactification of the space $\sigma_i - \sigma_{i-1}$. A similar integer coefficient complex will be described in Subsection 3.3.

3.2.4. EXAMPLE. Let us compute the group $\overline{H}_k(\sigma_i - (\sigma_{i-1} \cup S_i); \mathbb{Z}_2)$ involved in formula (6). It follows from the definition of S_i that this group coincides with the k-dimensional homology group of the chain (quotient) complex whose generators are the cells of the canonical decomposition of $\sigma_i - \sigma_{i-1}$ belonging to $\mathscr{J}$-blocks with noncomplicated $\mathscr{J}$, while the differentials are defined by the incidence coefficients of these cells. These cells belong to one of the following types.

1. If $\mathscr{J}$ is an (A, b)-configuration with $A = 2, \dots, 2$, $b = 0$, then the $\mathscr{J}$-block consists of a unique cell of dimension $3d-1$.

2. If $A = (2, \dots, 2)$, $b = 1$, then the $\mathscr{J}$-block consists of a unique cell of dimension $3d-2$.

3. If $A = (3, 2, \dots, 2)$, $b = 0$, then the $\mathscr{J}$-block consists of one cell of dimension $3d-1$ and three cells of dimension $3d-2$, and they correspond to the four possible generating families of the three-element set.

In Subsection 0.4, defining the homomorphism $h_i\colon X_i \to Y_i$, we introduced two families of generators of the group X_i: the $[i]$- and the $\langle i\rangle$-configurations, and two families of generators of Y_i: the i^*- and $\langle \tilde{i}\rangle$-configurations. In our present terminology, these are respectively cells of type 1, $(3d-1)$-dimensional cells of type 3, cells of type 2 and $(3d-2)$-dimensional cells of type 3. The homomorphism h_i is exactly the homomorphism determined by the incidence coefficients of these cells. Finally we obtain the following

PROPOSITION. *For $i \leqslant (3d+1)/5$, the group*

$$\overline{H}_k(\sigma_i - (\sigma_{i-1} \cup S_i); \mathbb{Z}_2)$$

is trivial when $k > 3d-1$, while if $k = 3d-1$, it is isomorphic to the kernel of the operator h_i.

Statement (B) of Theorem 2.5 in the $\mathbb{Z}_2$–case immediately follows from this proposition and from Theorem 3.1.2, which will be proved in Subsections 3.4–3.7 below.

3.3. The integer coefficient spectral sequence. The chain complex described in Subsection 3.2 computes the column $\underset{\leftarrow}{E}^{-i,*}$ of the stable spectral sequence with coefficients in $\mathbb{Z}_2$. Now we will describe a similar complex for the similar spectral sequence with integer coefficients. This complex has the same system of generators (corresponding to the cells of the canonical cell decomposition of $\sigma_i - \sigma_{i-1}$), but these generators are additionally supplied with orientations and their incidence coefficients can equal ± 1 depending on the choice of these orientations. Let us fix such a system of orientations.

3.3.1. Consider an arbitrary cell of the canonical decomposition of $\sigma_i - \sigma_{i-1}$. It is defined by (a) some (A, b)-configuration

$$\mathcal{J} = \{M_1, \dots, M_{\#A}, v_1, \dots, v_b\},$$

where each of the sets M_j is a subset of cardinality a_j in $\mathbb{R}^1$: $M_j = \{t_{j,1} < \cdots < t_{j,a_j}\}$ and (b) by some generating family

$$(T, V) = ((t_1, t_1'), \dots, (t_l, t_l'), v_1, \dots, v_b)$$

of the configuration $\mathcal{J}$.

An orientation of a cell consists of

(a) an orientation of the simplex in $\mathbb{R}^N$ spanning the points

$$\lambda(t_1, t_1'), \dots, \lambda(t_l, t_l'), \lambda(v_1, v_1), \dots, \lambda(v_b, v_b); \qquad (7)$$

(b) an orientation of the set of configurations equivalent to $\mathcal{J}$;

(c) an orientation of the subspace $\chi(\Gamma^d, \mathcal{J}) \subset \Gamma^d$ consisting of maps that respect $\mathcal{J}$.

In order to fix these orientations, let us order the sets M_j in accordance to decreasing numbers a_j and, for equal numbers a_j, in accordance to the increase of their smallest elements $t_{j,1}$.

3.3.2. An orientation of the simplex is an ordering of its vertices, i.e., of the points in (7). Let's order these points in the following way: first we take all the points $\lambda(t_{1,\alpha}; t_{1,\beta})$ in accordance to the lexicographic order of the pairs $(t_{1,\alpha}; t_{1,\beta})$ of the family T generating the set M_1: the pair $(t_{1,\alpha}; t_{1,\beta})$ appears earlier than $(t_{1,\gamma}; t_{1,\delta})$ if either

$$\min(t_{1,\alpha}; t_{1,\beta}) < \min(t_{1,\gamma}, t_{1,\delta})$$

or these two numbers are equal, but

$$t_{1,\alpha} + t_{1,\beta} < t_{1,\gamma} + t_{1,\delta}.$$

Then similarly ordered pairs from the generating family for $M_2, \dots, M_{\#A}$ follow and finally come the points $\lambda(v_j, v_j)$ in increasing order of v_j.

3.3.3. To orient the set of configurations equivalent to $\mathcal{J}$, we write the $\rho(\mathcal{J})$ points defining this configuration in increasing order. Then we choose a positive tangent coordinate system at the point $\mathcal{J}$ of this space in the following way. The first axis is the tangent direction to the line consisting of all configurations for which all the $\rho(\mathcal{J})$ points except the first one are the same as in $\mathcal{J}$, while the first increases. The second axis is defined by the increase of the next point in order, etc.

3.3.4. Suppose x, y, z are linear coordinates in $\mathbb{R}^3$. The configuration $\mathcal{J}$ defines $3(|A| - \#A + b)$ linear functions on the space Γ^d: on any map $\theta\colon \mathbb{R}^1 \to \mathbb{R}^3$, defined by the rule $\theta(t) = (x(t), y(t), z(t))$, these functions assume the values

$$\left.\begin{array}{c}
x(t_{1,2}) - x(t_{1,1}),\ y(t_{1,2}) - y(t_{1,1}),\ z(t_{1,2}) - z(t_{1,1}),\\
x(t_{1,3}) - x(t_{1,2}),\ y(t_{1,3}) - y(t_{1,2}),\ z(t_{1,3}) - z(t_{1,2}),\\
\cdots\\
x(t_{1,a_1}) - x(t_{1,a_1-1}),\ y(t_{1,a_1}) - y(t_{1,a_1-1}),\ z(t_{1,a_1}) - z(t_{1,a_1-1}),\\
x(t_{2,2}) - x(t_{2,1}), \dots\\
\cdots\\
x(t_{\#A,a_{\#A}}) - x(t_{\#A,a_{\#A}-1}), \dots\\
\left.\frac{\partial x}{\partial t}\right|_{v_1},\ \left.\frac{\partial y}{\partial t}\right|_{v_1},\ \left.\frac{\partial z}{\partial t}\right|_{v_1},\\
\cdots\\
\left.\frac{\partial x}{\partial t}\right|_{v_b},\ \left.\frac{\partial y}{\partial t}\right|_{v_b},\ \left.\frac{\partial z}{\partial t}\right|_{v_b}.
\end{array}\right\} \qquad (8)$$

If the number $i = |A|-\#A+b$ is in the stable range, $i \leqslant (3d+1)/5$, then these functions are linearly independent. The space $\chi(\Gamma^d, \mathcal{J})$ consisting of maps of the class Γ^d that respect the configuration $\mathcal{J}$ is exactly the set of zeros of all these functions. Choose a co-orientation of $\chi(\Gamma^d, \mathcal{J})$ in Γ^d (i.e., an orientation of its orthogonal complement) by means of an exterior $3i$-form equal to the exterior product of the linear forms (8), taken in the order indicated here:

$$d(x(t_{1,2}) - x(t_{1,1})) \wedge \cdots \wedge d\left(\frac{\partial z}{\partial t}\Big|_{v_b}\right).$$

Assume that some orientation is fixed in Γ^d; then there is a unique orientation of $\chi(\Gamma^d, \mathcal{J})$ such that any coordinate system in Γ^d will be positively oriented if its first $3i$ vectors are orthogonal to $\chi(\Gamma^d, \mathcal{J})$ and will define the coorientation determined above, while the last $3(d-i)$ vectors are tangent to $\chi(\Gamma^d, \mathcal{J})$ and are positive with respect to the orientation we are defining. The exact form of the orientation Γ^d involved in this construction is not important for us: if it is changed, then the orientations of all the cells change simultaneously and therefore their incidence coefficients do not change. Let us describe these incidence coefficients.

3.3.5. Suppose two cells of the canonical decomposition of $\sigma_i - \sigma_{i-1}$ belong to the same $\mathcal{J}$-block, while the generating family in the definition of one of them is obtained from the generating family of the other by throwing out one pair. Then the incidence coefficient of these cells is equal to $(-1)^{j-1}$, where j is the number of the pair that we have thrown out in the enumeration defined in Subsection 3.3.2.

3.3.6. If the smaller one of two incident cells is obtained from the larger one by a degeneracy (see 3.2.3), then the incidence coefficient depends of the type of this degeneracy. In the cases A2 and A3 from 3.2.3, this coefficient equals $\zeta(-1)^{\mu-\nu+\Delta+m}$, where μ is the number in the list (7) of vertices of the simplex for the given cell assigned to the point $\lambda(t_j, t_j')$ corresponding to the contracted pair (t_j, t_j'), ν is the number of the newborn point $\lambda(v_k, v_k)$ in a similar list for the configuration obtained, Δ is the dimension of these simplices for both cells (i.e., the number $l+b-1$ from formula (7)), m is the number of the largest of the points pasted together (among all the points that are part of the given configuration, see 3.3.3); $\xi = \pm 1$ is the coefficient responsible for the compatibility of the orientation of the spaces $\chi(\Gamma^d, \cdot)$ for the given configuration and the one obtained.

In the cases A4 and A5, the incidence coefficient is equal to $\varepsilon\xi(-1)^{\Delta+m}$, where Δ, m, ζ are the same as before; $\varepsilon = \pm 1$ is the parity of the renumbering of dotted arcs corresponding to each other before and after the contraction. For example, the contractions shown on Figure 13a correspond to $\varepsilon = \xi = -1$, the contraction on Figure 13b to $\varepsilon = \xi = 1$. See the example below. In general, if the given configuration is of type $(A, b) = ((2, \dots, 2), 0)$, then we always have $\varepsilon = \xi$.

In case B, the incidence coefficient is equal to $(-1)^{\Delta+m}$.

In the nonforbidden version of case D, the rule is the same as in cases A4 and A5.

EXAMPLE. For $i = 2$, the set $\sigma_i - \sigma_{i-1}$ has no cells of dimension $\geqslant 3d$, while the incidence coefficients of the cells of dimensions $3d-1$, $3d-2$ are shown on Figure 10.

The remaining part of this section is the proof of Theorem 3.1.2.

Once again assume that i is an arbitrary natural number.

3.4. The auxiliary filtration in the set $\sigma_i - \sigma_{i-1}$. Let us recall the notation $\rho(\mathcal{J})$ for the dimension of the space of configurations equivalent to $\mathcal{J}$.

On the set $\sigma_i - \sigma_{i-1}$, define the *auxiliary filtration*

$$\varnothing = F_{[i/2]} \subset F_{[i/2]+1} \subset \cdots \subset F_{2i} \equiv \sigma_i - \sigma_{i-1}$$

by taking its term F_α to be the union of all $\mathcal{J}$-blocks such that $\rho(\mathcal{J}) \leqslant \alpha$.

3.4.1. LEMMA. *The subset $S_i \subset \sigma_i - \sigma_{i-1}$ defined in Subsection 3.1 is exactly the term F_{2i-2} of the auxiliary filtration.* ::

Consider the *auxiliary spectral sequence* $\mathcal{E}^\gamma_{\alpha,\beta}$ converging to the group $\overline{H}_*(\sigma_i - \sigma_{i-1})$ and generated by our auxiliary filtration. By definition, its term $\mathcal{E}^1_{\alpha,\beta}$ is isomorphic to the group $\overline{H}_{\alpha+\beta}(F_\alpha - F_{\alpha-1})$.

3.4.2. LEMMA. *Any connected component of the subset $F_\alpha - F_{\alpha-1}$ coincides with a $\mathcal{J}$-block, where $\mathcal{J}$ is an (A, b)-configuration such that $|A|-\#A+b = i$, $\rho(\mathcal{J}) = \alpha$. In particular, the group $\mathcal{E}^1_{\alpha,\beta}$ splits into the direct sum of groups $\overline{H}_{\alpha+\beta}$ (the $\mathcal{J}$-block) over all such (A, b)-configurations $\mathcal{J}$ (one taken from each of the equivalence classes).* ::

To prove Theorem 3.1.2, it remains to show that if the configuration $\mathcal{J}$ is complicated, then $\overline{H}_k(\mathcal{J}\text{-block}) = 0$ for $k \geqslant 3d-2$.

3.5. The fibration structure on $\mathcal{J}$-blocks. By construction, any $\mathcal{J}$-block is the total space of the locally trivial fibration whose base is the set of all pairs of the form "(A, b)-configuration equivalent to $\mathcal{J}$; map

$\theta \in \Gamma^d$ respecting this configuration", while a typical fiber is the union of the inner points of all possible standard simplices in $\mathbb{R}^N$ determined by all the generating families of the configuration $\mathcal{J}$. It immediately follows from Lemma 2.1.2 that the dimension of the base of this fibration is no greater than $3d - 3i + \rho(\mathcal{J})$. If the configuration $\mathcal{J}$ is complicated, then the last number is no greater than $3d - i - 2$ and it remains to prove that the one-point compactification of the fiber of the fibration is acyclic in dimensions greater than $i - 1$. Let us study the fiber in more detail.

Obviously the dimension of this fiber is equal to $\sum \binom{a_i}{2} + b - 1$. A dense set within the fiber is the open simplex spanning all possible points $\lambda(t, \tau)$, $\lambda(v, v)$ such that the points t, τ are contained in one of the $\#A$ groups of points determining the A-part of the configuration $\mathcal{J}$, while v is one of the points determining its b-part. Besides this open simplex, the fiber contains some of its (nonclosed) faces. Enumeration of these faces leads to the following formalism.

3.6. The generating complex of (A, b)-configurations.

3.6.1. To each set M of cardinality $a < \infty$ corresponds its generating complex $K(M)$ defined in the following way.

Consider a simplex of dimension $\binom{a}{2} - 1$ whose vertices correspond to two-element subsets of our set M. A face of this simplex is called *generating* if any two elements of the set can be joined by a chain consisting of pairs corresponding to vertices of this face. Obviously the dimension of the generating face may vary from $a - 2$ to $\binom{a}{2} - 1$.

The obvious triangulation of our simplex determines an (acyclic) chain complex of dimension $\binom{a}{2} - 1$; the *generating complex* $K(M)$ is defined as its quotient complex by the subcomplex spanning all possible nongenerating faces.

EXAMPLE. Suppose M consists of three elements κ, λ, μ. Then the complex $K(M)$ consists of one nonclosed triangle whose vertices are the pairs (κ, λ), (λ, μ), (μ, κ) and of three open intervals: its sides.

Now suppose we are given an (A, b)-configuration $\mathcal{J}$. By its *generating complex* we mean the join of $\#A$ generating complexes corresponding to all the sets constituting its A-configuration and b acyclic (in dimensions greater than zero) complexes corresponding to its b singular points. Obviously the generating complexes of all the (A, b)-configurations with identical A, b are isomorphic to each other.

3.6.2. TAUTOLOGICAL LEMMA. *For any (A, b)-configuration $\mathcal{J}$, the closed homology group of the fiber from the fibration given in 3.5 is isomorphic to the homology group of the generating complex of the (A, b)-configurations.*

It remains to prove that the generating complex of any complicated (A, b)-configuration is acyclic in dimensions greater than $i - 1$. But $i = |A| - \#A + b$ and therefore (using the formula for the homology of joins) it suffices to prove the following statement.

3.6.3. THEOREM. *For any set M with a elements, we have $H_l(K(M)) = 0$ when $l > a - 2$.*

3.7. Proof of Theorem 3.6.3. Suppose $\Sigma(M)$ is the subcomplex of the simplex constituted by all its nongenerating faces. Since this simplex is acyclic, we have $H_l(K(M)) \cong H_{l-1}(\Sigma(M))$, and it remains to prove that the complex $\Sigma(M)$ is acyclic in dimensions larger than $a - 3$.

Each generator of this complex (i.e., each nongenerating face of the simplex) determines a nontrivial[2] partition of the set M: in one of the subsets of this partition we put all the points of M that can be joined by chains of pairs corresponding to vertices of this face.

3.7.1. DEFINITIONS. By a *maximal partition* of the set M, we mean any of its partitions into two parts. The partition M is said to be *subjected* to another partition, if each part of the first partition belongs to some part of the second one. The *intersection* of several partitions is, by definition, the partition that is subjected to all of them and in its turn is such that any other partition subjected to all of them is subjected to it.

3.7.2. LEMMA. *Any partition of the set M is the intersection of several maximal partitions. Any strictly decreasing (in the sense of being subjected) chain of nontrivial partitions of the set M contains no more than $a - 2$ elements.* ::

To any nontrivial partition I of the set M corresponds a subcomplex $\Sigma(I) \subset \Sigma(M)$. Namely, consider the set of all two-element subsets of the set M belonging to some part or other of the partition I. The *generators* of the subcomplex $\Sigma(I)$ are by definition all the faces of a $\binom{a}{2} - 1$ dimensional simplex whose vertices correspond to such pairs.

3.7.3. LEMMA. *For any nontrivial partition I, the complex $\Sigma(I)$ is acyclic: $H_i(\Sigma(I)) = G$ for $i = 0$ and $= 0$ for $i > 0$.* ::

Indeed, the generators of this complex correspond to all possible faces of a certain face of the original simplex. ::

[2]The *trivial* partitions of M are the partitions into the set M itself or into all its separate points.

3.7.4. LEMMA. *The complex $\Sigma(M)$ is the union of its subcomplexes $\Sigma(I)$. The intersection of any number of subcomplexes $\Sigma(I_\alpha)$, $\alpha \in \mathfrak{A}$ is the subcomplex $\Sigma(\bigcap I_\alpha)$, if the partition $\bigcap I_\alpha$ is nontrivial, and the empty complex if it is trivial.* ::

Theorem 3.6.3 now follows from Lemmas 3.7.2–3.7.4 and the following homological lemma.

3.7.5. LEMMA. *Suppose a finite simplicial complex A is the union of a certain family of subcomplexes A_α such that*

(a) *all the complexes A_α are acyclic: $H_i(A_\alpha) = G$ for $i = 0$ and $= 0$ for $i > 0$;*

(b) *the intersection of any number of complexes A_α is also an element of the family A_α, or is empty;*

(c) *any sequence of strictly decreasing subcomplexes $A_{\alpha_1} \supset A_{\alpha_2} \supset \dots$ contains no more than l nonempty terms.*

Then the complex A is acyclic in dimensions greater than $l - 1$.

This is "a statement well known to the experts." This concludes the proof of Theorems 3.6.3 and 3.1.2, and therefore it also proves statement (A) of Theorem 2.5.

NOTE. In fact, the following strengthening of Theorem 3.6.3 holds: the complex $K(M)$ is acyclic in dimensions not equal to $a - 2$, and $H_{a-2}(K(M)) = G^{(a-1)!}$. For the proof, see [V4].

§4. Algorithms for computing the invariants and their values

In this section we describe two algorithms, ready for computer implementation.

The first algorithm computes stable invariants of i-th order for any $i > 0$ (in particular, the terms $\underset{\leftarrow}{E}_r^{-i,i}$, $r = 2, 3, \dots, \infty$ of the principal spectral sequence converging to the cohomology of the space $\Gamma^d \setminus \Sigma$, $d \geqslant (5i - 1)/3$).

The second algorithm computes, for any such invariant (given by the corresponding family of output data of the first algorithm) its value on an arbitrary noncompact knot $\theta \in K$, coded by means of a finite diagram (see [CF], [L],[FC]).

4.1. The truncated spectral sequence. For any $i > 0$ and any coefficient group G, let us define the groups $X_i = X_i(G)$ and $Y_i = Y_i(G)$ as free G-modules whose generators are the $[i]$- and $\langle i \rangle$-configurations (respectively the $\widetilde{\langle i \rangle}$- and i^*-configurations) introduced in Subsection 0.4. Cells of the canonical decomposition of $\sigma_i - \sigma_{i-1}$ correspond to all these configurations; the incidence coefficients of these cells, described in Subsection 3.3, define the homomorphism

$$h_i = h_i(G) : X_i \to Y_i.$$

By Theorem 3.1.2, the terms $\underset{\leftarrow}{E}_1^{-i,i}$ and $\underset{\leftarrow}{E}_1^{-i,i+1}$ of the principal stable spectral sequence are isomorphic respectively to the kernel of the homomorphism h_i and to a certain subgroup of the group $Y_i/h_i(X_i)$.

Replace by zero all the terms $\underset{\leftarrow}{E}_r^{p,q}$ of the stable spectral sequence such that $p + q \geqslant 2$. The *truncated spectral sequence* $\widetilde{E}_r^{p,q}$, thus obtained, possesses the following properties:

(i) $\widetilde{E}_r^{p,q} = 0$ for $p \geqslant 0$ or $p+q \notin \{0, 1\}$; $\widetilde{E}_0^{-i,i} \equiv X_i$, $\widetilde{E}_0^{-i,i+1} \equiv Y_i$;

(ii) for all $i, r \geqslant 1$, the term $\underset{\leftarrow}{E}_r^{-i,i}$ of the stable spectral sequence is canonically isomorphic to the corresponding term of the truncated one, while the term $\underset{\leftarrow}{E}_r^{-i,i+1}$ is canonically isomorphic to a certain subgroup of the group $\widetilde{E}_r^{-i,i+1}$.

Further we describe the algorithm for computing precisely this truncated spectral sequence $\widetilde{E}_r^{p,q}$.

4.2. Actuality table. Any ith order invariant is described by the *actuality table* defined below: this table is filled in for each (basis) element of the group $\widetilde{E}_1^{-i,i}$ in the process of computing the spectral sequence (the filling out procedure may stop if the next differential of this element turns out to be nontrivial). Let us describe this table.

Before they are filled in, actuality tables are identical for all invariants of the ith order. Each table has i levels. The lth level consists of $2l!/2^l l!$ spaces, which are in one-to-one correspondence with all equivalence classes of $[l]$-configurations. In each such space we put a picture (or a code) representing some immersed curve in $\mathbb{R}^3$, respecting the corresponding $[l]$-configuration and coinciding outside of some compact set with the standard linearly embedded line and not possessing any "supplementary" self-intersections except the l intersections that must appear in this configuration; we also require the tangents to the branches of the configuration at each of its l self-intersection points not to be colinear. For example, the first level consists of one space: it may contain the curve from Figure 4. The second level contains three spaces and the curves from Figure 6 may be drawn in them. In the general case, the choice of these

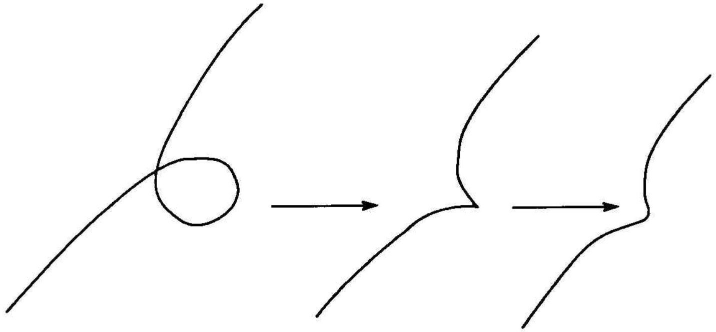

FIGURE 14

curves are left to the programmer.[3]

The table is filled in from top to bottom beginning with the ith level. In the process each space is filled in by a number (or, more generally, by an element of the coefficient group of the cohomology under consideration). This number is called the *actuality index* of the corresponding self-intersecting curve.

For a space of the upper level, these indices are defined in the following way. The given element of the group $\tilde{E}_1^{-i,i}$ for which we are constructing the actuality table is a formal linear combination of $[i]$- and $\langle i \rangle$-configurations (see Subsection 0.4). The coefficient with which some $[i]$-configuration appears in this linear combination is exactly the actuality index placed in the corresponding column of the ith level.

The spaces of the $(i-1)$th level are filled in when we compute the differential $d_1 : E_1 \to E_1$ of the truncated spectral sequence, the spaces of the $(i-2)$th level when we compute d_2, etc. This process will be described in Subsection 4.4, but now we will show how to use the table.

4.3. The algorithm for computing the values of the invariants.

4.3.1. *Coorientation of the discriminant.* The hypersurface $\Sigma \subset \Gamma^d$ in its nonsingular points has the standard transversal orientation, which will now be described. A nonsingular point $\theta \in \Sigma$ is a map $\mathbb{R}^1 \to \mathbb{R}^3$ that identifies certain points $t < t' \in \mathbb{R}^1$, and the derivatives of θ at these

[3]With one exception. Assume that some pair of points determining our configuration is not divided by points of other pairs. Then the loop in $\mathbb{R}^3$ whose end points are the points of this pair must not be linked with other parts of the curve (our curve can be continuously moved in such a way that all the intermediate curves respect certain $[l]$-configurations equivalent to the given one and have no extra self-intersections, and at the end of this move the self-intersection corresponding to our pair of points degenerates into a singular point, see Figure 14). All the pictures in Figures 4 and 6 satisfy this requirement.

points are not colinear. See Figure 4. For any point $\theta_1 \in \Gamma^d$ close to the point θ define the number $w(\theta_1)$ as the determinant of the triple of vectors

$$\left.\frac{\partial \theta_1}{\partial \tau}\right|_t, \left.\frac{\partial \theta_1}{\partial \tau}\right|_{t'}, (\theta_1(t') - \theta_1(t))$$

in the standard coordinates x, y, z. Obviously $w(\theta) = 0$ and the hypersurfaces Σ and $\{\theta_1 | w(\theta_1) = 0\}$ have the same tangent plane at the point θ. The vector issuing from the point θ and transversal to the set Σ is called *positive* if the derivative of the function $w(\cdot)$ along this vector is positive and *negative* if it is negative.

This rule determines the co-orientation of the hypersurface Σ at all its nonsingular points and moreover the coorientation of any nonsingular local branch of Σ at points of self-intersection of Σ.

This co-orientation is related in the following way to the orientation of the manifold σ_1 defined in Subsection 3.3. Suppose we have fixed an arbitrary orientation of the space Γ^d; then we can orient the surface Σ near any of its nonsingular points as the boundary of the positive (with respect to our coorientation) part of its complement. But, near nonsingular points, the projection $\sigma_1 \to \Sigma$ is a local diffeomorphism and therefore gives an orientation of σ_1. This orientation always coincides with the orientation defined in Subsections 3.3.3 and 3.3.4 (which, incidentally, also depended on the choice of the orientation of Γ^d).

4.3.2. Suppose we are given a noncompact knot $\theta : \mathbb{R}^1 \to \mathbb{R}^3$, $\theta \in \Gamma^d$, where d is sufficiently large, and an invariant of the ith order γ determined by its actuality table. The computation of the values of the invariant γ on the knot θ consists of i steps.

First step. Join the trivial embedding $\mathbb{R}^1 \to \mathbb{R}^3$ to the knot θ by a curve in general position in the space of normed immersions $\mathbb{R}^1 \to \mathbb{R}^3$ (i.e., immersions with fixed asymptotic behavior at infinity; see §1). This curve may have several transversal intersections with the discriminant Σ at nonsingular points of the latter. These intersection points $\theta_1, \dots, \theta_s$ are immersions $\mathbb{R}^1 \to \mathbb{R}^3$ possessing precisely one self-intersection. The value of the invariant γ on the knot θ is equal to the sum of the actuality indices of these points θ_j (the computation of these indices will be described in all the succeeding steps of the algorithm) taken with coefficients equal to $+1$ or -1 depending on whether Σ is intersected in the positive or in the negative direction.

Second step. Let us begin to compute the actuality index of each of the points θ_j. By reparametrization of $\mathbb{R}^1$ we can assume that the map θ_j

identifies the same pair of points t_1, t'_1 as the etalon map shown on the first level of the actuality table. For this etalon map the actuality index is already known (and is written in the table); if the etalon map satisfies requirements from the previous footnote (for example, if it is the curve from Figure 4), then this index is equal to 0. Suppose $K(t_1, t'_1)$ is the space of all normed maps $\mathbb{R}^1 \to \mathbb{R}^3$ identifying the points t_1, t'_1. Join the etalon map with the map θ_j by an arbitrary curve in general position in space $K(t_1, t'_1)$. This curve contains several maps $\theta_{j,1}, \dots, \theta_{j,s_j}$, each of which identifies, besides the points t_1, t'_1, some other pair of points t_2, t'_2 that does not coincide with t_1 and t'_1. The required actuality index of θ_j is equal to the index of the etalon map plus the sum of indices of the maps $\theta_{j,1}, \dots, \theta_{j,s_j}$ (the latter indices are determined in subsequent steps of the algorithm) taken with coefficients $\varepsilon \cdot (-1)^{n(t_2)+n(t'_2)}$, where ε is equal to $+1$ or -1 depending on whether the discriminant is intersected in the positive or in the negative direction, while $n(t_2)$ and $n(t'_2)$ are the numbers of the points t_2, t'_2 in the family $\{t_1, t'_1, t_2, t'_2\}$ written out in increasing order.

At the lth step of the algorithm $(l < i)$, we must determine the actuality indices of a certain family of maps $\theta_{j_1, \dots, j_{l-1}}$, which identify $l-1$ nonintersecting pairs of points in $\mathbb{R}^1$ each. To do this, we find (on the $(l-1)$th level of the actuality table) the space corresponding to the $[l-1]$-configuration respected by the map $\theta_{j_1, \dots, j_{l-1}}$. Reparametrize the line $\mathbb{R}^1$ so that this map and the map shown in this space respect one and the same configuration. The subsequent construction is the same as before: we take a homotopy of the etalon immersed curve into $\theta_{j_1, \dots, j_{l-1}}$ in the space $K(t_1, t'_1) \cap \cdots \cap K(t_{l-1}, t'_{l-1})$, take the actuality indices of the points $\theta_{j_1, \dots, j_l}$ met along the way with coefficients $\varepsilon \cdot (-1)^{n(t_l)+n(t'_l)+l}$, where ε is determined as before, while $n(t_l), n(t'_l)$ are the numbers of the points t_l, t'_l as they appear in the sequence $\{t_1, t'_1, \dots, t'_{l-1}, t_l, t'_l\}$ written in increasing order.

Finally, the actuality index computed at the last ith step of the process for any map $\theta_{j_1, \dots, j_i}$ is equal to the index written in the table on the ith level in the space corresponding to the $[i]$-configuration respected by this map. (This is all in accordance with the general outline described previously, if we assume that the table has an $(i+1)$th level that is filled by actuality indices equal to zero.)

To fill in the actuality table we will need the following object.

4.4. The extended actuality table. This table looks the same as the main one described in Subsection 4.2, but contains supplementary spaces corresponding to all possible $\langle l \rangle$-configurations, $l \leqslant i$; in any such space, before the table is filled in, we fix an immersed curve in $\mathbb{R}^3$ that respects the corresponding $\langle l \rangle$-configuration and has no extra self-intersection; here at every self-intersection point (at triple points as well as double points) the tangent directions to its branches must be linearly independent in $\mathbb{R}^3$; see Figure 15. Moreover, at each triple point where the corresponding map θ identifies the points $t < t' < t''$, the frame

$$\left(\left.\frac{\partial \theta}{\partial \tau}\right|_t, \left.\frac{\partial \theta}{\partial \tau}\right|_{t'}, \left.\frac{\partial \theta}{\partial \tau}\right|_{t''} \right)$$

must be positively oriented.

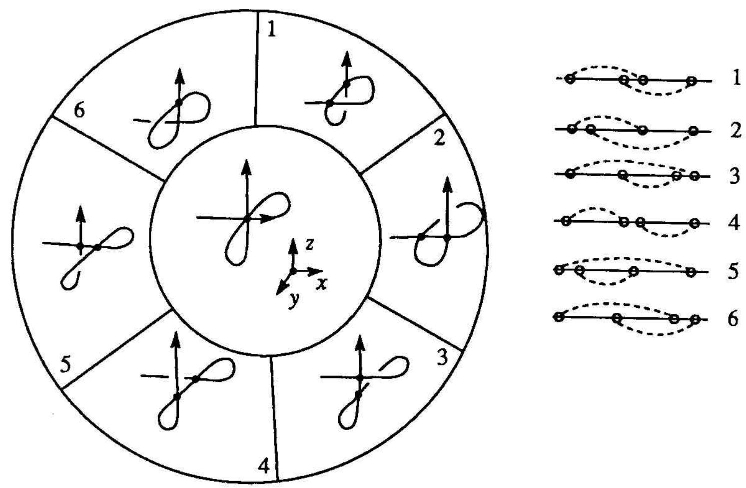

FIGURE 15

EXAMPLE. The first level contains no new spaces, while the second one contains a unique new space in which the immersed curve with triple point shown on Figure 15 and having no other self-intersections is placed.

We will fill in the extended actuality table for each (basis) element γ_0 of the group $\tilde{E}_1^{-i,i}$: its data that do not enter in the main table are not directly needed for computing the values of the invariants, however, when we fill in the lth level of the main table, it is in general necessary to know the $(l+1)$th level of the extended table as well as that of the main one.

The upper (ith) level of the extended table is filled in just as before: in the column corresponding to some $\langle i\rangle$-configuration we write the coefficient with which this configuration appears in the given element $\gamma_0 \in \widetilde{E}_1^{-i,i} \subset X_i$.

4.5. The differential d_1 of the spectral sequence and the process of filling in the $(i-1)$th level of the actuality table.

NOTATION. By Ξ_l we denote the set $(\sigma_l - \sigma_{l-1}) - S_l$ (see 3.1), i.e. the union of all $\mathcal{J}$-blocks of the space $\sigma_l - \sigma_{l-1}$ over all noncomplicated configurations $\mathcal{J}$.

By construction, $\overline{H}_k(\Xi_i) = \widetilde{E}_1^{-i,3d-1-k+i}$; in particular, this group is trivial when k is not equal to $3d-1$ or $3d-2$.

Suppose γ_0 is an arbitrary nonzero element of group $\widetilde{E}_1^{-i,i} \equiv \overline{H}_{3d-1}(\Xi_i)$. Let us compute the corresponding element $d_1\gamma_0 \in \widetilde{E}_1^{-i+1,i}$.

4.5.1. *The geometric boundary of* γ_0. By construction, γ_0 is a linear combination of oriented cells of the canonical decomposition of Ξ corresponding to $[i]$- and $\langle i\rangle$-configurations, and the boundary of this linear combination in Ξ_i is equal to zero. Let us describe its boundary in Ξ_{i-1}.

The boundary in Ξ_{i-1} of any of its cells is determined in the following way. If this cell C corresponds to some $[i]$-configuration $\mathcal{J}$, $C = C(\mathcal{J})$, then its boundary in Ξ_{i-1} contains i components corresponding to all pairs of points determined by the configuration $\mathcal{J}$. Consider one of these pairs; suppose this is the mth pair with respect to the natural ordering of the pairs (see 3.3.2). Remove this pair from the configuration $\mathcal{J}$ and denote the $[i-1]$-configuration thus obtained by $\mathcal{J}_m$; consider the corresponding cell of the canonical decomposition of the space Ξ_{i-1}. By construction, this cell is the subset in $\Gamma^d \times \mathbb{R}^N$ consisting of pairs of the form (θ, u), where u is a point of the nonclosed $(i-2)$-dimensional simplex in $\mathbb{R}^N$ corresponding to some $[i-1]$-configuration $\widetilde{\mathcal{J}}_m$ equivalent to $\mathcal{J}_m$, while θ is a map of the class Γ^d respecting this configuration. In this last cell, consider the subset consisting of all (θ, u) such that θ additionally identifies some pair of points that does not enter in $\mathcal{J}_m$, and adding this pair of points to $\mathcal{J}_m$, we obtain an $[i]$-configuration equivalent to $\mathcal{J}$. This subset is a hypersurface in our cell; in general it possesses singularities and self-intersections. Such a hypersurface will be called a singular hypersurface of the *first kind.*

Besides this, the boundary of the cell $C(\mathcal{J})$ may contain several other surfaces that lie in cells corresponding to $\langle i-1\rangle$-configurations. Namely, assume that among the i characteristic pairs of the given $[i]$-configuration $\mathcal{J}$ there is a triple of pairs (t_1, t_1'), (t_2, t_2'), (t_3, t_3') such that the points t_1 and t_2', t_2 and t_3', t_3 and t_1' lie next to each other in $\mathbb{R}^1$. Consider the $\langle i-1\rangle$-configuration $\mathcal{J}!$ obtained from $\mathcal{J}$ by replacing these three pairs by one triple consisting of the centers of the corresponding segments. In the cell $C(\mathcal{J}!)$, consider the hypersurface consisting of all pairs (θ, u) such that the derivatives of θ at the points of the characteristic triple are coplanar. Such hypersurfaces will be called singular surfaces of the *second kind.* The boundary of the given cell $C = C(\mathcal{J})$ in Ξ_{i-1} is the union of surfaces of the first and second kind taken with the appropriate orientations.

If the cell C corresponds to an $\langle i\rangle$-configuration, then its boundary in Ξ_{i-1} consists of $i-2$ hypersurfaces of the *third kind,* corresponding to all possible pairs of points participating in the definition of this configuration (but not in its triple) and belonging to cells corresponding to $\langle i-1\rangle$-configurations obtained from the given one by forgetting this pair.

Finally the *geometric boundary* $\partial_1\gamma_0$ of our cycle γ_0 in Ξ_{i-1} is defined by linearity; since γ_0 is a cycle in Ξ_i, it follows that $\partial_1\gamma_0$ is a cycle in Ξ_{i-1}[4]. In order to find the element $d_1\gamma_0$, it remains to construct a cycle in Ξ_{i-1} homological to $\partial_1\gamma_0$ and determined by a linear combination of $(3d-2)$-dimensional cells of the canonical decomposition of Ξ_{i-1}. This is done in Subsections 4.5.2 and 4.5.3.

4.5.2. *The homological boundary of* γ_0. For any $[l]$- (or any $\langle l\rangle$-) configuration $\mathcal{J}$ denote by $\Sigma(\mathcal{J})$ the union of all singular hypersurfaces of the first (resp. second and third) kind in the corresponding cell $C(\mathcal{J})$ of the space Ξ_l. Denote by $\Sigma(l)$ the union of the sets $\Sigma(\mathcal{J})$ over all $(3d-1)$-dimensional cells of Ξ_l. By construction, the support of the chain $\partial_1\gamma_0$ belongs to $\Sigma(i-1)$.

DEFINITION. An *elementary component* in Ξ_l is any connected component of the complement of $\Sigma(l)$ in any $(3d-1)$-dimensional cell of the canonical decomposition of Ξ_l. The standard orientation of this component is given by the orientation of the corresponding cell. See Subsection 3.3.

Let us try to span the cycle $\partial_1\gamma_0 \subset \Xi_{i-1}$ by a chain in Ξ_{i-1} consisting of elementary components (taken with the appropriate multiplicities).

DEFINITION. A chain in Ξ_{i-1} given by the linear combination of elementary components is said to be *compatible* with the cycle $\partial_1\gamma_0$ if in the

[4]This conclusion necessitates some justification, since the union of the sets S_l is not closed in σ. However, the intersection of the closure of the set S_i with Ξ_{i-1} is a set of codimension not less than 2 in Ξ_{i-1} and our argument is valid nevertheless.

intersection with any open $(3d-1)$-dimensional cell C of the canonical decomposition of Ξ_{i-1} the boundary of this chain is equal to $\partial_1\gamma_0$ (i.e. under the restriction homomorphism $\overline{H}_*(\Xi_{i-1}) \to \overline{H}_*(C)$ the images of the element $\partial_1\gamma_0$ and of the boundary of this chain coincide).

DEFINITION. The *principal component* of the complement to $\Sigma(l)$ in a $(3d-1)$-dimensional cell $C \subset \Xi_l$ is by definition the component containing the curve given in the corresponding space of the actuality table.

LEMMA. *The group of chains in* Ξ_{i-1} *compatible with* $\partial_1\gamma_0$ *is isomorphic (noncanonically) to the group* X_{i-1}.

Indeed, the coefficient, with which the principal component of the complement to $\Sigma(i-1)$ in the cell $C(\mathcal{J})$ appears in a compatible chain, uniquely determines the coefficients that appear at all the other components contained in this cell. Namely, when we pass from one such component to another through a smooth part of $\Sigma(i-1)$ of the first or third kind in the positive direction (for the definition see 4.3.1) this coefficient increases by the product of the two numbers $I\chi$; here I is the actuality index of any normed immersion $\mathbb{R}^1 \to \mathbb{R}^3$ belonging to this part of the surface and having i self-intersections or having $i-2$ self-intersections and one triple point (this index can be found in the ith level of the actuality table); let us define the number χ. To any point of the cell C there corresponds a map $\theta\colon \mathbb{R}^1 \to \mathbb{R}^3$ respecting some $[i-1]$- or $\langle i-1\rangle$-configuration depending on the type of the cell. At a nonsingular point of the surface $\Sigma(i-1)$ of the first or third kind, such a map identifies additionally one more pair of points τ, τ', i.e. respects a certain $[i]$-(respectively $\langle i\rangle$)-configuration. Suppose $n(\tau)$ and $n(\tau')$ are the numbers of these points in the list of all $2i(2i-1)$ points determining this configuration and ordered in increasing order. Then the number χ equals $(-1)^{n(\tau)+n(\tau')+i}$.

The computation of the jump in the coefficient when we pass through a surface of the second kind in a $\langle i-1\rangle$-cell can be reduced to passing through surfaces of the first kind. Namely suppose $C(\mathcal{J})$ is a cell in Ξ_{i-1} corresponding to a $\langle i-1\rangle$-configuration $\mathcal{J}$, suppose (θ, u) is a generic point of the surface of the second kind in $C(\mathcal{J})$. Then all three derivatives of θ at the points of the identified triple are nonzero and are pairwise noncollinear. Such a map θ may be slightly moved in two ways so as to obtain maps θ_1, θ_2, which preserve all double points located far from our triple, making this triple point split up into three self-intersection points; see Figure 16. The passage through the surface $\Sigma(i-1)$ at the point (θ, u) may be carried out by a one-parameter family of maps θ^t, $t \in (-\epsilon, \epsilon)$,

which does not change the two local branches of the curve $\theta(\mathbb{R}^1)$, and rotates the third one so that its tangent is always located in the fixed plane orthogonal to the plane spanning the tangents to the first two branches. At the same time we will slightly change the shifted map θ_1 so that only the third local branch of the curve $\theta_1^t(\mathbb{R}^1)$ is modified; namely, it must always intersect one in first two branches at the same point while its tangent at this point must repeat the motion of the tangent to the corresponding branch θ^t of the Figure 16. Then all the maps θ_1^t respect a certain $[i-1]$-configuration $\mathcal{J}_1$. This entire deformation carries out the passage through a singular surface of the first kind inside the corresponding cell $C(\mathcal{J}_1) \subset \Xi_{i-1}$. Suppose X_1 is the jump of coefficients in the chain compatible with $\partial_1\gamma_0$ corresponding to this passage. Similarly we define X_2 by using the map θ_2. The required jump in the cell $C(\mathcal{J})$ in the motion along the curve $\{\theta^t, u\}$ is equal to the sum $X_1 + X_2$. ::

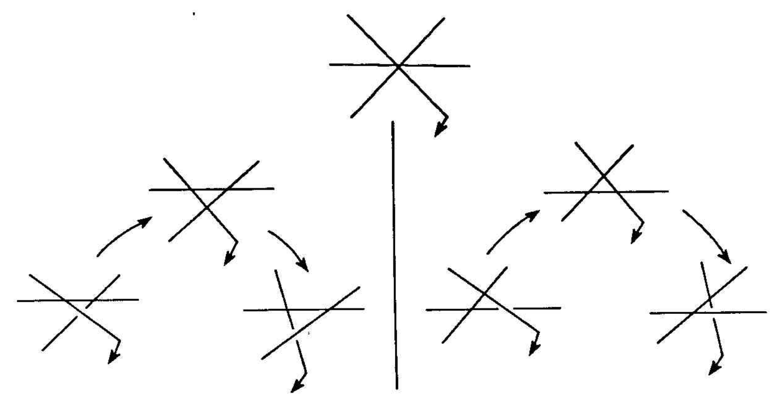

FIGURE 16

Let us choose a chain in Ξ_{i-1} compatible with $\partial_1\gamma_0$ so that the principal component of any $(3d-1)$-dimensional cell enters this chain with coefficient 0. By construction, the boundary of this (and any other) compatible chain is the sum of the cycle $\partial_1\gamma_0$ and a certain linear combination of $(3d-2)$-dimensional cells of the canonical decomposition of the space Ξ_{i-1}. Denote this linear combination by $\tilde{d}_1\gamma_0$. This is an element of the group $Y_{i-1} \equiv \tilde{E}_0^{-i+1,i}$. This element depends on the choice of the compatible chain (i.e., on the choice of curves fixed in the actuality table). However, the class of this element in the group $\tilde{E}_1^{-i+1,i} \equiv Y_{i-1}/h_{i-1}(X_{i-1})$ is defined in an invariant way. This class is precisely $d_1\gamma_0$.

4.5.3. *The algorithm for the computation of* $\tilde{d}_1\gamma_0$. Take an arbitrary $(3d-2)$-dimensional cell $\tilde{C}$ of the canonical decomposition of Ξ_{i-1} corresponding to some $\langle \widetilde{i-1} \rangle$-configuration and compute the coefficient that this cell has in $\tilde{d}_1\gamma_0$. Consider the corresponding $\langle i-1 \rangle$-configuration and the appropriate space on the $(i-1)$th level of the extended actuality table. Take the immersed curve in $\mathbb{R}^3$ appearing in this space. There exists exactly six different ways to make a small move of this curve so as to obtain a curve respecting some $[i-1]$-configuration. Here the triple point will split up into two double points (see Figure 15). Choose two such moves out of the six so that the dotted arcs of the $[i-1]$-configuration obtained corresponding to these two pairs are born from the dotted arcs involved in the original $\langle \widetilde{i-1} \rangle$ configuration. For example, if the cell $\tilde{C}$ is given by the $\langle \widetilde{i-1} \rangle$-configuration shown on the first summand on the right side of the upper line on Figure 10 (respectively the second and third summands), then on Figure 15 the corresponding ways will be 1 and 4 (resp. 2 and 5, 3 and 6). These two moves determine two immersed curves in $\mathbb{R}^3$ respecting two different $[i-1]$-configurations and not possessing extra self-intersection points (and therefore belonging to some components of the complement to $\Sigma(i-1)$ in the corresponding cells of the canonical decomposition of Ξ_{i-1}). Now let us compute the coefficients with which these components appear in our compatible chain: to do this, join the two curves obtained by paths in general position in the corresponding cells with the etalon curves and take the sum over all the intersection points of these paths with $\Sigma(i-1)$ of the numbers indicated in the proof of the lemma given in Subsection 4.5.2. Finally, the required coefficient that determines the multiplicity with which the cell $\tilde{C}$ appears in $\tilde{d}_1\gamma_0$ is equal to the sum of the two coefficients that we have just computed multiplied by the incidence coefficients of the cell $\tilde{C}$ with the corresponding two $[i-1]$-cells.

Now suppose $\tilde{C}$ is a $(3d-2)$-dimensional cell in Ξ_{i-1} corresponding to some i^*-configuration. If during the programming process we have worked in agreement with the footnote in Subsection 4.2, then this cell appears in $\tilde{d}_1\gamma_0$ with a zero coefficient.

4.5.4. *Filling in the* $(i-1)$th *level of the actuality table.* Suppose we have computed the elements $\tilde{d}_1(\cdot) \in Y_{i-1}$ for all the generators of the group $\tilde{E}_1^{-i,i}$ and therefore the homomorphism $\tilde{d}_1 : E_1^{-i,i} \to Y_{i-1}$ as well. Define the group $\tilde{E}_2^{-i,i}$ as the subgroup of $\tilde{E}_1^{-i,i}$ consisting of elements that this homomorphism takes to elements of the subgroup $h_{i-1}(X_{i-1}) \subset Y_{i-1}$. In $\tilde{E}_1^{-i,i}$ choose a new system of generators so that it includes a certain system of generators of $\tilde{E}_2^{-i,i}$.

For any such generator $\gamma_0 \in \tilde{E}_2^{-i,i}$, consider the corresponding element $\tilde{d}_1\gamma_0$ and an arbitrary element $\alpha \in X_{i-1}$ such that $h_{i-1}(\alpha) + \tilde{d}_1\gamma_0 = 0$; α is a linear combination of cells corresponding to $[i-1]$-configurations and $\langle i-1 \rangle$-configurations. Then the geometrical boundary $\partial_1\gamma_0$ of the element γ_0 is spanned in Ξ_{i-1} by a chain compatible with it in which the principal component of the complement to $\Sigma(i-1)$ in any cell appears with the same coefficient with which the cell appeared in the linear combination α. This coefficient must be placed in the space corresponding to the cell in the $(i-1)$th level of the extended actuality table.

The generators of the group $\tilde{E}_1^{-i,i}$ that do not belong to $\tilde{E}_2^{-i,i}$ will not be required for further calculations of the invariants of the ith order; nevertheless, we must not forget about them. For each of these generators, we will place in the computer memory the corresponding values of $\tilde{d}_1(\cdot)$: they will be needed when we compute the homomorphisms $d_2(\tilde{E}_2^{-i-1,i+1})$, $d_3(\tilde{E}_3^{-i-2,i+2})$, etc.

4.6. The differential d_2 and subsequent ones. They are computed exactly in the same way as d_1. Let us neverthelsss point out the main steps of the computation.

Suppose we have already computed the subgroups

$$\tilde{E}_r^{-j,j} \subset \tilde{E}_{r-1}^{-j,j} \subset \cdots \subset \tilde{E}_0^{-j,j} \equiv X_j$$

for all $j \leqslant i$; in particular $\tilde{E}_r^{-i,i}$ is the kernel of the homomorphism

$$d_{r-1} : \tilde{E}_{r-1}^{-i,i} \to \tilde{E}_{r-1}^{-i+r-1,i-r+2}.$$

For the given generator γ_{r-1} of this group, the levels $i, i-1, \ldots, i-r+1$ of the extended actuality table have been filled in. Let us compute $d_r(\gamma_{r-1})$.

Let us define a tentative compatible chain in Ξ_{i-r}: the linear combination of elementary components of the complement to $\Sigma(i-r)$ in the open $(3d-1)$-dimensional canonical cells, the principal component entering this combination with coefficient 0, while the coefficients of any pair of neighboring components differ by the actuality index of the component in Ξ_{i-r+1} (taken with appropriate sign, as described in the proof of the lemma from 4.5.2), corresponding to any part of their common boundary (these indices have been computed during the previous step).

The boundary of this compatible chain is the sum of the geometrical boundary $\partial_r(\gamma_{r-1})$ and of a certain cycle consisting of $(3d-2)$-dimensional canonical cells. This cycle is denoted by $\tilde{d}_r(\gamma_{r-1})$, while $d_r(\gamma_{r-1})$ denotes the class of this cycle in the quotient group

$$Y_{i-r}/\{h_{i-r}(X_{i-r}), \tilde{d}_1(\tilde{E}_1^{-i+r-1,i-r+1}, \dots, \tilde{d}_{r-1}(\tilde{E}_{r-1}^{-i+1,i-1})\}. \qquad (9)$$

The kernel of the map d_r of the group $\tilde{E}_r^{-i,i}$ into (9) is denoted by $\tilde{E}_{r+1}^{-i,i}$.

Choose a new system of generators in the group X_i; here the generators that do not enter in $\tilde{E}_r^{-i,i}$ are not changed, while the generators of $\tilde{E}_r^{-i,i}$ are replaced so that part of them will at the same time be a system of generators for $\tilde{E}_{r+1}^{-i,i}$. This replacement determines an obvious transformation of the levels of the actuality table that we have already filled in: for example, the data of such a table for the new generator $\gamma+\gamma'$ is obtained via component by component addition of the corresponding data of the tables for γ and γ'.

For each generator γ_r of the group $\tilde{E}_{r+1}^{-i,i}$, choose an arbitrary element

$$\alpha = (\alpha_{i-r}, \alpha_{i-r+1}, \dots, \alpha_{i-1}) \in X_{i-r} \oplus \tilde{E}_1^{-i+r-1,i-r+1} \oplus \dots \oplus \tilde{E}_{r-1}^{-i+1,i-1}$$

such that

$$-\tilde{d}_r(\gamma_r) = h_{i-r}(\alpha_{i-r}) + \tilde{d}_1(\alpha_{i-r+1}) + \dots + \tilde{d}_{r-1}(\alpha_{i-1});$$

the existence of such an element follows from the definition of $\tilde{E}_{r+1}^{-i,i}$. Add (component by component) the actuality indices written in the corresponding spaces of the $i, i-1, \dots, i-r$ levels of the actuality table for the elements $\alpha_{i-r}, \dots, \alpha_{i-1}, \gamma_r$; here we assume that only zeros are written in the $(i-r)$th level of γ_r, while for any element $\alpha_j \in X_j$ the actuality table possesses all levels greater than j and all the spaces of all these levels are also filled in by zeros only. The table thus obtained is declared to be the actuality table of the element γ_r; in it the $(i-r)$th level has been filled in.

4.7. Proof of the theorem from Subsection 0.5. In $\mathbb{R}^3$ construct an arbitrary plane containing the chosen direction $\partial/\partial x + \partial/\partial y + \partial/\partial z$. Reflection in this plane acts in the space of noncompact knots K and preserves the discriminant Σ. We could have chosen the space Γ^d so that it would be invariant with respect to this action (even if to do this we would have to move the space $I_5(\tilde{\Gamma}^d) \subset \tilde{\Gamma}^{5d+4}$ rather than the space $I_3(\tilde{\Gamma}^d) \subset \tilde{\Gamma}^{3d+2}$). Then this reflection acts on the set $\sigma = \sigma(d)$ as well and respects its filtration by the sets σ_i. The induced action of this reflection on the group $\bar{H}_*(\sigma_i - \sigma_{i-1}) \sim E_1^{-i,*}$ is the identity if i is even and coincides with multiplication by -1 if i is odd: this easily follows from our method of orienting the canonical cells in $\sigma_i - \sigma_{i-1}$; see Subsection 3.3..

Therefore, replacing any invariant of the ith order by its half-sum (with i even) or half-difference (with i odd) with its mirror image under this reflection, we obtain an invariant of the same order that determines the same element in the group $E_1^{-i,i}$.

This element is the required one.

§5. The simplest invariants and their values for standard knots

5.1. Theorem. *For any group of coefficients* G *in our stable spectral sequence, we have* $E_\infty^{-1,1} = 0$, $E_\infty^{-2,2} = G$, $E_\infty^{-3,3} = G$, $E_\infty^{-4,4} = G^3$.

Proof. Direct calculations.

The corresponding basis invariants for the case $G = \mathbb{Q}$ (or $\mathbb{Z}[1/2]$) are described in Subsection 5.2; their values for table prime knots with $\leqslant 7$ crossings and for the two simplest nonprime knots appear on Figure 2 in square brackets (first the value of the second order invariant, then of the third order one, and then of three fourth order invariants). To obtain a basis in the space of invariants of order $\leqslant 4$ over $\mathbb{Z}$, it is sufficient to add to our second invariant of order 4 the basis invariants of order 2 and 3 taken with arbitrary half-integer coefficients; but the basis thus obtained will not be compatible with reflections (see Subsection 0.5).

5.2. Actuality tables. The second and third levels of the actuality table for all five invariants appear in Figure 17. Here we omit the positions corresponding to those $[i]$-configurations for which one of the characteristic pairs of points is not separated by points of other pairs (see the footnote in Subsection 4.2): all such positions contain only zero values of actuality coefficients.

The notation under the first picture on Figure 17 means that the corresponding actuality coefficient, defined by the third order basis invariant, is equal to 1, while the coefficients of the three basis invariants of fourth order are equal to $0, 1/2$, and 0. The coefficients of the invariants of orders $2, 3, 4$ for the only picture of second order are respectively equal to $1, -1, (0, 0, 0)$. The fourth level of the actuality table for the order 4 invariants is encoded below; here in the notation $[(13)(24)(57)(68)] \to (1, 0, -1)$ the numbers $1, 0, -1$ are the actuality coefficients defined by the three basis invariants; the notation in the square

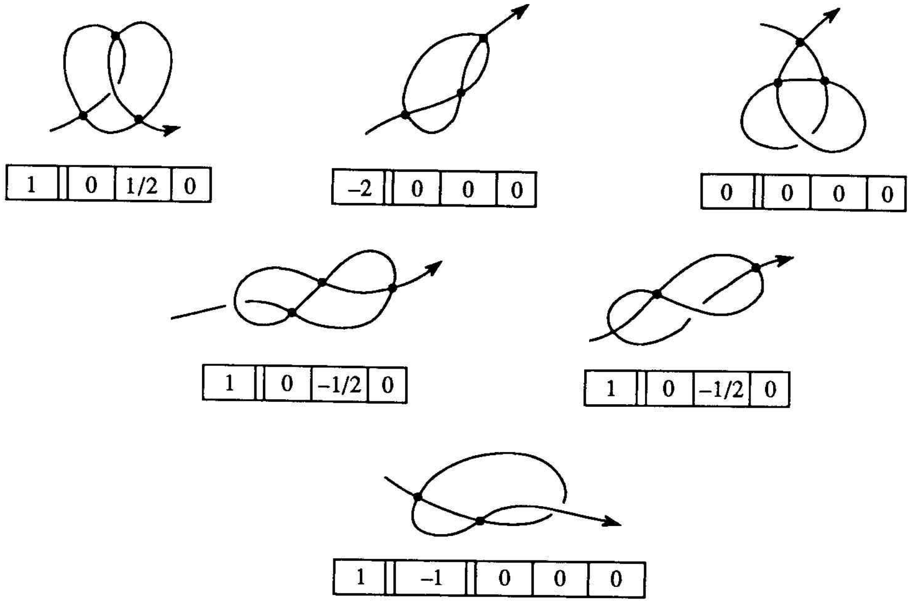

FIGURE 17

brackets means that these coefficients are assigned to any map $\mathbb{R}^1 \to \mathbb{R}^3$ having exactly 4 simple self-intersections such that the first point (of the 8 points of $\mathbb{R}^1$ participating in these self-intersections) is pasted with the third, the second – with the fourth and so on.

$[(13)(24)(57)(68)] \to (1, 0, -1)$;
$[(13)(26)(47)(58)] \to (-1, 0, -1)$;
$[(13)(27)(46)(58)] \to (0, 0, 1)$;
$[(14)(25)(37)(68)] \to (-1, 0, -1)$;
$[(14)(26)(38)(57)] \to (-1, 0, -1)$;
$[(14)(27)(36)(58)] \to (0, -1, 0)$;
$[(15)(24)(37)(68)] \to (1, 0, 0)$;
$[(15)(26)(38)(47)] \to (1, 1, 0)$;
$[(15)(27)(36)(48)] \to (1, 1, 0)$;
$[(15)(28)(37)(46)] \to (1, 0, 0)$;
$[(16)(24)(38)(57)] \to (0, 0, 1)$;
$[(16)(25)(38)(47)] \to (0, -1, 0)$;
$[(16)(28)(35)(47)] \to (0, 0, 1)$;
$[(17)(24)(36)(58)] \to (0, 0, 1)$;
$[(17)(25)(38)(46)] \to (0, 0, 1)$;
$[(17)(28)(35)(46)] \to (1, 0, -1)$.
$[(13)(25)(47)(58)] \to (0, 0, 1)$;
$[(13)(26)(48)(57)] \to (1, 0, 0)$;
$[(13)(28)(46)(57)] \to (1, 0, -1)$;
$[(14)(26)(37)(58)] \to (1, 1, 0)$;
$[(14)(27)(35)(68)] \to (0, 0, 1)$;
$[(14)(28)(36)(57)] \to (0, 0, 1)$;
$[(15)(26)(37)(48)] \to (-2, -1, -1)$;
$[(15)(27)(38)(46)] \to (-1, 0, -1)$;
$[(15)(28)(36)(47)] \to (-1, 0, -1)$;
$[(16)(24)(37)(58)] \to (-1, 0, -1)$;
$[(16)(25)(37)(48)] \to (1, 1, 0)$;
$[(16)(27)(35)(48)] \to (-1, 0, -1)$;
$[(17)(24)(35)(68)] \to (1, 0, -1)$;
$[(17)(25)(36)(48)] \to (-1, 0, -1)$;
$[(17)(26)(35)(48)] \to (1, 0, 0)$;

All the other types of 4-fold self-intersections have only zero coefficients.

§6. Conjectures, problems, additions

6.1. Stabilization conjecture. The groups $E_\infty^{i,i}$, $i \geqslant 2$ of the stable spectral sequence determine a complete system of invariants (i.e. the corresponding classes of zero-dimensional cohomology of the space of knots distinguish any pair of nonequivalent knots).

It is obvious that any two nonequivalent knots are distinguished by appropriate elements coded in the terms $E_\infty^{-i,i}(d)$, $i = i(d)$, of our spectral sequences converging to the cohomology of the spaces $\Gamma^d \setminus \Sigma$ for d such that both these knots can be approximated by polynomial maps of degree d; the previous conjecture says that if d is sufficiently large, these terms are contained in the stable domain: $i(d) \leqslant (3d+1)/5$.

If this conjecture is true, then the following integer-valued functions I, I' are defined: to distinguish any pair of knots presented by diagrams with $\leqslant s$ double points (respectively presented by polynomial maps of degree $\leqslant d$), it is sufficient to compute and compare these invariants of order $\leqslant I(s)$ (respectively, of order $\leqslant I'(d)$).

Problem: estimate the asymptotics of the functions I, I'.

6.2. PROBLEM. Study the terms $E_r^{-i,i}(d)$ that are contained in the nonstable domain. This problem is intimately related with Conjecture 6.1, but may have independent interest.

6.3. PROBLEM. Compute as far as possible the stable spectral sequence. It is possible that the calculation of several subsequent invariants will make it possible to guess what the general form of the term $\underset{\leftarrow}{E}_\infty^{-i,i}$ is.

Here are two tentative problems in this connection:

Is it true that the main diagonal of the stable spectral sequence degenerates in the first term: $\underset{\leftarrow}{E}_\infty^{-i,i} = \underset{\leftarrow}{E}_1^{-i,i}$ for all i?

It is true that the stable $\mathbb{Z}_2$ spectral sequence $\underset{\leftarrow}{E}_r^{p,q}\langle\mathbb{Z}_2\rangle$ is equal to $\underset{\leftarrow}{E}_r^{p,q}\langle\mathbb{Z}\rangle \otimes \mathbb{Z}_2$?

6.4. PROBLEM. Compute the higher (corresponding to $p+q>0$) terms of the spectral sequence. What is the geometrical interpretation of the corresponding homology classes of the space of knots? Is the conjecture from Subsection 2.6 true?

6.5. Problem. How are our invariants related to other invariants of knots (see [CF,FC,L])?

6.6. Generalization to links and embeddings in $\mathbb{R}^k$. Our spectral sequence can obviously be generalized to the case of links in $\mathbb{R}^3$, as well as embeddings $S^1 \to \mathbb{R}^k$, $k \geqslant 4$. In the latter case there are no problems with the proof of convergence: the corresponding stable spectral sequence converges precisely to the cohomology of the space of nonsingular embeddings (and its support looks like the one shown in Figure 2 in [V3]).

Let us describe the most general case when our method is applicable. Suppose M is a smooth manifold. Consider the space $\mathcal{J}_2^l(M, \mathbb{R}^k)$ of all possible bijets of maps $M \to \mathbb{R}^k$ (i.e. nonordered pairs of l-jets of maps $M \to \mathbb{R}^k$ at two different points of M). Suppose Ξ is a closed semi-algebraic subset in $\mathcal{J}_2^l(M, \mathbb{R}^k)$, invariant with respect to the obvious action of the diffeomorphism group of M. The set Ξ is said to be *transitive* if, for any smooth map $\varphi : M \to \mathbb{R}^k$, from the fact that the bijet of φ at the pair of points $a, b \in M$ and at the pair of points b, c belong to Ξ, it follows that the bijet of φ at the pair a, c also belongs to Ξ. Important examples: the set of bijets corresponding to the self-intersections and self-tangency of $\varphi(M)$ are transitive, while the set corresponding to the points of nontransversality of two branches of a 2-dimensional immersed surface in $\mathbb{R}^4$ is nontransitive. If the codimension of a transitive subset Ξ in $\mathcal{J}_2^l(M, \mathbb{R}^k)$ is no less than $2\dim M + 1$, it is possible to construct a spectral sequence similar to ours and presenting invariants of the maps $M \to \mathbb{R}^k$ not having multisingularities of type Ξ (or monosingularities obtained from them by degeneracy). If the same codimension is no less than $2\dim M + 2$, then the set of all such maps is connected and there are no invariants, but our spectral sequence in all dimensions converges exactly to the corresponding cohomology (and no problems with its convergence arise, since the term $\underset{\leftarrow}{E}_1 = \underset{\leftarrow}{E}_1^{p,q}$ of the sequence has only a finite number of nonzero cells on any diagonal of the form $p+q = \text{const}$).

6.7. Mirror image knots. Obviously, if a knot coincides with its mirror image, then all our invariants of odd order assume zero values on this knot. Is the converse true?

6.8. Remark. In our description of the algorithms computing the invariants and their values in §4, we did not formalize one step: two immersions $\mathbb{R}^1 \to \mathbb{R}^3$ having l self-intersections with a similar alternation of identified points are joined by a path in the space of immersions having l

self-intersections, and all the immersions with $l+1$ self-intersections we meet on the way are listed. The description of some algorithm writing out this list is a simple combinatorial programming problem, but how does one make this algorithm as economical as possible?

Note added in proof. Joan Birman communicated to me that our invariant of the 3rd order is independent of the Jones polynomial invariants.

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