

COMPLEXES OF CONNECTED GRAPHS

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Graphs with the given k vertices generate an (acyclic) simplicial complex. We describe the homology of its quotient complex, formed by all connected graphs, and demonstrate its applications to the topology of braid groups, knot theory, combinatorics, and singularity theory. The multidimensional analogues of this complex are indicated, which arise naturally in the homotopy topology, higher dimensional Chern-Simons theory and complexity theory.

1. Introduction

Let A be a set of k elements. Denote by $\Delta(A)$ the simplex with $\binom{k}{2}$ vertices, being in one to one correspondence with the two element subsets of A . Any facet of this simplex can be depicted by a graph with k vertices corresponding to the elements of A : this graph contains the segment between some two elements $a, b \in A$ iff the vertex corresponding to the set (a, b) belongs to our facet.

A graph is called *connected* if any two points in A can be joined by a chain of its segments.

Denote by $M(A)$ the set of all facets in $\Delta(A)$ corresponding to all nonconnected graphs. Obviously, this is a subcomplex of the simplex $\Delta(A)$.

Definition. The *complex of connected graphs* associated with A is the quotient complex $\Delta(A)/M(A)$; the notation of this quotient complex is $K(A)$.

Of course, all these objects corresponding to different k element sets A are isomorphic. We shall denote by Δ_k , M_k and K_k the complexes $\Delta(A)$, $M(A)$ and $K(A)$ where A is the set of naturals $1, 2, \dots, k$.

Theorem 1. For any k element set A , and coefficient group G , the homology group $H_i(K(A), G)$ is trivial for $i \neq k - 2$, and

$$H_{k-2}(K(A), G) = G^{(k-1)!} \quad (1)$$

This theorem is a special case of a general theorem of Folkman [F]. We give an independent proof, based on the geometrical considerations, namely, on the theory of plane arrangements.

Here is an explicit realization of the group $H_{k-2}(K(A))$. Let us distinguish one point $*$ $\in A$.

Theorem 2. For a basis in the group $H_{k-2}(K(A))$ we can take the classes of $(k-1)!$ trees, homeomorphic to a segment (i.e., having only the vertices of orders 1 or 2), and such that an endpoint of this segment (a vertex of order 1) coincides with the distinguished point $*$ in A .

Denote by $\nabla(A)$ the simplex with $2^{k-1} - 1$ vertices, being in one-to-one correspondence with the nonordered partitions of A onto 2 nonempty subsets. A facet of this simplex is called *noncomplete* if there exist two elements $a, b \in A$ such that, for any vertex of this facet, a and b lie in the same part of the corresponding partition of A . The set of all noncomplete facets is, obviously, a subcomplex in $\nabla(A)$; we shall denote this complex by $\Omega(A)$.

Definition. The complex $C(A)$ of complete partitions associated with A is the quotient complex $\nabla(A)/\Omega(A)$.

Again, the complexes ∇_k, Ω_k, C_k are the complexes $\nabla(A), \Omega(A), C(A)$ for $A = (1, 2, \dots, k)$.

Theorem 3. For any finite set A and any coefficient group G , there is a natural isomorphism

$$H_*(C(A), G) = H_*(K(A), G). \quad (2)$$

The number $(k-1)!$ from the formula (1) is well-known to the specialists in the braid groups. Indeed, the Poincaré polynomial of the cohomology of the colored braid group with k strings, $I(k)$, equals

$$(1+t)(1+2t)\dots(1+(k-1)t); \quad (3)$$

in particular $\dim H_{k-1}(I(k)) = (k-1)!$; see [A1]. This is not an occasional coincidence: the complex of connected graphs appears naturally in the description of the homotopy type (and, in particular, of the homology groups) of the classifying space of the group $I(k)$. This description depends on the fact, that the classifying space $K(I(k), 1)$ can be represented as the complement to a certain collection of planes in the space C^k , and on a general formula (see [GM], [V1], [V2], [ZZ]) expressing the stable homotopy type of the complement of arbitrary collection of affine planes in $\mathbb{R}^n$ in terms of the combinatorial characteristics of this collection (namely, of the dimensions of the planes and all their intersections).

Indeed, let $V_1, \dots, V_s$ be a collection of affine planes in $\mathbb{R}^n$ (such collections are called *affine plane arrangements*), and V the union of these planes. Let Λ be any affine plane which is the intersection of several planes V_i . Consider the simplex $\Delta(\Lambda)$ whose vertices correspond formally to all the planes V_i containing Λ . A facet of this simplex, i.e., a collection of such planes, is called *marginal*, if the intersection of these planes is strictly

greater than Λ . The quotient space of the simplex $\Delta(\Lambda)$ by the union of all marginal facets will be denoted by $K(\Lambda)$.

Theorem 4 (see [V1], [V2], [ZZ]). The one point compactification of the variety V is homotopy equivalent to the wedge of $\dim(\Lambda)$ fold suspensions of the spaces $K(\Lambda)$ taken over all planes Λ which are the intersections of some planes of the set $(V_1, \dots, V_s)$.

Corollary 1. The stable homotopy type of the complement $\mathbb{R}^n - V$ of the arrangement V is completely determined by its combinatorial (dimensional) characteristics.

Indeed, this complement is Spanier-Whitehead dual to the one point compactification of V ; hence the stable homotopy types of these spaces are completely determined one through the other; see [W].

Corollary 2. The cohomology group $H^i(\mathbb{R}^n - V)$ is isomorphic to

$$\oplus \tilde{H}_{n-i-1-\dim(\Lambda)}(K(\Lambda)) \quad (4)$$

(the summation over all planes Λ which are the intersections of several planes V_i).

Indeed, this follows from Theorem 4 by the Alexander duality.

The formula (4) was previously obtained by Goresky and MacPherson, see [GM].

Corollary 3. For any k element set A , the homology group $H_i(K(A), G)$ is trivial for $i \neq k-2$; see Theorem 1.

Indeed, the lower estimate follows from the fact that any connected graph with k vertices has at least $k-1$ segments. To prove the upper estimate, consider the hyperplane arrangement in $\mathbb{R}^k$, given by all "diagonal" planes $A_{i,j}$ distinguished by the equalities $x_i = x_j$, $1 \leq i < j \leq k$. Let Λ be the main diagonal of this arrangement, i.e., the line $x_1 = \dots = x_k$. The complex $K(\Lambda)$ is naturally isomorphic to the complex K_k . On the other hand, by Theorem 4, the homology group $H_l(K(\Lambda))$ is a direct summand in the $l+1$ dimensional homology group of the one point compactification of our arrangement in $\mathbb{R}^k$. Since this compactification is an $n-1$ dimensional complex, the desired estimate follows.

Another important application of the complex $K(A)$ appears in knot theory. In [V2]-[V4], I constructed a spectral sequence providing the cohomology classes of the spaces of knots (i.e., of nonsingular embeddings $S^1 \rightarrow \mathbb{R}^n$, $n \geq 3$). The complex $K(A)$ is an essential part of this spectral sequence; see Section 4.

The link theory provides a similar complex: the complex of connected graphs with colored vertices.

Theorem 0 (see [BW]). Suppose that $t > 2$. Then the homology groups $H_t(K(A, t))$ are trivial for all i except maybe for i of the form $k-1-q(t-2)$, $q \geq 1$.

(The fact is that $H_i(K(A, t)) = 0$ for $i > k - (t - 1)$ can be proved again in the same way as Corollary 3 to Theorem 4.)

The complex $K(A, t)$ has also the versions with "colored vertices": the simplest (and very absorbing) problem where such complexes appear is the topological classification of triplets of closed curves in $\mathbb{R}^2$ having no common points.

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2. Complex of connected graphs and complex of complete partitions. Proofs of Theorems 1, 2, 3

Proof of Theorem 3.

Lemma 1. For any $i \geq 0$ and any coefficient group G , $\tilde{H}_i(M(A), G) = H_{i+1}(K(A), G)$, $\tilde{H}_i(\Omega(A), G) = H_{i+1}(C(A), G)$, where $\tilde{H}$ denotes the homology reduced modulo a point.

This follows immediately from the acyclicity of simplices.

Therefore we have only to prove that $H_*(M(A), G) = H_*(\Omega(A), G)$. To do this, we construct a cellular complex

$$\sigma(A) \subset M(A) \times \Omega(A)$$

which is homology equivalent to both $M(A)$ and $\Omega(A)$ (here we do not distinguish between the complexes $M(A)$, $\Omega(A)$ and the topological spaces defined as the unions of corresponding facets of the simplices).

Consider an arbitrary partition π of A into n nonempty subsets, $2 \leq n \leq k-1$. Let $\alpha(\pi)$ be the simplex in $M(A)$ whose vertices correspond to all two element subsets $(a, b) \in A$ such that a and b are in the same part of the partition. Let $\beta(\pi)$ be the simplex in $\Omega(A)$, whose vertices correspond to all partitions of A into 2 nonempty subsets such that π is a subpartition of these partitions. The product $\alpha(\pi) \times \beta(\pi)$ is a cell in $M(A) \times \Omega(A)$. Define the subset $\sigma(A) \subset M(A) \times \Omega(A)$ as the union of such products $\alpha(\pi) \times \beta(\pi)$ over all nontrivial partitions π of the set A . Now, Theorem 3 follows from the following lemma.

Lemma 2. The obvious projections of the set $\sigma(A)$ onto $M(A)$ and $\Omega(A)$ induce the isomorphisms of the homology groups.

Proof. Consider the increasing filtration of the complex $M(A)$ by its skeletons, and the filtration of $\sigma(A)$ by the preimages of these skeletons under

Suppose that the set A of the vertices of our graphs is partitioned into d parts: $A = A_1 \cup \dots \cup A_d$, $\text{card } A_i = k_i$, $\text{card } A = k_1 + \dots + k_d = k$. A graph with the vertices in A is called *concordant with the partition* if the endpoints of any of its segments belong to different parts of this partition.

Again, all the concordant graphs generate an (acyclic) complex which is naturally isomorphic to the simplicial complex of the obvious triangulation of a simplex with $k_1 k_2 + k_1 k_3 + \dots + k_{d-1} k_d$ vertices. The connected concordant graphs (i.e., the concordant graphs connecting all points of the set A) constitute a quotient complex of this complex. We denote this quotient complex by $K(A; A_1, \dots, A_d)$.

The usual complex $K(A)$ is a special case of this one, which corresponds to the partition into separate points.

Theorem 5. The complex $K(A; A_1, \dots, A_d)$ is acyclic in the dimensions not equal to $k-2$.

The proof is almost the same as that of Corollary 3. For more about the topological applications of this complex, see Section 6.

There exist natural multidimensional analogues of the complexes of connected graphs: the complexes $K(A, t)$, $t > 2$, playing the same role in the higher dimensional Chern-Simons theory and equations of multidimensional simplices (see [FNRS], [MS]) which the standard complexes $K(A)$ and $K(A; A_1, \dots, A_d)$ play (by means of the braid and knot theories) in the usual variants of these theories. These complexes were for the first time investigated by A. Björner and V. Welker [BW] in a connection with the problems of the complexity theory; see also [BLX].

The complex $K(A)$ considered above appears in $[V2]-[V4]$ from the resolution of the discriminant variety, i.e., of the (closure of) set of all maps $S^1 \rightarrow \mathbb{R}^n$ having selfintersections. The "selfintersection" is a bisingularity: it is defined by a condition imposed on a map at two different points of the issue manifold. Now suppose that we define the discriminant by a condition imposed at r points, $r \geq 3$ (the simplest problem where such discriminants arise is the study of the space of maps $S^1 \rightarrow \mathbb{R}^n$, $n \geq 2$, having no triple selfintersections, or, more generally, the selfintersections of multiplicity r). The homologies of such spaces are provided by a spectral sequence similar to the one from $[V2]-[V4]$; but the role of the complex $K(A)$ here is played by the complexes $K(A, t)$ described as follows.

For any $t \geq 2$ consider the simplex $\Delta(A, t)$ with $\binom{k}{t}$ vertices being in one-to-one correspondence with the subsets of cardinality t in A . A facet of this simplex, i.e., a collection of such subsets, is called *connected*, if any two points in A can be joined by a chain in A , any two neighboring elements of which belong to some subset of our collection.

Again, the connected facets form a quotient complex in $\Delta(A, t)$; denote it by $K(A, t)$. For $t = 2$, this is exactly the usual complex of connected graphs considered above.

the obvious projection $\sigma(A) \rightarrow M(A)$. Over any open simplex in $M(A)$, this projection is a trivial fiber bundle whose fiber is a simplex. Therefore, the homological spectral sequences, generated by these two filtrations, are isomorphic beginning with the term E^1 and this isomorphism is induced by the projection. Thus, $H_*(\sigma(A)) = H_*(M(A))$; for the complex $\Omega(A)$ the proof is exactly the same, and Lemma 2 is proved.

Remark. In fact the projections of $\sigma(A)$ onto $M(A)$ and $\Omega(A)$ induce also the homotopy equivalences of all these spaces.

Proof of Theorem 1. By Corollary 3 and Theorem 3, this theorem is equivalent to the following:

Lemma 3. *The Euler characteristic of the complex $\Omega(A)$ is equal to $1 - (-1)^k(k - 1)!$.*

Proof of Lemma 3. Let F_p be the union of all simplices $\beta(\pi) \subset \Omega(A)$ considered in the proof of Theorem 3 and such that π is a partition of A into n parts, $n \leq p$. The sets F_p define an increasing filtration of the space $\Omega(A)$, $F_2 \subset F_3 \subset \dots \subset F_{k-1} = \Omega(A)$. Consider the spectral sequence $E_{p,q}^r$ calculating the homology of $\Omega(A)$ and generated by this filtration.

By definition, $E_{p,q}^1 = H_{p+q}(F_p, F_{p-1})$.

Sublemma. *For any $p = 2, 3, \dots, k - 1$, the group $E_{p,q}^1$ splits into a direct sum of subgroups corresponding to different partitions of A into p nonempty subsets, and any of these groups is isomorphic to the group $H_{p+q}(C_p) = H_{p+q-1}(\Omega_p; a \text{ point})$.*

Consider any such partition π of A and the corresponding simplex $\beta(\pi) \subset \Omega(A)$. This simplex has $2^{p-1} - 1$ vertices corresponding to all partitions of A into two subsets such that π is subordinate to these partitions; thus this simplex can be identified with the simplex $\nabla([\pi])$ where $[\pi]$ is the set of parts of the partition π . A facet of our simplex $\beta(\pi)$ does not belong to F_{p-1} iff the corresponding facet in $\nabla([\pi])$ corresponds to a complete partition; in this case this facet cannot be a facet of some other simplex $\beta(\pi')$ where π' is a partition of A into p parts, $\pi' \neq \pi$, and the sublemma follows.

Now suppose that Lemma 3 is proved for all complexes $\Omega(B)$, $\text{card}(B) < k = \text{card}(A)$. Then it follows from the sublemma that the wanted Euler characteristic equals

$$\sum_{n=2}^{k-1} (-1)^n (n-1)! \langle k?n \rangle$$

where $\langle k?n \rangle$ is the number of nonordered partitions of a k element set into n nonempty parts. Since $\langle k?1 \rangle = \langle k?k \rangle = 1$, Lemma 3 is equivalent to the equality

$$\sum_{n=1}^k (-1)^n (n-1)! \langle k?n \rangle = 0.$$

This equality follows immediately from the obvious combinatorial identity $\langle k?n \rangle = n \langle k-1?n-1 \rangle + \langle k-1?n \rangle$, and Theorem 1 is completely proved.

Proof of Theorem 2. We shall prove this theorem for the complex K_k , so that $A = (1, 2, \dots, k)$.

Definition. A k -tree is a tree with k vertices $(1, \dots, k)$. A k -tree is called *normed* if its vertex (1) is of order 1, and is called *linear*, if it is homeomorphic to a segment (i.e., the orders of all its vertices are no more than 2).

Lemma 4. *Any k -tree is homologous in the complex K_k to a linear combination of normed k -trees.*

Lemma 5. *Any normed k -tree is homologous in the complex K_k to a linear combination of normed linear trees.*

These two lemmas together with Theorem 1 imply Theorem 2, because the number of all linear normed k -trees equals exactly $(k-1)!$.

Proof of Lemma 4. Consider any two segments of our tree having the common vertex (1) . Consider a graph with k segments obtained from our tree by adding the segment joining the other endpoints of these two segments. Obviously, the boundary of this graph (more rigorously, of the facet, corresponding to this graph) is a linear combination of our tree, of two trees having one less order of the vertex (1) , and $k-3$ graphs containing a triangle (and hence nonconnected). This implies the lemma.

Proof of Lemma 5. Let us start from the vertex (1) of our normed tree and go along this tree until we meet the first time a vertex of order > 2 . (Before this moment our path is uniquely determined). Choose any two segments of this graph having the endpoints at this vertex and not coinciding with the segment by which we came to it. Consider the graph with k segments obtained from our tree by adding the segment connecting two other endpoints of these two segments. Again, the boundary of this graph is homologous in the complex K_k to the linear combination of our k -tree and two trees with one less order of our vertex. This proves Lemma 5.

3. The topology of the complement of a plane arrangement

Proof of Theorem 4. Colored braid group

3.1. Notations. Let $V_1, \dots, V_s$ be a finite set of affine planes in $\mathbb{R}^n$, and V the union of all V_i . Denote by S the set of naturals $1, \dots, s$. For any subset $J \subset S$, V_J is the intersection of planes V_j , $j \in J$. The dimension of this intersection is denoted by $|J|$. For any J, J' is the maximal subset in S such that $V_J = V_{J'}$. $\bar{V}_J$ and $\bar{V}$ are the notations for the one point compactifications of the complexes V_J and V .

A set J is called *geometrical* if $J = J'$.

For any geometrical set $J \subset S$, denote by $\Delta(J)$ the simplex whose vertices are in one-to-one correspondence with the elements of J . (In the Introduction this simplex was denoted by $\Delta(V_J)$; this change cannot lead to a misunderstanding.) Let $M(J)$ be the subcomplex in $\Delta(J)$ consisting of all marginal facets, and recall the notation $K(V_J)$ for the quotient complex $\Delta(J)/M(J)$.

3.2. Geometrical resolution of the complex V . Without loss of generality, we shall assume that the dimensions of all planes V_j are positive. Consider a space $\mathbb{R}^n$, where n is sufficiently large, and some s affine embeddings $I_j : V_j \rightarrow \mathbb{R}^n$, $j \in S$. For any point $x \in V$, consider all its images in $\mathbb{R}^n$ under all maps I_j such that $x \in V_j$. Denote by $\#x$ the number of such j and by x' the convex hull of these images in $\mathbb{R}^n$.

Lemma 6. *If n is sufficiently large and the system of embeddings I_j is generic, then for any point $x \in V$ the polyhedron x' is a simplex with $\#x$ vertices, and the intersection of simplices x', y' is empty if $x \neq y$.*

The proof is trivial.

We shall suppose that the maps I_j satisfy this lemma.

Denote by V' the union of all simplices x' over all $x \in V$ and by $\bar{V}'$ the one point compactification of V' . These spaces $V', \bar{V}'$ will be called the geometrical resolutions of V and $\bar{V}$. The natural projection $\pi : V' \rightarrow V$ (which maps any simplex x' in the point x) is obviously proper and can be extended to a continuous map $\bar{V}' \rightarrow \bar{V}$ which will be denoted by the same letter π .

Lemma 7. *The projection $\pi : \bar{V}' \rightarrow \bar{V}$ induces a homotopy equivalence of these spaces.*

This is (a special case of) the principal fact of the theory of simplicial resolutions, see f.i. [D].

Hence we have only to prove the following theorem.

Theorem 4'. *For any affine plane arrangement V , the one point compactification of its resolution V' is homotopy equivalent to the wedge indicated in Theorem 4.*

3.3. Proof of Theorem 4' This proof is based on a variant of stratified Morse theory, see [GM].

Definition. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ is called a *generic quadratic function* if it can be expressed in the form $x_1^2 + \dots + x_n^2$ in some affine coordinate system in $\mathbb{R}^n$, whose origin does not belong to V , and any level set $f^{-1}(t)$ of this function is tangent to at most one plane V_j .

Let us fix such a function f . For any set J , denote by t_J the only number t such that $f^{-1}(t)$ is tangent to V_J ; all numbers t_J are called singular values, and the other values are regular.

For any value $t \in \mathbb{R}^1$ denote by $V'(t)$ the space

$$\pi^{-1}(V \cap f^{-1}([t, \infty)).$$

Lemma 8.

- (a) *If t is greater than all singular values t_J , then the quotient space $V'/V'(t)$ is homotopy equivalent to $\bar{V}'$;*
- (b) *If t is less than all values t_J , then this quotient space is a point;*
- (c) *if the segment $[t, s]$ does not contain singular values, then the obvious factorization mapping*

$$(V'/V'(s)) \rightarrow (V/V'(t))$$

is a homotopy equivalence;

- (d) *if the segment $[t, s]$ contains exactly one singular value t_J , $t < t_J < s$, and the set J is geometrical, then the space $V'/V'(s)$ is homotopy equivalent to the wedge of spaces $V'/V'(t)$ and $\Sigma^{J \setminus J}(\Delta(J)/M(J))$, where Σ^i is the notation of the i -fold reduced suspension.*

Theorem 4' follows immediately from this lemma.

Items a, b and c of this lemma are obvious; let us prove d. For any geometrical set J , define the *proper inverse image* of the plane V_J as the closure in $\mathbb{R}^N$ of the union of simplices x' over all points $x \in V_J$ which do not belong to subplanes V_I of lower dimensions in V_J . Denote this closure by V'_J , and by $V'_J(t)$ its intersection with $V'(t)$.

Any space V'_J is naturally homeomorphic to the complex $\Delta(J) \times V_J = \Delta(J) \times \mathbb{R}^{J \setminus J}$.

Now let J be the geometrical set considered in Lemma 8(d). Then, the space $V'_J/V'_J(s)$ is naturally homeomorphic to the space $(\Delta(J) \times V_J)/(\Delta(J) \times (V_J \cap f^{-1}([s, \infty))) \simeq \Sigma^{J \setminus J} \Delta(J)$.

Let W be the union of inverse images of all other planes V_I , $I' \neq J$, having nonempty intersections with the disk $f^{-1}([0, s])$. The intersection of varieties V'_J and W can be naturally identified with the complex $M(J) \times V_J \simeq M(J) \times \mathbb{R}^{J \setminus J}$. Let ϕ be the identical imbedding of this intersection into W . Let ψ be the map from the quotient space $(M(J) \times V_J)/(M(J) \times V_J(s))$ into $W/(W \cap V'(s))$ induced by ϕ . The space $V'/V'(s)$ can be considered as the space

$$(W/(W \cap V'(s))) \cup_{\psi} (V'_J/V'_J(s)),$$

where $\cup_{\psi}$ is the topological operation "paste together by the map ψ "; see f. i. [FV].

But the quotient space $W/(W \cap V'(s))$ is naturally homotopy equivalent to the space $V'/V'(t) = W/(W \cap V'(t))$: this homotopy equivalence

is realized by the obvious factorization map which contracts all points z at which $f(\pi(z)) \in [t, s]$.

The composition of the map ψ and this factorization is a map into one point. Since the operation $\cup_\psi$ is homotopy invariant (see [FV], section I.2.11), the space $V'/V'(s)$ is homotopy equivalent to the composite space $(V'/V'(t)) \cup_{\psi'} (V'_j/V'_j(s))$ where the map ψ' is defined on the same subspace as ψ and takes this subspace into one point $[V'(t)] \in V'/V'(t)$. Hence the space $V'/V'(s)$ is homotopy equivalent to the wedge of spaces $V'/V'(t)$ and

$$\begin{aligned} & V'_j/(V'(s) \cup (M(J) \times V_j)) \\ &= (\Delta(J) \times V_j)/((\Delta(J) \times V_j(s)) \cup (M(J) \times V_j)) \\ &= (\Delta(J)/M(J)) \wedge (V_j/V_j(s)) = \Sigma^{|J|}(\Delta(J)/M(J)). \end{aligned}$$

Q.E.D.

Theorems 4' and 4 are completely proved.

3.4. Important example: the colored braid group

Definition. The ordered configuration space $F(\mathbb{C}^1, k)$ is the space of all ordered subsets of cardinality k in $\mathbb{C}^1$.

This space can be considered as a subset in $\mathbb{C}^k$; namely, as the complement of the union of all planes $A_{i,j}$ distinguished by the equations $x_i = x_j$, $i \neq j$, in the coordinates $x_1, \dots, x_k$ in $\mathbb{C}^k$.

Definition. The colored braid group of k strings, $I(k)$, is the fundamental group of the space $F(\mathbb{C}^1, k)$.

Theorem (see [A1]). The space $F(\mathbb{C}^1, k)$ is a classifying space of the group $I(k) : F(\mathbb{C}^1, k) = K(I(k), 1)$. In particular, the cohomology of the group $I(k)$ coincides with that of the space $F(\mathbb{C}^1, k)$.

The study of the topology of the space $F(\mathbb{C}^1, k)$ is a special case of the problem considered in Theorem 4: the complex hyperplanes in $\mathbb{C}^k$ can be considered as real planes of codimension 2 in $\mathbb{R}^{2k}$. Let us apply the general assertion of Theorem 4 to this space. The "deepest" stratum of the arrangement $\bigcup A_{i,j}$ is the complex line $\Lambda = (x_1 = \dots = x_k)$; it is contained in all the planes $A_{i,j}$. It is easy to see that the complex $K(\Lambda)$ related to this stratum is exactly the complex K_k of connected graphs with k vertices. Moreover, the strata of any dimension r of our arrangement are in one-to-one correspondence with the nonordered decompositions of the k element set into exactly r subsets. This implies, in particular, the equality $H_{k-1}(I(k)) = \mathbb{Z}^{(k-1)!}$, as well as some combinatorial identities which follow from the comparison of two descriptions of the groups $H_i(I(k))$, $i < k-1$: the first following from Theorem 4, and the second obtained in [A1] (and expressed by formula (3)).

4. Applications to the cohomology of spaces of knots

In [V2]–[V4], a system of knot invariants was constructed (and, moreover, a way to construct the higher dimensional cohomology classes of the space of knots was outlined). The crucial point in this construction is a simplicial resolution of the discriminant variety, i.e., of the set of maps $S^1 \rightarrow \mathbb{R}^n$ having selfintersections or points of vanishing derivative. This resolution is a topological space together with a projection onto the discriminant variety; the inverse image of any discriminant point ϕ (i.e., of a singular map $\phi : S^1 \rightarrow \mathbb{R}^n$) is a simplex, whose vertices are in the one-to-one correspondence with all the pairs of points S^1 glued together by the map ϕ , and all the points where $d\phi = 0$. In particular, over a map ϕ with a k -fold selfintersection point a simplex appears, whose vertices correspond to the subsets of cardinality 2 in a set of k points.

The space of the resolution has a natural filtration, which is defined by the degrees of degeneracy of corresponding singular maps. For instance, the k -fold selfintersection has filtration $k-1$: coincidence of k points takes $k-1$ independent conditions. Thus, our simplex lies in the term F_{k-1} of the filtration. A facet of this simplex lies in the term F_{k-i} of the filtration iff the k vertex graph formed by the vertices of this facet has at least i connected components, in particular it lies in F_{k-2} iff this graph is nonconnected. Thus, calculating the group $H_*(F_{k-1}/F_{k-2})$ of our filtration (or, equivalently, the column $E_{k-1,*}^1$ of the spectral sequence generated by this filtration) involves the calculation of the homology of the complex K_k .

5. Topology of the Maxwell set

Another application of the simplicial resolutions appears from the topological study of (the complement of) the Maxwell set of a complex singularity.

The Maxwell set of a singularity $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ is (the closure of) the set of parameters of a versal deformation of the singularity, which correspond to the functions having two critical points with the same critical value, see [AGLV]. This set can be resolved in almost the same way as the discriminant in the knot space, and this resolution makes it possible to calculate the homology of the complement of the Maxwell set, see [N]. Again, the complexes K_k appear naturally in these resolutions.

6. The topological applications of the complex of the connected graphs with colored vertices

The complex $K(A; A_1, \dots, A_d)$ (see Theorem 5 in the Introduction) appears naturally in the homotopy classification of links.

Indeed, consider the space $L(d)$ of all smooth maps of the disjoint union of d circles $S_{(1)}^1, \dots, S_{(d)}^1$ into $\mathbb{R}^3$. Obviously this set is contractible.

Define the discriminant $\Sigma(d)$ of this space as the set of maps which send two points of some two different circles into the same point in $\mathbb{R}^3$.

Definition. A link with d strings is a smooth imbedding of the disjoint union of d circles into $\mathbb{R}^3$. Two links are *homotopy equivalent* if they lie in the same component of the complement of discriminant in the space $L(d)$.

(The last definition is obviously equivalent to the original definition due to Milnor [M].) In particular, the numerical homotopy link invariants are exactly the zero-dimensional cohomology classes of the complement of the discriminant.

As in [V2]–[V4], the topology of the space $L(d) - \Sigma(d)$ reduces to that of the discriminant $\Sigma(d)$. The cohomology classes of this space are deduced from a simplicial resolution of the discriminant. In this resolution, over a map which glues together some k_1 points from the first circle, k_2 points from the second and so on, exactly the simplex considered in the definition of the complex $K(A; A_1, \dots, A_d)$ appears. This simplex lies in the $k - 1$ st term of the natural filtration of the resolution, while the union of its facets corresponding to the nonconnected graphs lies in the term F_{k-2} .

The simplest (of filtration 1) homotopy invariants obtained from this spectral sequence are just the linking numbers of different components of the link; the p fold degrees and products of these linking numbers are the simplest examples of the invariants of filtration p . The first invariant from our spectral sequence, which cannot be reduced to the linking numbers, has filtration 5 (in the sense of [V2]–[V4], is an invariant of the fifth order); this invariant distinguishes between the Borromean rings and the trivial three component link.

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