CLASSIFICATION OF LATTICE-REGULAR LATTICE CONVEX POLYTOPES.

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ABSTRACT. In this paper for any dimension n we give a complete list of lattice convex polytopes in $\mathbb{R}^n$ that are regular with respect to the group of affine transformations preserving the lattice.

CONTENTS

INTRODUCTION.

Consider an n-dimensional real vector space. Let us fix a full-rank lattice in it. A *convex polytope* is a convex hull of a finite number of points. A nonempty hyperplane

section of a convex polytope P is called its *face* if this section is at the boundary of P. Zero- and one-dimensional faces are called *vertices* and *faces*.

Consider an arbitrary n-dimensional convex polytope P. An arbitrary unordered $(n+1)$-tuple of faces containing the whole polytope P, some its hyperface, some hyperface of this hyperface, and so on (up to a vertex of P) is called a *face-flag* for the polytope P.

The convex polygon is said to be lattice if all its vertices are lattice points. The affine transformation is called *lattice-affine* if it preserves the lattice. Two convex lattice polytopes are said to be *lattice-congruent* if there exist a lattice-affine transformation taking one polytope to the other. A lattice polytope is called *lattice-regular* if for any two its face-flags there exist a lattice-affine transformation preserving the polytope and taking one face-flag to the other.

In this paper we give a complete description of lattice-regular convex lattice polytopes in $\mathbb{R}^n$ for an arbitrary n (Theorem 2.2 in Section 2).

The study of convex lattice polytopes is actual in different branches of mathematics, such as lattice geometry(see, for example [3], [4], [8], [18]), geometry of toric varieties (see [7], [10], [17]) and multidimensional continued fractions (see [1], [11], [9], [12], [16]). Mostly, it is naturally to study such polytopes with respect to the lattice-congruence equivalence relation.

Now we formulate two classical examples of unsolved problems on convex lattice polytopes. The first one comes from the geometry of toric varieties.

Problem 1. Find a complete invariant of lattice-congruence classes of convex lattice (two-dimensional) polygons.

Only some estimates are known at this moment (see for example [2] and [4]).

The second problem comes from lattice geometry and theory of multi-dimensional continued fractions. A lattice symplex is called *empty* if the intersection of this (solid) symplex with the lattice coincides with the set of its vertices.

Problem 2. Find a description of lattice-congruence classes of empty symplices.

The answer to the second problem in the two-dimensional case is simple. All empty triangles are lattice-congruent. Tree-dimensional case is much more complicated. The key to the description gives White's theorem (1964) shown in [20] (for more information see [18], [16], and [9]).

The problems similar to the shown above are complicated and seems not to be solved in the nearest future. Nevertheless, specialists of algebraic geometry or theory of multi-dimensional continued fractions usually do not need the complete classifications but just some special examples.

In the present paper we make the first steps in the study of the lattice polytopes with non-trivial group of *lattice-symmetries* (i.e. the group of lattice-affine transformations, preserving the polytope). We describe the "maximally" possible lattice-symmetric polytopes: the lattice-regular polytopes.

Let us formulate statement for the second step in the study of the lattice polytopes with non-trivial group of lattice-symmetries. A convex lattice pyramid P with the base

B is said to be *lattice-regular* if B is a lattice-regular polytope, and the group of lattice-symmetries of the base B (in the hyperplane containing B) extending to the group of lattice-symmetries of the whole pyramid P.

Problem 3. Find a description of lattice-regular convex lattice pyramids.

This paper is organized as follows. We give a well-known classical description of Euclidean and abstract regular polytopes in terms of Schläfli symbols in Section 1. In Section 2 we give necessary definitions of lattice geometry and formulate a new theorem on lattice-affine classification of lattice-regular convex polytopes. Further in Section 2 we prove this theorem for the two-dimensional case. We study the cases (in any dimension) of lattice-regular symplices, cubes, and generalized octahedra in Sections 4, 5, and 6 respectively. Finally in Section 7 we investigate the remaining cases of low-dimensional polytopes and conclude the proof of the main theorem.

1. EUCLIDEAN AND ABSTRACT REGULAR POLYTOPES.

For the proof of the main theorem on lattice-regular polytopes we use the classification of abstract convex polytopes. We start this section with the description of Euclidean regular polytopes and Schläfli symbols for them, and then continue with the case of abstract regular polytopes.

1.1. **Euclidean regular polytopes.** Consider a regular n-dimensional polytope P. For any two $(k+1)$-dimensional faces of P the numbers of adjacent k-dimensional faces to each of these two faces coincide. Denote this number by p_k. By definition, let $\{p_1, \ldots, p_{n-1}\}$ be *Schläfli symbol* for the polygon P. Traditionally, the string $p, p, \ldots, p$ of the length s in Schläfli symbol is replaced by the symbol p^s.

Theorem A. On classification of regular Euclidean polytopes. *Any regular convex Euclidean polytope is congruent to some polytope of the following list.*

List of regular Euclidean polytopes.
Dimension 1: *a segment with Schläfli symbol $\{\}$.*
Dimension 2: *a regular m-gon (for any $m \geq 3$) with Schläfli symbol $\{m\}$.*
Dimension 3: *a regular tetrahedron ($\{3, 3\}$), a regular octahedron ($\{3, 4\}$), a regular cube ($\{4, 3\}$), a regular icosahedron ($\{3, 5\}$), a regular dodecahedron ($\{5, 3\}$).*
Dimension 4: *a regular symplex $\{3, 3, 3\}$, a regular cube ($\{4, 3, 3\}$), a regular generalized octahedron (or cross polytope, or hyperoctahedron; with Schläfli symbol$\{3, 3, 4\}$), a regular 24-cell (or hyperdiamond, or icositetrachoron; with Schläfli symbol ($\{3, 4, 3\}$), a regular 600-cell (or hypericosahedron, or hexacosichoron; with Schläfli symbol $\{3, 3, 5\}$), a regular 120-cell (or hyperdodecahedron, or hecatonicosachoronor; with Schläfli symbol $\{5, 3, 3\}$).*
Dimension n (n>4): *a regular symplex ($\{3^{n-1}\}$), a regular cube ($\{4, 3^{n-2}\}$), a regular generalized octahedra ($\{3^{n-2}, 4\}$).*

Remark 1.1. The cases of dimension one, two, and three were already known to the ancient mathematicians. The cases of higher dimensions were studied by Schläfli (see in [19]).

1.2. **Abstract regular polytopes.**

In this subsection we consider arbitrary convex polytopes. Consider two n-dimensional polytopes. An isomorphism of two n-dimensional polytopes is said to be *combinatorical* if it takes any face of one polytope to some face of the same dimension of the other polytope. Two polytopes are called *combinatorically isomorphic* if there exist a combinatorical isomorphism between them.

A convex polytope is called *combinatorical regular* if for any two its face-flags there exist a combinatorical isomorphism taking the polytope to itself and one face-flag to the other.

Theorem B. (McMullen [13].) *A polytope is combinatorical regular iff it is combinatorically isomorphic to a regular polytope.*

The proof of this statement essentially uses the work of Coxeter [5].

Remark 1.2. Theorem B implies the classification of real affine and projective polytopes (see [14]). Both classifications coincide with the classification of Euclidean regular polytopes. For further investigations of abstract polytopes see for example the work of L. Danzer and E. Schulte [6] and the book on abstract regular polytopes by P. McMullen and E. Schulte [15].

2. Definitions and formulation of the main result.

Let us fix some basis of lattice vectors $\overline{e}_i$ for $i = 1, \ldots, n$ generating the lattice in $\mathbb{R}^n$. Denote by O the origin in $\mathbb{R}^n$.

Consider arbitrary non-zero integers n_i, where $i = 1, \ldots, k$, are arbitrary integers. By $\gcd(n_1, \ldots, n_k)$ we denote the greater common divisor of the integers n_i, where $i = 1, \ldots, k$. We write that $a \equiv b \pmod c$ if the reminders of a and b modulo c coincide.

2.1. **Some definitions of lattice geometry.**

Let Q be an arbitrary lattice polytope with the vertices $A_i = O + \overline{v}_i$ (where $\overline{v}_i$ — lattice vectors) for $i = 1, \ldots, m$, and t be an arbitrary positive integer. The polygon P with the vertices $B_i = O + t\overline{v}_i$ for $i = 1, \ldots, m$ is said to be the *t-multiple* of the polygon Q.

Definition 2.1. A lattice polytope P is said to be *elementary* if for any integer $t > 1$ and any lattice polytope Q the polytope P is not lattice-congruent to the t-multiple of the lattice polytope Q.

Further in the proofs we will use the following definition. Consider a k-dimensional lattice polytope P. Let its Euclidean volume equal V. Denote the Euclidean volume of the minimal k-dimensional symplex in the k-dimensional plane of the polytope by V_0. The ratio V/V_0 is said to be the *lattice volume* of the given polytope (if $k = 1$, or 2 — the *lattice length* of the segment, or the *lattice area* of the polygon respectively).

2.2. Notation for particular lattice polytopes. We will use the following notation.

Symplices. For any $n > 1$ we denote by $\{3^{n-1}\}_p^L$ the n-dimensional symplex with the vertices:

$$V_0 = O, \qquad V_i = O + \overline{e}_i, \text{ for } i = 1, \ldots, n-1, \quad \text{and} \quad V_n = (p-1)\sum_{k=1}^{n-1} \overline{e}_k + p\overline{e}_n.$$

Cubes. Any lattice cube is generated by some lattice point P and a n-tuple of linearly independent lattice vectors $\overline{v}_i$:

$$\left\{ P + \sum_{i=1}^n \alpha_i \overline{v}_i \,\middle|\, 0 \le \alpha_i \le 1, i = 1, \ldots, n \right\}$$

We denote by $\{4, 3^{n-2}\}_1^L$ for any $n \ge 2$ the lattice cube with a vertex at the origin and generated by all basis vectors.

By $\{4, 3^{n-2}\}_2^L$ for any $n \ge 2$ we denote the lattice cube with a vertex at the origin and generated by the first $n-1$ basis vectors and the vector $\overline{e}_1 + \overline{e}_2 + \ldots + \overline{e}_{n-1} + 2\overline{e}_n$.

By $\{4, 3^{n-2}\}_3^L$ for any $n \ge 3$ we denote the lattice cube with a vertex at the origin and generated by the vectors: $\overline{e}_1$, and $\overline{e}_1 + 2\overline{e}_i$ for $i = 2, \ldots, n$.

Generalized octahedra. We denote by $\{3^{n-2}, 4\}_1^L$ for any $n \ge 2$ the lattice generalized octahedron with the vertices $O \pm \overline{e}_i$ for $i = 1, \ldots, n$.

By $\{3^{n-2}, 4\}_2^L$ for any positive n we denote the lattice generalized octahedron with the vertices $O \pm \overline{e}_i$ for $i = 1, \ldots, n-1$, and $O \pm \left(\overline{e}_1 + \overline{e}_2 + \ldots + \overline{e}_{n-1} + 2\overline{e}_n\right)$.

By $\{3^{n-2}, 4\}_3^L$ for any positive n we denote the lattice generalized octahedron with the vertices O, $O - \overline{e}_1$, $O - \overline{e}_1 - \overline{e}_i$ for $i = 2, \ldots, n$, and e_i for $i = 2, \ldots, n$.

A segment, octagons, and 24-sells. Denote by $\{\}^L$ the lattice segment with the vertices O and $O + \overline{e}_1$.

By $\{6\}_1^L$ we denote the hexagon with the vertices $O \pm \overline{e}_1$, $O \pm \overline{e}_2$, $O \pm (\overline{e}_1 - \overline{e}_2)$.

By $\{6\}_2^L$ we denote the hexagon with the vertices $O \pm (2\overline{e}_1 + \overline{e}_2)$, $O \pm (\overline{e}_1 + 2\overline{e}_2)$, $O \pm (\overline{e}_1 - \overline{e}_2)$.

By $\{3, 4, 3\}_1^L$ we denote the 24-sell with 8 vertices of the form

$$O \pm 2(\overline{e}_2 + \overline{e}_3 + \overline{e}_4), \quad O \pm 2(\overline{e}_1 + \overline{e}_2 + \overline{e}_4), \quad O \pm 2(\overline{e}_1 + \overline{e}_3 + \overline{e}_4), \quad O \pm 2\overline{e}_4,$$

and 16 vertices of the form

$$O \pm (\overline{e}_2 + \overline{e}_3 + \overline{e}_4) \pm (\overline{e}_1 + \overline{e}_2 + \overline{e}_4) \pm (\overline{e}_1 + \overline{e}_3 + \overline{e}_4) \pm \overline{e}_4.$$

By $\{3, 4, 3\}_2^L$ we denote the 24-sell with 8 vertices of the form

$$O \pm 2(\overline{e}_1 + \overline{e}_2 + \overline{e}_3 + \overline{e}_4), \quad O \pm 2(\overline{e}_1 - \overline{e}_2 + \overline{e}_3 + \overline{e}_4),$$
$$O \pm 2(\overline{e}_1 + \overline{e}_2 - \overline{e}_3 + \overline{e}_4), \quad O \pm 2(\overline{e}_1 + \overline{e}_2 + \overline{e}_3 - \overline{e}_4),$$

and 16 vertices of the form

$$O \pm (\overline{e}_1 + \overline{e}_2 + \overline{e}_3 + \overline{e}_4) \pm (\overline{e}_1 - \overline{e}_2 + \overline{e}_3 + \overline{e}_4) \pm (\overline{e}_1 + \overline{e}_2 - \overline{e}_3 + \overline{e}_4) \pm (\overline{e}_1 + \overline{e}_2 + \overline{e}_3 - \overline{e}_4).$$

2.3. Theorem on enumeration of convex elementary lattice-regular lattice polytopes.
Now we formulate the main statement of the work.

Theorem 2.2. *Any elementary lattice-regular convex lattice polytope is lattice-congruent to some polytope of the following list.*

List of the polygons.

Dimension 1: *the segment* $\{\}^L$.

Dimension 2: *the triangles* $\{3\}_1^L$ *and* $\{3\}_2^L$; *the squares* $\{4\}_1^L$ *and* $\{4\}_2^L$; *the octagons* $\{6\}_1^L$ *and* $\{6\}_2^L$.

Dimension 3: *the tetrahedra* $\{3,3\}_i^L$, *for* $i = 1, 2, 4$; *the octahedra* $\{3,4\}_i^L$, *for* $i = 1, 2, 3$; *the cubes* $\{4,3\}_i^L$, *for* $i = 1, 2, 3$.

Dimension 4: *the symplices* $\{3,3,3\}_1^L$ *and* $\{3,3,3\}_5^L$; *the generalized octahedra* $\{3,3,4\}_i^L$, *for* $i = 1, 2, 3$; *the 24-sells* $\{3,4,3\}_1^L$ *and* $\{3,4,3\}_2^L$; *the cubes* $\{4,3,3\}_i^L$, *for* $i = 1, 2, 3$.

Dimension n (n>4): *the symplices* $\{3^{n-1}\}_i^L$ *where positive integers* i *are divisors of* n; *the generalized octahedra* $\{3^{n-2}, 4\}_i^L$, *for* $i = 1, 2, 3$; *the cubes* $\{4, 3^{n-2}\}_i^L$, *for* $i = 1, 2, 3$.

All polytopes of this list are lattice-regular. Any two polytopes of the list are not lattice-congruent to each other.
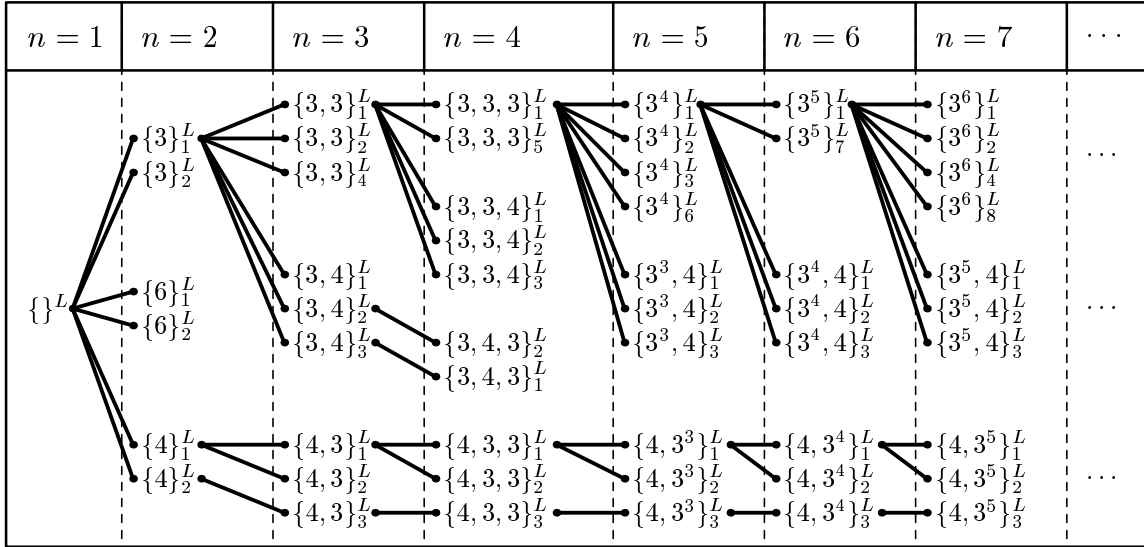

FIGURE 1. The adjacency diagram for the elementary lattice-regular convex lattice polytopes.

On Figure 1 we show the adjacency diagram for the elementary lattice-regular convex lattice polygons of dimension not exceeding 7. Lattice-regular lattice polygons of different

(six) types are shown on Figure 2 in the next section. Lattice-regular lattice three-dimensional polygons of different (nine) types are shown on Figures 3, 4, and 5 further in Sections 4, 5, and 6 respectively.

3. Two-dimensional case.

In this section we prove Theorem 2.2 for the two-dimensional case.

Proposition 3.1. *Any elementary lattice-regular (two-dimensional) lattice convex polygon is lattice-congruent to one of the following polygons (see Figure 2):*
1) $\{3\}_1^{\mathbb{Z}}$; **2)** $\{3\}_2^{\mathbb{Z}}$; **3)** $\{4\}_1^{\mathbb{Z}}$; **4)** $\{4\}_2^{\mathbb{Z}}$; **5)** $\{6\}_1^{\mathbb{Z}}$; **6)** $\{6\}_2^{\mathbb{Z}}$.
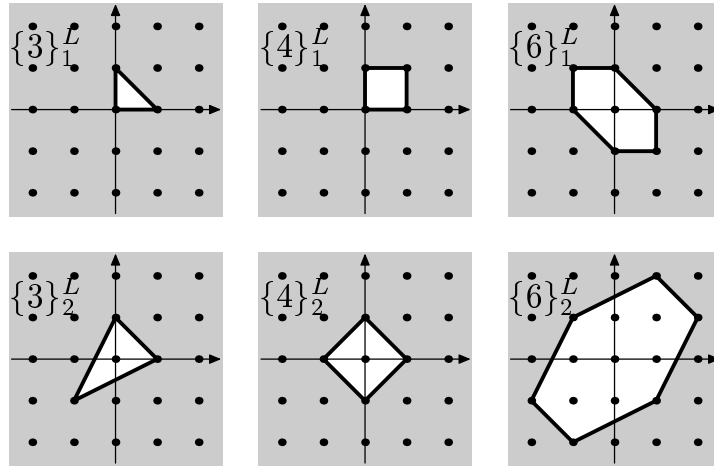

FIGURE 2. The lattice-regular polygons with edges of unit length.

Proof. Suppose that the lattice polygon $A_1 A_2 \ldots A_n$, where $n \geq 3$, is primitive and lattice-regular.

Let us prove that all edges of $A_1 A_2 \ldots A_n$ are of unite lattice lengths. Since the polygon is lattice-regular, all its edges are lattice-congruent, and hence they are of the same lattice length. Suppose that the lattice lengths of all edges equal some positive integer k. Then our polygon is lattice-congruent to the k-tuple of the polygon $A_1' A_2' \ldots A_n'$, where $A_1' = A_1$, and $A_l' = A_l + 1/k(\overline{A_l A_{l+1}})$ for $l = 2, \ldots, n-1$. Therefore, $k = 1$.

Denote by B_i the midpoint of the edge $A_i A_{i+1}$ for $i = 1, \ldots, n-1$, and by B_n the midpoint of the edge $A_n A_1$. Suppose that n is even ($n = 2\tilde{n}$). Denote by M the midpoint of the segment $A_1 A_{\tilde{n}+1}$. Note that the point M is the common intersection of the segments $A_i A_{\tilde{n}+i}$ for $i = 1, \ldots, \tilde{n}$. Suppose that n is odd ($n = 2\tilde{n} + 1$). Denote by M the intersection point of the segments $A_1 B_{\tilde{n}}$ and $A_2 B_{\tilde{n}+1}$. Note that the point M is the common intersection of the segments $A_i B_{\tilde{n}+i}$, $A_{\tilde{n}+i+1} B_i$ for $i = 1, \ldots, \tilde{n}$, and the segment $A_{\tilde{n}} B_{2\tilde{n}+1}$.

For any integer i such that $1 \leq i \leq n$ the following holds. The transformation that preserves the points M and B_i, and taking the point A_i to the point A_{i+1} (or A_n to A_1 in the case of $i = n$) is lattice-affine and preserve the polygon $A_1 A_2 \ldots A_n$.

Suppose that the polygon $A_1A_2 \ldots A_n$ contains some lattice point not contained in the union of its vertices and segments MB_i for $i = 1, \ldots, n$. Then by symmetry reasons the triangle A_1MB_1 contains at least one lattice point, that is not contained in the edges A_1B_1 and MB_1. Denote one of such points by P. Let Q be the point symmetric to the point P about the line MB_1. The segment PQ is parallel to the segment, and hence the lattice point $A_1 + \overline{PQ}$ is contained in the interior of the segment A_1A_2. Then the lattice length of the edge A_1A_2 is not unit. We come to the contradiction with the above.

Therefore, all inner lattice points of the polygon $A_1A_2 \ldots A_n$ are contained in the union of the segments MB_i for $i = 1, \ldots, n$ and vertices. Now we study all different cases of configurations of lattice points on the segment MB_1.

Case 1. Suppose that MB_1 does not contain lattice points. Then by symmetry reasons the polygon $A_1A_2 \ldots A_n$ does not contain lattice points different to its vertices. Hence the vectors $\overline{A_2A_1}$ and $\overline{A_2A_3}$ generate the lattice. Consider the linear system of coordinates such that the points A_1, A_2, and A_3 have the coordinates $(0, 1)$, $(0, 0)$, and $(1, 0)$ in it respectively.

If $n = 3$, then the triangle $A_1A_2A_3$ is lattice-congruent to the triangle $\{3\}_1^L$.

Let $n > 3$. Since the vectors $\overline{A_1A_2}$ and $\overline{A_2A_3}$ generate the lattice, and the vectors $\overline{A_2A_3}$ and $\overline{A_3A_4}$ generate the lattice, the point A_4 has the coordinates $(a, 1)$ for some integer a. Since the segment A_1A_4 does not contain lattice points distinct to the endpoints, $A_4 = (1, 1)$. By the same reasons $A_n = (1, 1)$. Therefore, $n = 4$, and the lattice polygon $A_1A_2A_3A_4$ is lattice-congruent to the lattice-regular quadrangle $\{4\}_1^L$.

Case 2. Suppose that the point M is lattice and the segment MB_1 does not contain lattice points distinct to M. Then the vectors $\overline{MA_1}$ and $\overline{MA_2}$ generate the lattice. Consider the linear system of coordinates such that the points A_1, M, and A_2 have the coordinates $(0, 1)$, $(0, 0)$, and $(1, 0)$ in it. Since the vectors $\overline{A_1M}$ and $\overline{MA_2}$ generate the lattice, and the vectors $\overline{A_2M}$ and $\overline{MA_3}$ generate the lattice, the point A_3 has the coordinates $(-1, a)$ for some integer a.

If $a \geq 2$, then the polygon is not convex or it contains straight angles.

If $a = 1$, then the vectors $\overline{A_1A_2}$ and $\overline{A_2A_3}$ generate the lattice. Since the vectors $\overline{A_3M}$ and $\overline{MA_4}$ generate the lattice, and the vectors $\overline{A_2A_3}$ and $\overline{A_3A_4}$ generate the lattice, the new coordinates of the point A_4 are $(0, -1)$. Since $A_4 = A_1 + 2\overline{A_1M}$, we have

$$A_5 = A_2 + 2\overline{A_2M} = (0, -1), \qquad A_6 = A_3 + 2\overline{A_3M} = (1, -1), \quad \text{and} \quad n = 6.$$

Therefore, the lattice-regular polygon $A_1A_2A_3A_4A_5A_6$ is lattice-congruent to the lattice-regular hexagon $\{6\}_1^L$.

If $a = 2$, then $A_3 = A_1 + 2\overline{A_1M}$. Hence $A_3 = A_2 + 2\overline{A_2M} = (0, -1)$, and $n = 4$. Therefore, the polygon $A_1A_2A_3A_4$ is lattice-congruent to the lattice-regular quadrangle $\{4\}_2^L$.

If $a = 3$, then A_3 is contained in the line MB_1. Hence $n = 3$. Therefore, the lattice triangle $\triangle A_1A_2A_3$ is lattice-congruent to the lattice-regular triangle $\{3\}_2$.

Since $A_n = (a, -1)$, the edges A_nA_1 and A_1A_2 intersect for the case of $a > 3$.

Case 3. Suppose that the segment MB_1 contains the unique lattice point P distinct from the endpoints of the segment MB_1. Then the vectors $\overline{PA_1}$ and $\overline{PA_2}$ generate the lattice. Consider the linear system of coordinates such that the points A_1, P, and A_2 have the coordinates $(0,1)$, $(0,0)$, and $(1,0)$ in it. Since the polygon is lattice-regular, the point $M+\overline{PM}$ is also a lattice point, and hence $M = (-1/2, -1/2)$. Denote the point (-1,-2) by M'. (Note that the point M is the midpoint of the segment $M'B$).

The vectors $\overline{A_1M'}$ and $\overline{M'A_2}$ generate a sublattice of index 3. The vectors $\overline{A_2M'}$ and $\overline{M'A_3}$ generate a sublattice of index 3. The segment A_2A_3 is of unit lattice length. Therefore, the point A_3 has the coordinates $(2a-1, 6a+2)$ for some integer a.

If $a \geq 0$, then the polygon is not convex, or it contains straight angles. Since $A_n = (6a+2, 2a-1)$, the edges A_nA_1 and A_1A_2 intersect for the case of $a < 0$.

Case 4. Suppose that the point M is lattice and the segment MB_1 contains the unique interior lattice point P. Then the vectors $\overline{PA_1}$ and $\overline{PA_2}$ generate the lattice. Consider the linear system of coordinates such that the points A_1, P, and A_2 have the coordinates $(0,1)$, $(0,0)$, and $(1,0)$ in it. Since the polygon is lattice-regular, the point $M + \overline{PM}$ is also lattice, and hence $M = (-1,-1)$.

The vectors $\overline{A_1M'}$ and $\overline{M'A_2}$ generate a sublattice of index 3. The vectors $\overline{A_2M'}$ and $\overline{M'A_3}$ generate a sublattice of index 3. The segment A_2A_3 is of unit lattice length. Therefore, the point A_3 has the coordinates $(a-1, 2a+2)$ for some integer a, such that $a \not\equiv 1 (\mathrm{mod}\, 3)$.

If $a \geq 0$, then the polygon is not convex, or it contains straight angles, but this is impossible.

If $a = -1$, then the vectors $\overline{A_1A_2}$ and $\overline{A_2A_3}$ generate a sublattice of index 3. Since the vectors $\overline{A_3M'}$ and $\overline{M'A_4}$ generate a sublattice of index 3, and the vectors $\overline{A_2A_3}$ and $\overline{A_3A_4}$ generate a sublattice of index 3, the point $A_4 = (-3,-2)$. Since $A_4 = A_1+2\overline{A_1M}$, we have

$$A_5 = A_2 + 2\overline{A_2M} = (-2,-3), \qquad A_6 = A_3 + 2\overline{A_3M} = (0,-2), \quad \text{and} \quad n = 6.$$

Therefore, the lattice polygon $A_1A_2A_3A_4A_5A_6$ is lattice-congruent to the lattice-regular hexagon $\{6\}_2^L$.

Since $A_n = (2a+2, a-1)$, the edges A_nA_1 and A_1A_2 are intersecting for the case of $a < -1$.

The remaining cases. Suppose that the segment MB_1 contains at least two interior lattice points. Let P_1 and P_2 be two distinct interior lattice points of the segment MB_1. Let also the segment MP_2 contains the point P_1.

Consider an lattice-affine transformation ξ taking the point M to itself, and the segment A_1A_2 to the segment A_2A_3. The points $Q_1 = \xi(P_1)$ and $Q_2 = \xi(P_2)$ are contained in the segment MB_2. Since the lines P_1Q_1 and P_2Q_2 are parallel and the triangle P_2MQ_2 contains the segment P_1Q_1, the lattice point $S = P_2+\overline{Q_2P_2}$ is contained in the interior of the segment P_1Q_1. Hence the lattice point S of the polygon $A_1A_2 \ldots A_n$ is not contained in the union of segments MB_i for $i = 1, \ldots, n$. We come to the contradiction.

We have studied all possible cases of configurations of lattice points contained inside lattice polygons. The proof of Proposition 3.1 is completed.   □

4. Lattice-regular lattice symplices.

In this section we study all lattice-regular lattice symplices for all integer dimensions.

Let us fix some basis of lattice vectors $\overline{e}_i$, for $i = 1, \ldots, n$ generating the lattice in $\mathbb{R}^n$ and the corresponding coordinate system. Denote by O the origin in $\mathbb{R}^n$.

Proposition 4.1. $\mathbf{Sym}_1$. *All elementary lattice-regular one-dimensional lattice symplices are lattice segments of unit lattice length.*

$\mathbf{Sym}_n$ **(for $n > 1$).** *i) The symplex $\{3^{n-1}\}_n^L$ where p is a positive divisor of n is elementary and lattice-regular;*
ii) any two symplices listed in (i) are not lattice-congruent to each other;
iii) any elementary lattice-regular n-dimensional lattice symplex is lattice-congruent to one of the symplices listed in (i).

The three-dimensional tetrahedra $\{3,3\}_1^L$, $\{3,3\}_2^L$, and $\{3,3\}_4^L$ are shown on Figure 3.

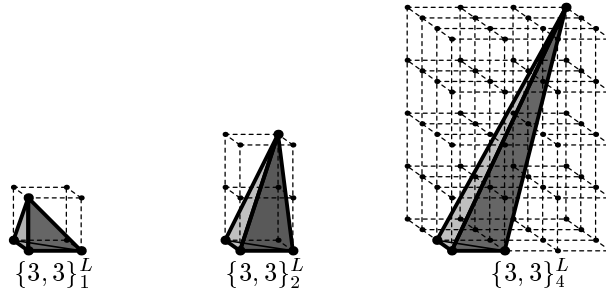$\{3,3\}_1^L$ $\{3,3\}_2^L$ $\{3,3\}_4^L$

FIGURE 3. Three-dimensional elementary lattice-regular convex lattice tetrahedra.

Proof. We start the proof of studying of some low-dimensional cases. The one-dimensional case is trivial and is omitted here. The two-dimensional case was described in Proposition 3.1. Let us study the three-dimensional case.

Three-dimensional case. Consider an arbitrary elementary lattice-regular three-dimensional lattice tetrahedron S. Since its faces are lattice-regular, by Proposition 3.1 the faces are lattice-congruent either to $\{3\}_1^L$ or to $\{3\}_3^L$.

Suppose that the faces of S are lattice-congruent to $\{3\}_1^L$. Then there exist a positive integer b, nonnegative integers a_1, a_2 less than b, and a lattice-affine transformation taking the the tetrahedron S to the tetrahedron S' with the vertices

$$V_0 = O, \quad V_1 = O + \overline{e}_1, \quad V_2 = O + \overline{e}_2, \quad V_3 = O + a_1\overline{e}_1 + a_2\overline{e}_2 + b\overline{e}_3.$$

Since S' is also a lattice-regular tetrahedron, the group of its symmetries is isomorphic to the group of permutations of order 4. This group is generated by the following transpositions of vertices: V_1 and V_2, V_2 and V_3, and V_0 and V_2. The first two transpositions

are linear, and the third one is linear after shifting by the vector $-\overline{e}_1$. Direct calculations shows, that the matrices of the corresponding linear transformations are the following:

$$\begin{pmatrix} 0 & 1 & \frac{a_1-a_2}{b} \\ 1 & 0 & \frac{a_2-a_1}{b} \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a_1 & -\frac{a_1(a_2+1)}{b} \\ 0 & a_2 & \frac{1-a_2^2}{b} \\ 0 & b & -a_2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & \frac{2a_2+a_1-1}{b} \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the listed transformations are lattice-linear we have only the following possibilities, all these matrices are integer. Therefore, $a_1 \equiv a_2 \equiv b-1 \pmod{b}$. Since, positives a_1 and a_2 was chosen to be smaller than b, we have $a_1 = a_2 = b-1$. Since the matrix A_3 is integer, the coefficient $3-4/b$ is also integer. So we have to check only the following cases for a_1, a_2, and b: $b = 1$, and $a_1 = a_2 = 0$; $b = 2$, and $a_1 = a_2 = 1$; $b = 4$, and $a_1 = a_2 = 3$. These cases corresponds to the tetrahedra $\{3,3\}_1^L$, $\{3,3\}_2^L$, and $\{3,3\}_4^L$ respectively. Since the lattice volume of $\{3,3\}_p^L$ equals p, the above tetrahedra are not lattice-congruent.

Let us prove that the faces of S are not lattice-congruent to $\{3\}_3^L$ by reductio ad absurdum. Suppose it is so. Let V_0, V_1, V_2, and V_3 be the vertices of S. Since the faces $V_0V_2V_3$ and $V_1V_2V_3$ are congruent to $\{3\}_3^L$, the face $V_0V_2V_3$ contains a unique lattice point in its interior (we denote it by P_1), and the face $V_1V_2V_3$ contains a unique lattice point in its interior (we denote it by P_2). Consider a lattice-symmetry of S permuting V_0 and V_1 and preserving V_2 and V_3. This symmetry takes the face $V_0V_2V_3$ to the face $V_1V_2V_3$, and hence it maps the point P_1 to P_2. Thus, the lattice vector $\overline{P_1P_2}$ is parallel to the vector V_0V_1. Hence, the point $V_0 + \overline{P_1P_2}$ is interior lattice point of the segment V_0V_1. Therefore, the segment V_0V_1 is not of unit lattice length and the face $V_0V_1V_2$ is not lattice-congruent to $\{3\}_3^L$. We come to the contradiction.

This completes the proof of Proposition 4.1 for the three-dimensional case.

In the higher dimensional case we first study two certain families of symplices.

The first type of n-dimensional symplices. Consider any symplex (for $n > 3$) with the vertices

$$V_0 = O, \qquad V_k = O + \overline{e}_i, \text{ for } k = 1, \dots, n-1, \quad \text{and} \quad V_n = O + \sum_{i=1}^{n} a_i \overline{e}_i,$$

we denote it by $S^n(a_1, \dots, a_n)$. We suppose that all a_i are nonnegative integers satisfying $a_k < a_n$ for $k = 1, \dots, n-1$. Let us find the conditions on a_i for the symplex to be lattice-regular.

If $S^n(p; a_1, \dots, a_n)$ is a lattice-regular symplex then the group of its symmetries is isomorphic to the group of permutations of order $n+1$. This group is generated by the following transpositions of vertices: the transposition exchanging V_k and V_{k+1} for $k = 1, \dots, n-1$, and the transposition exchanging V_0 and V_2. The first $n-1$ transpositions are linear (let their matrices be A_k for $k = 1, \dots, n-1$), and the last one is linear after shifting by the vector $-\overline{e}_1$ (denote the corresponding matrix by A_n). Let us describe the matrices of these transformations explicitly.

The matrix A_k for $k = 1, \ldots n{-}2$ coincides with the matrix transposing the vectors $\overline{e}_k$ and $\overline{e}_{k+1}$, except the last column. The n-th column contains the coordinates of the vector

$$\tfrac{a_k - a_{k+1}}{a_n}(\overline{e}_k - \overline{e}_{k+1}) + \overline{e}_n.$$

The matrix A_{n-1} coincides with the matrix of identity transformation except the last two columns. These columns contain the coefficients of the following two vectors respectively:

$$\sum_{j=1}^{n} a_j \overline{e}_j, \quad \text{and} \quad \sum_{j=1}^{n-2} \left(\tfrac{-a_j(a_{n-1}+1)}{a_n}\overline{e}_j \right) + \tfrac{1-a_{n-1}^2}{a_n}\overline{e}_{n-1} - a_{n-1}\overline{e}_n.$$

The matrix A_n coincides with the unit matrix except the second row. This row is as follows: $(-1, \ldots, -1, \tfrac{a_1 + \ldots + a_{n-1} - 1 + a_2}{a_n})$.

The determinants of all such matrices equal -1. So the corresponding affine transformations are lattice iff all the coefficients of all the matrices are lattice. The matrices A_k for $k \le n{-}2$ are lattice iff

$$a_1 \equiv a_2 \ldots \equiv a_{n-1} (\mathrm{mod}\, a_n).$$

Since $a_k < a_n$, we have the equalities. Suppose $a_1 = \ldots = a_{n-1} = p{-}1$ for some positive integer p. The matrix A_{n-1} is integer, iff $1{-}p^2 \equiv p(p{+}1) \equiv 0 (\mathrm{mod}\, a_n)$. Therefore, $r{+}1$ is divisible by a_n, and hence $a_n = p$. The matrix A_n is integer, iff $n(p{-}1){-}1$ is divisible by p, or equivalently $n{+}1$ is divisible by p.

So, we have already obtained that the symplex $S^n(a_1, \ldots, a_n)$ where $n > 3$ and $0 \le a_k < a_n$ for $k = 1, \ldots, n{-}1$ is lattice-regular iff it is coincides with some $\{3^{n-1}\}_p^L$ for some p dividing n. Since the lattice volume of S_p^n equals p, the above symplices are not lattice-congruent.

The second type of n-dimensional symplices. Here we study symplices (for $n > 3$) with the vertices

$$V_0 = O, \qquad V_k = O + \overline{e}_i, \text{ for } k = 1, \ldots, n{-}2,$$
$$V_{n-1} = O + (p-1)\sum_{i=1}^{n-2} \overline{e}_i + p\overline{e}_{n-1}, \quad \text{and} \quad V_n = O + \sum_{i=1}^{n} a_i \overline{e}_i,$$

denote such symplices by $S^n(p; a_1, \ldots, a_n)$. We also suppose, that all a_i are nonnegative integers satisfying $a_k < a_n$ for $k = 1, \ldots, n{-}1$, and $p \ge 2$. Let us show that all these symplices are not lattice-regular. Consider an arbitrary symplex $S^n(p; a_1, \ldots, a_n)$, satisfying the above conditions.

Consider the symmetry exchanging V_{n-1} and V_n. This transformation is linear. Its matrix coincide with the matrix of identity transformation, except for the last two columns. These columns contains the coefficients of the following two vectors respectively:

$$\sum_{j=1}^{n-2} \left(\tfrac{a_j+1}{p} - 1 \right) \overline{e}_j + \tfrac{a_{n-1}}{p}\overline{e}_{n-1} + \tfrac{a_n}{p}\overline{e}_n, \quad \text{and}$$
$$\sum_{j=1}^{n-2} \left(\tfrac{p(a_{n-1}-a_j+p-1)-a_{n-1}-a_j a_{n-1}}{pa_n}\overline{e}_j \right) + \tfrac{p^2-a_{n-1}^2}{pa_n}\overline{e}_{n-1} - \tfrac{a_{n-1}}{p}\overline{e}_n.$$

If this transformation is lattice-linear, then $a_2{+}1$ is divisible by p.

Consider the symmetry exchanging V_1 and V_n. This transformation is linear. Its matrix coincide with the matrix of identity transformation, except for the first column and the last two columns. These columns contains the coefficients of the following three vectors respectively:

$$\sum_{j=1}^{n} a_j \overline{e}_j, \qquad -\sum_{j=1}^{n} \left(\frac{a_j (p-1)}{p} \overline{e}_j \right) + \frac{p-1}{p} \overline{e}_1 + \overline{e}_{n-1}, \quad \text{and}$$

$$\frac{a_{n-1}p - pa_1 - a_{n-1} - p}{pa_n} \sum_{j=1}^{n} a_j \overline{e}_j + \frac{p + a_1}{pa_n} \overline{e}_1 + \overline{e}_n.$$

If this transformation is lattice-linear then a_2 is divisible by r.

Since a_2 and a_2+1 are divisible by r and $r \geq 2$, the symplex $S^n(p; a_1, \ldots, a_n)$ is not lattice-regular.

Conclusion of the proof.

Statement (i) is already proven. Since p is a lattice volume of S_p^n, Statement (ii) holds. We prove Statement (iii) of the proposition by the induction on the dimension n. For $n = 1, 2, 3$ the statement is already proven. Suppose that it is true for an arbitrary $n \geq 3$. Let us prove the statement for $n+1$.

Consider any lattice-regular $(n+1)$-dimensional lattice symplex S. Since it is lattice-regular, all its faces are lattice-regular. By the induction assumption there exist a positive integer p dividing $n+1$ such that the faces of S are lattice-congruent to $\{3^{n-1}\}_p^L$. Therefore, S is lattice-affine equivalent to the symplex $S^{n+1}(p; a_1, \ldots, a_{n+1})$, where $p \geq 1$, and $a_k < a_n$ for $k = 1, \ldots, n-1$. By the above cases the lattice-regularity implies, that $p = 1$, and that there exist a positive integer p' dividing $n+2$ such that the symplex $S^{n+1}(1; a_1, \ldots, a_{n+1})$ coincides with $\{3^n\}_{p'}^L$. This concludes the proof of the Statement (iii) for the arbitrary dimension.

Proposition 4.1 is proven. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. LATTICE-REGULAR LATTICE CUBES.

In this section we describe all lattice-regular lattice cubes for all integer dimensions.

Proposition 5.1. Cube$_1$. *All elementary lattice-regular one-dimensional lattice cubes are lattice segments of unit lattice length.*

Cube$_2$. *All elementary lattice-regular two-dimensional lattice cubes are lattice-congruent to $\{4\}_1^L$, or to $\{4\}_2^L$. The cubes $\{4\}_1^L$ and $\{4\}_2^L$ are not lattice-congruent.*

Cube$_n$ (for $n > 2$). *All elementary lattice-regular n-dimensional lattice cubes are lattice-congruent to $\{4, 3^{n-2}\}_1^L$, $\{4, 3^{n-2}\}_2^L$, or $\{4, 3^{n-2}\}_3^L$. The cubes $\{4, 3^{n-2}\}_1^L$, $\{4, 3^{n-2}\}_2^L$, and $\{4, 3^{n-2}\}_3^L$ are not lattice-congruent to each other.*

The three-dimensional cubes $\{4, 3^{n-2}\}_1^L$, $\{4, 3^{n-2}\}_2^L$, and $\{4, 3^{n-2}\}_3^L$ are shown on Figure 4.

We use the following two facts.

Lemma 5.2. *Any elementary lattice-regular n-dimensional lattice cube contains at most one lattice point in its interior. If the cube contains an interior lattice point, this point coincides with the intersection point of the diagonals of the cube.*
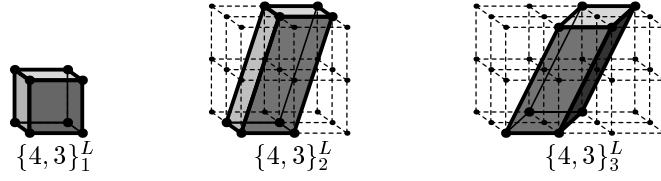
FIGURE 4. Three-dimensional elementary lattice-regular convex lattice cubes.

Proof. Suppose the converse is true. There exist a primitive n-dimensional cube with an interior lattice point A distinct to the intersections of the diagonals. Then there exist a lattice-linear reflection ξ of the cube above some $n-1$ dimensional plane, that do not preserve A. So the line of vector $\overline{A\xi(A)}$ coincide with the line containing one of the generating vectors of the cube. Hence, the one-dimensional faces of the cube are not elementary. Therefore, the cube is not elementary. We come to the contradiction. $\square$

Let us give the following important definition. Consider some hyperplane containing a sublattice of the lattice of corank 1 and a lattice point in the complement to this hyperplane. Let the Euclidean distance from the given point to the given hyperplane equals l. The minimal value of nonzero Euclidean distances from the points of the lattice to the hyperplane is denoted by l_0. The ratio l/l_0 is said to be the *lattice distance* from the given lattice point to the given lattice hyperplane.

Corollary 5.3. *The lattice distances from any vertex of any elementary lattice-regular n-dimensional lattice cube to any its $(n-1)$-dimensional face (that does not containing the given vertex) equals either 1, or 2.* $\square$

Proof of Proposition 5.1. The one-dimensional case is trivial. The two-dimensional case was described in Proposition 3.1. Let us study higher-dimensional cases.

The lattice cubes $\{4, 3^{n-2}\}_1^L$, $\{4, 3^{n-2}\}_2^L$, *and* $\{4, 3^{n-2}\}_3^L$. First, let us study the cases of the polytopes $\{4, 3^{n-2}\}_1^L$, $\{4, 3^{n-2}\}_2^L$, and $\{4, 3^{n-2}\}_3^L$ for any $n \geq 3$. Since lattice volumes of $\{4, 3^{n-2}\}_1^L$, $\{4, 3^{n-2}\}_2^L$, and $\{4, 3^{n-2}\}_3^L$ are $n!$, $2n!$, and $2^{n-1}n!$ respectively, the listed cubes are not lattice-congruent to each other. Let us prove that these polytopes are lattice-regular for any $n \geq 3$.

Since the vectors of $\{4, 3^{n-2}\}_1^L$ generate lattice, it is lattice-regular.

Now we study the case of $\{4, 3^{n-2}\}_2^L$. Denote by $\overline{v}_i$ the vector $\overline{e}_i$ for $i = 1, \ldots, n-1$ and by $\overline{v}_n$ the vector $\overline{e}_1 + \overline{e}_2 + \ldots + \overline{e}_{n-1} + 2\overline{e}_n$. The group of lattice symmetries of the cube $\{4, 3^{n-2}\}_2^L$ is generated by the linear operators A_k transposing the vectors $\overline{v}_i$ and $\overline{v}_{i+1}$ for $i = 1, \ldots, n-1$, and the last one: the composition of the symmetry A_n sending $\overline{v}_1$ to $-\overline{v}_1$ and preserving $\overline{v}_i$ for $i = 2, \ldots, n$ and the lattice shift on the vector $\overline{v}_1$.

Let us check that all the linear transformations A_k are lattice-linear. We show explicitly the matrices of A_k in the basis $\overline{e}_i$ for $i = 1, \ldots, n$. The matrix of A_k for $k = 1, \ldots, n-2$ coincides with the the matrix of the transposition of the vectors $\overline{e}_k$ and $\overline{e}_{k+1}$. The matrix of A_{n-1} coincides with the matrix of identity transformation except the last two columns.

These columns contain the coefficients of the vectors $\overline{v}_n$ and $\overline{e}_{n-1} + \overline{e}_n - \overline{v}_n$ respectively. The matrix of A_n coincides with the matrix of identity transformation except the first row, which is $(-1, 0, \ldots, 0, 1)$. Since all these matrices are in $SL(n, \mathbb{Z})$, the cube $\{4, 3^{n-2}\}_2^L$ is lattice-regular.

Let us consider now the case of $\{4, 3^{n-2}\}_3^L$. Put by definition $\overline{v}_1 = \overline{e}_1$, and $\overline{v}_i = \overline{e}_1 + 2\overline{e}_i$ for $i = 2, \ldots, n$. The group of lattice symmetries of the cube $\{4, 3^{n-2}\}_3^L$ is generated by the linear operators A_k transposing the vectors $\overline{v}_i$ and $\overline{v}_{i+1}$ for $i = 1, \ldots, n-1$, and the last one: the composition of the symmetry A_n sending $\overline{v}_1$ to $-\overline{v}_1$ and preserving $\overline{v}_i$ for $i = 2, \ldots, n$, and the lattice shift on the vector $\overline{v}_1$.

Let us check that all the linear transformations A_k are lattice-linear. The matrix of A_1 coincides with the matrix of identity transformation except the second row, which is $(2, -1, \ldots, -1)$. The matrix of A_k for $k = 2, \ldots, n-1$ coincides with the matrix transposing the vectors $\overline{e}_k$ and $\overline{e}_{k+1}$. The matrix of A_n coincides with the matrix of identity transformation except the first row, which is $(-1, 1, \ldots, 1)$. Since all these matrices are in $SL(n, \mathbb{Z})$, the cube $\{4, 3^{n-2}\}_3^L$ is lattice-regular.

Conclusion of the proof of Proposition 5.1. Now we prove, that any elementary lattice-regular n-dimensional lattice cube for $n \geq 3$ is lattice-congruent to $\{4, 3^{n-2}\}_1^L$, $\{4, 3^{n-2}\}_2^L$, or $\{4, 3^{n-2}\}_3^L$. by the induction on n.

The base of induction. Any face of any three-dimensional lattice-regular cube is lattice-regular.

Suppose, that the faces of three-dimensional lattice-regular cube C are lattice-congruent to $\{4\}_1^L$. Then C is lattice-congruent to the cube generated by the origin and the vectors $\overline{e}_1$, $\overline{e}_2$, and $a_1\overline{e}_1 + a_2\overline{e}_2 + a_3\overline{e}_3$. By Corollary 5.3 we can choose a_3 equals either 1 or 2. If $a_3 = 1$ then C is lattice-congruent to $\{4, 3\}_1^L$. If $a_3 = 2$, then we can choose a_1 and a_2 being 0, or 1. Direct calculations show, that the only possible case is $a_1 = 1$, and $a_2 = 0$, i.e. the case of $\{4, 3\}_2^L$.

Suppose now, that the faces of three-dimensional lattice-regular cube C are lattice-congruent to $\{4\}_2^L$. Then C is lattice-congruent to the cube generated by the origin and the vectors $\overline{e}_1$, $\overline{e}_1 + 2\overline{e}_2$, and $a_1\overline{e}_1 + a_2\overline{e}_2 + a_3\overline{e}_3$. By Corollary 5.3 we can choose a_3 equals either 1 or 2. Then we can also choose a_1 and a_2 being 0, or 1. Direct calculations show, that the only possible case $a_1 = 1$, $a_2 = 0$, and $a_3 = 2$ corresponds to $\{4, 3\}_3^L$.

The step of induction. Suppose that any elementary lattice-regular $(n-1)$-dimensional lattice cubes $(n > 3)$ are lattice-congruent to $\{4, 3^{n-3}\}_1^L$, $\{4, 3^{n-3}\}_2^L$, or $\{4, 3^{n-3}\}_3^L$. Let us prove that any elementary lattice-regular n-dimensional lattice cubes are lattice-congruent to $\{4, 3^{n-2}\}_1^L$, $\{4, 3^{n-2}\}_2^L$, or $\{4, 3^{n-2}\}_3^L$.

Any face of any lattice-regular cube is lattice-regular. Suppose, that the faces of $(n-1)$-dimensional lattice-regular cube C are lattice-congruent to $\{4, 3^{n-3}\}_1^L$. Then C is lattice-congruent to the cube C' generated by the origin and the vectors $\overline{v}_i = \overline{e}_i$ for $i = 1, \ldots, n-1$ and the vector $\overline{e}_n = a_1\overline{e}_1 + \ldots + a_n\overline{e}_n$.

By Corollary 5.3 we can choose a_n equals either 1 or 2. If $a_n = 1$, then the lattice volume of C is $n!$ and it is lattice-congruent to $\{4, 3^{n-2}\}_1^L$. If $a_n = 2$, then we can choose a_i being 0, or 1 for $i = 1, \ldots, n-1$. Consider a symmetry of C' transposing the vectors $\overline{v}_k$

and $\overline{v}_{k+1}$ for $k = 1, \ldots, n-2$. This transformation is linear and its matrix coincides with the matrix of the transposition of the vectors $\overline{e}_k$ and $\overline{e}_{k+1}$, except the last column. The n-th column contains the coordinates of the vector

$$\tfrac{a_k - a_{k+1}}{2}\left(\overline{e}_k - \overline{e}_{k+1}\right) + \overline{e}_n.$$

Since the transformation is lattice,

$$a_k \equiv a_{k+1}(\mathrm{mod}\,2) \text{ for } k = 1, \ldots, n-2.$$

Since any a_i is either zero or unit, the above imply $a_1 = a_2 = \ldots = a_{n-1}$. If $a_1 = 0$, then the vector $\overline{v}_n$ is not of the unit lattice length, but the vector $\overline{v}_1$ is of the unit length, so C' is not lattice-regular. If $a_1 = 1$ then C' coincides with $\{4, 3^{n-2}\}_2^L$.

Suppose, that the faces of $(n-1)$-dimensional lattice-regular cube C are lattice-congruent to $\{4, 3^{n-3}\}_2^L$. Then C is lattice-congruent to the cube C' generated by the origin and the vectors $\overline{v}_i = \overline{e}_i$, for $i = 1, \ldots, n-2$, $\overline{v}_{n-1} = \overline{e}_1 + \ldots + \overline{e}_{n-2} + 2\overline{e}_{n-1}$, and $\overline{v}_n = a_1\overline{e}_1 + \ldots + a_n\overline{e}_n$.

Consider a symmetry of C' transposing the vectors $\overline{v}_{n-1}$ and $\overline{v}_n$. This transformation is linear and its matrix coincides with the matrix of identity transformation except the last two columns. These columns contain the coefficients of the following two vectors respectively:

$$\sum_{j=1}^{n} \tfrac{a_j}{2}\overline{e}_j - \tfrac{1}{2}\sum_{j=1}^{n-2} \overline{e}_j, \quad \text{and} \quad \sum_{j=1}^{n-2}\left(\tfrac{a_{n-1} - 2a_j - a_{n-1}a_j + 2}{2a_n}\overline{e}_j\right) + \tfrac{4 - a_{n-1}^2}{2a_n}\overline{e}_{n-1} - \tfrac{a_{n-1}}{2}\overline{e}_n.$$

Since the described transformation is lattice, the integers a_{n-1} and a_n are even, and a_{n-2} is odd.

Consider now a symmetry of C' transposing the vectors $\overline{v}_{n-2}$ and $\overline{v}_{n-1}$. This transformation is linear and its matrix coincides with the matrix of identity transformation except the last three columns. These columns contain the coefficients of the following three vectors respectively:

$$\overline{v}_{n-1}, \quad -\sum_{j=1}^{n-3} \overline{e}_j - \overline{e}_{n-1}, \quad \text{and} \quad \tfrac{a_{n-1} - a_{n-2}}{a_n}\sum_{j=1}^{n-3} \overline{e}_j + 2\tfrac{a_{n-1} - a_{n-2}}{a_n}\overline{e}_{n-1} + \overline{e}_n.$$

Since the described transformation is lattice, the integer $a_{n-1} - a_{n-2}$ is even, and thus a_{n-2} is even. We come to the contradiction with the divisibility of a_{n-2} by 2. So C is not lattice-regular.

Suppose, that the faces of $(n-1)$-dimensional lattice-regular cube C are lattice-congruent to $\{4, 3^{n-3}\}_3^L$. Then C is lattice-congruent to the cube C' generated by the origin and the vectors $\overline{v}_1 = \overline{e}_1$, $\overline{v}_i = \overline{e}_1 + 2\overline{e}_i$ for $i = 2, \ldots, n-1$, and $\overline{v}_n = a_1\overline{e}_1 + \ldots + a_n\overline{e}_n$. By Corollary 5.3 we can choose a_n equals either 1 or 2. Then we choose a_i being 0, or 1 for $i = 1, \ldots n-1$.

Consider a symmetry of C' transposing the vectors $\overline{v}_{n-1}$ and $\overline{v}_n$. This transformation is linear and its matrix coincides with the matrix of identity transformation except the

last two columns. These columns contain the coefficients of the following two vectors respectively:

$$-\tfrac{1}{2}\overline{e}_1 + \sum_{j=1}^{n} \tfrac{a_j}{2}\overline{e}_j, \quad \text{and} \quad \tfrac{(1-a_1)(2+a_{n-1})}{2a_n}\overline{e}_1 - \sum_{j=2}^{n-2}\left(\tfrac{a_j(a_{n-1}+2)}{2a_n}\overline{e}_j\right) + \tfrac{4-a_{n-1}^2}{2a_n}\overline{e}_{n-1} - \tfrac{a_{n-1}}{2}\overline{e}_n.$$

Since the described transformation is lattice, the integer a_1 is odd, and the integers a_i for $i = 1, \ldots n$ are even. Thus

$$a_n = 2, \qquad a_1 = 1, \quad \text{and} \quad a_2 = \ldots = a_{n-1} = 0.$$

Therefore C' coincides with $\{4, 3^{n-2}\}_3^L$.

We have already studied all possible n-dimensional cases. This proves the statement for the dimension n.

All statements of Proposition 5.1 are proven. $\qquad\square$

6. LATTICE-REGULAR LATTICE GENERALIZED OCTAHEDRA.

In this section we describe all lattice-regular lattice generalized octahedra for all integer dimensions greater than 2.

Proposition 6.1. *All elementary lattice-regular n-dimensional lattice generalized octahedra for $n \geq 3$ are lattice-congruent to $\{3^{n-2}, 4\}_1^L$, $\{3^{n-2}, 4\}_2^L$, or $\{3^{n-2}, 4\}_3^L$. The generalized octahedra $\{3^{n-2}, 4\}_1^L$, $\{3^{n-2}, 4\}_2^L$, and $\{3^{n-2}, 4\}_3^L$ are not lattice-congruent to each other.*

We show the (three-dimensional) octahedra $\{3, 4\}_1^L$, $\{3, 4\}_2^L$, and $\{3, 4\}_3^L$ on Figure 5.
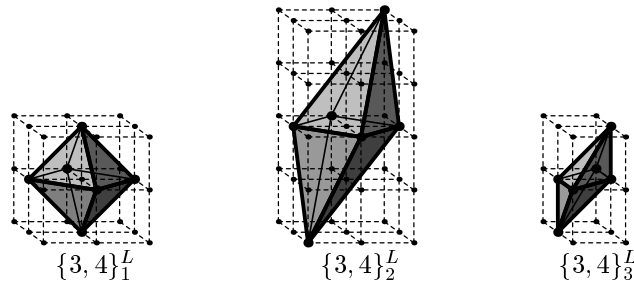

$\{3,4\}_1^L$ $\qquad\qquad$ $\{3,4\}_2^L$ $\qquad\qquad$ $\{3,4\}_3^L$

FIGURE 5. Three-dimensional elementary lattice-regular convex lattice tetrahedra.

Proof. Consider an arbitrary elementary n-dimensional lattice-regular generalized octahedron P. Let the vertices $V_1, \ldots, V_{2n}$ of P be enumerated in such a way that for any positive integer $i \leq n$ there exist a lattice symmetry exchanging V_i and V_{i+n} and preserving any other vertex. So, $V_i V_j V_{i+n} V_{j+n}$ is a lattice-regular square for $i \neq j$. Therefore, the midpoints of the segments $V_i V_{i+n}$ coincide for $i = 1, 2, \ldots, n$. Denote the common midpoint of the segments $V_i V_{i+n}$ by A.

Suppose the point A is lattice. Consider the lattice cube with the vertices $A \pm \overline{AV_1} \pm \overline{AV_2} \pm \ldots \pm \overline{AV_n}$, we denote it by $C(P)$. The cube $C(P)$ is also lattice-regular.

Note that the lattice-regular generalized octahedra P' and P'' have lattice-congruent cubes $C(P')$ and $C(P'')$ iff P' and P'' are lattice-congruent.

Since P is elementary, the segment AV_1 is of unit lattice length. Therefore, the cube $C(P)$ is lattice-congruent to the 2-multiple of some $\{4, 3^{n-2}\}_k^L$, for $k = 1, 2, 3$. If $C(P)$ is lattice-congruent to the 2-multiple of $\{4, 3^{n-2}\}_1^L$, or to the 2-multiple of $\{4, 3^{n-2}\}_2^L$, then P is lattice-congruent to $\{3^{n-2}, 4\}_1^L$, or to $\{3^{n-2}, 4\}_2^L$, respectively. If $C(P)$ is lattice-congruent to the 2-multiple of $\{4, 3^{n-2}\}_3^L$, then P is not elementary.

Suppose now, the common midpoint A of the diagonals is not lattice. If the lattice length of V_1V_{n+1} equals $2k+1$ for some positive k, then the generalized octahedron P is not elementary. Suppose the segment V_1V_{n+1} is of unite lattice length. Consider a 2-multiple to the polygon P and denote in by $2P$. Since the segment V_1V_{n+1} is of unit lattice length, the cube $C(2P)$ is the 2-multiple of some $\{4, 3^{n-2}\}_k^L$, for $k = 1, 2, 3$. If $C(2P)$ is lattice-congruent to the 2-multiple of $\{4, 3^{n-2}\}_1^L$, or of $\{4, 3^{n-2}\}_2^L$, then P is not a lattice polytope. If $C(2P)$ is lattice-congruent to the 2-multiple of $\{4, 3^{n-2}\}_3^L$, then P is lattice-congruent to $\{3^{n-2}, 4\}_3^L$.

The generalized octahedra $\{3^{n-2}, 4\}_1^L$, $\{3^{n-2}, 4\}_2^L$, and $\{3^{n-2}, 4\}_3^L$ are lattice-regular, since so are the cubes $C(\{3^{n-2}, 4\}_1^L)$, $C(\{3^{n-2}, 4\}_2^L)$, and $C(2\{3^{n-2}, 4\}_3^L)$.

The generalized octahedra $\{3^{n-2}, 4\}_1^L$, $\{3^{n-2}, 4\}_2^L$, and $\{3^{n-2}, 4\}_3^L$ are not lattice-congruent to each-other, since the corresponding elementary cubes $C(\{3^{n-2}, 4\}_1^L)$, $C(\{3^{n-2}, 4\}_2^L)$, and $C(2\{3^{n-2}, 4\}_3^L)$ are not lattice-congruent. $\square$

7. Proof of Theorem 2.2.

In this section we obtain proof of Theorem 2.2 by combining the results of propositions from the previous sections and describing the remaining low-dimensional cases.

Consider any convex lattice-regular lattice polytope. Since it is lattice-regular and convex it is combinatorially regular. Therefore, by Theorem B it is combinatorially isomorphic to one of the Euclidean polytopes of Theorem A. In Section 3 we gave the description of the two-dimensional case. In Sections 4, 5, and 6 we studied the cases of lattice-regular polytopes combinatorially isomorphic to regular symplices ($\{3^{n-1}\}$), regular cubes ($\{4, 3^{n-2}\}$), and regular generalized octahedra ($\{3^{n-2}, 4\}$) respectively.

Now we will study the remaining special cases of three- and four-dimensional regular polytopes.

7.1. Three-dimensional icosahedra and dodecahedra. We have already classified all lattice-regular elementary tetrahedra, cubes, and octahedra. There is no lattice-regular dodecahedron, since there is no lattice-regular pentagon. There is no lattice-regular icosahedron, since there is no lattice-affine transformation with a fixed point of order 5.

So, the classification in the three-dimensional case is completed.

7.2. Four-dimensional 24-sells, 120-sells, and 600-sells. *The case of 24-sell.* Suppose that P is a lattice-regular 24-sell. It contains 16 vertices such that the subgroup of the group of all lattice-symmetries of P preserving these 16 vertices is isomorphic to the group of the symmetries of the four-dimensional cube. So we can naturally define a combinatorial-regular cube associated with this 16 vertices. Any two-dimensional face of this cube is an Euclidean parallelogram, since such face is the diagonal section containing four lattice points of some lattice-regular octahedron. If all two-dimensional faces are parallelograms, then all three-dimensional faces are parallelepipeds and these 16 vertices are vertices of a four-dimensional parallelepiped. Denote this parallelepiped by C. Since all transformations of C are lattice-affine, the polytope C is a lattice-regular cube. (Note that for any 24-sell there exist exactly three such (distinct) cubes).

Suppose that C is a lattice-regular cube generated by the origin and some vectors $\overline{v}_i$ for $i = 1, 2, 3, 4$. Consider also the coordinates $(*, *, *, *)_v$ corresponding to this basis. Let the point $(a_1, a_2, a_3, a_4)_v$ of P connected by edges with the vertices of C is in the plane with the unit last coordinate. Then, the point $(2-a_1, a_2, a_3, 2-a_4)_v$ is also a vertex. Thus, the point (a_1, a_2, a_3, a_4-1) is also a vertex. Note that the points $(a_1, a_2, a_3, a_4)_v$ and $(a_1, a_2, a_3, a_4-2)_v$ are symmetric about the center of the cube: $(1/2, 1/2, 1/2, 1/2)_v$. So $a_4-1/2 = 1/2-a_4-2$. Thus, $a_4 = 3/2$. Similar calculations show that $a_1 = a_2 = a_3 = 1/2$. Therefore, the eight points do not contained in C coincide with the following points:

$$O + 1/2(\overline{v}_1+\overline{v}_2+\overline{v}_3+\overline{v}_4) \pm \overline{v}_i, \quad \text{for } i = 1, 2, 3, 4.$$

Consider a lattice-regular primitive 24-sell P and C one of the corresponding cubes. Since the edges of C are the edges of P, the cube C is also primitive. Let us study all three possible cases of lattice-affine types of C.

Suppose C coincides with $\{4, 3, 3\}_1^L$. Then the remaining points

$$O + 1/2(\overline{e}_1+\overline{e}_2+\overline{e}_3+\overline{e}_4) \pm \overline{e}_i, \quad \text{for } i = 1, 2, 3, 4$$

are not lattice. Therefore, the case of $\{4, 3, 3\}_1^L$ is impossible.

In the case of $\{4, 3, 3\}_2^L$ and $\{4, 3, 3\}_3^L$ all the vertices are lattice and in our notation coincide with $\{3, 4, 3\}_1^L$ and $\{3, 4, 3\}_2^L$ respectively. Straightforward calculations shows, that both resulting 24-sells are lattice-regular.

The case of 120-sell. The 120-sell is non-realizable as an lattice-regular lattice polytope, since its two-dimensional faces should be lattice-regular lattice pentagons. By the above, lattice-regular lattice pentagons are not realizable.

The case of 600-sell. Consider an arbitrary polytope with topological structure of 600-sell having one vertex at the origin O. Let OV be some edge of this 600-sell. The group of symmetries of an abstract 600-sell with fixed vertex O is isomorphic to the group of symmetries of an abstract icosahedron. The group of symmetries of an abstract 600-sell with fixed vertices O and V_1 is isomorphic to the group of symmetries of an abstract pentagon. So, there exist a symmetry A of the 600-sell of order 5 preserving the vertices O and V_1. If the polytope P is lattice-regular, then this symmetry is lattice-linear. Since A^5 is the identity transformation and the space is four-dimensional, the characteristic polynomial of A in the variable x is either $x-1$ or $x^4+x^3+x^2+x+1$. Since $A(\overline{OV_1}) = \overline{OV_1}$,

the characteristic polynomial of A is divisible by $x-1$. If the characteristic polynomial is $x-1$, then the operator A is the identity operator of order 1 and not of order 5. Therefore, there is no lattice-regular lattice polytope with the combinatorial structure of the 600-sell.

7.3. Conclusion of the proof of Theorem 2.2. We have studied all possible combinatorical cases of lattice-regular polytopes. The proof of Theorem 2.2 is completed. □

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