

# Homology Isomorphism of the Complex of 2-Connected Graphs and the Graph-complex of Trees

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In this paper we prove that the homology of the complex of 2-connected graphs is naturally isomorphic to the homology of the graph-complex of trees. Both complexes are connected with combinatorics of knot spaces. More precisely the first complex appears in the spectral approach to the calculation of the homology of the space of knots in  $\mathbf{R}^n$ ,  $n \geq 3$  (see [V2, V4]). The homology of the second complex has a natural interpretation in the bialgebra of Chinese diagrams (see [BN]). This bialgebra turned out to be a very useful tool in the investigation of the space of finite order knot invariants. The isomorphism in question provides a connection between the two mentioned approaches.

Let  $D$  be a set of cardinality  $n$ . We shall consider loopless graphs without multiple edges on the vertex set  $D$ . A connected graph  $\Gamma$  is called *2-connected* if any graph obtained from  $\Gamma$  by deleting some vertex and its adjacent edges is connected.

We assign a simplex with  $n(n-1)/2$  vertices to the set  $D$ . Any pair of distinct points in our set corresponds to a vertex of the simplex. Then each face (subsimplex) of the simplex is assigned to some graph on the set  $D$  (the edges of the graph correspond to the vertices of the subsimplex).

The union of all faces that correspond to the graphs that are not connected (resp. not 2-connected) is a simplicial complex. Factorizing the complex of all faces by one of these two subcomplexes, we obtain the complexes  $\Delta_D$  (resp.  $\Delta_D^2$ ) of the connected (resp. 2-connected) graphs.

We consider these complexes as pointed CW-complexes, where the marked point  $*$  is always the image of a subcomplex of factorization. We define factorization by the empty set as the adding of one point  $*$ .

Both these two complexes have been introduced by V. Vassiliev.

Evidently, the geometry of these complexes depends only on the cardinality  $n$ . Set  $M_n = \{1, 2, \dots, n\}$  and denote  $\Delta_{M_n}$  (resp.  $\Delta_{M_n}^2$ ) by  $\Delta_n$  (resp.  $\Delta_n^2$ ).

The homotopical type of the complexes has a nice description as follows.

**THEOREM 1** [B&BWA, V1]. The complex  $\Delta_n$  is homotopy-equivalent to a wedge of  $(n-1)!$  spheres of dimension  $n-2$ .  $\square$

**THEOREM 2** [BBLSW, T]. The complex  $\Delta_n^2$  is homotopy-equivalent to a wedge of  $(n-2)!$  spheres of dimension  $2n-4$ .  $\square$

We refer the reader to [K, W] for various definitions of the graph-complex of trees, but we will describe its homology, that is not trivial only in one gradation.

For the set  $D$  we consider the set of all trees, that have  $n$  univalent vertices (leaves) in correspondence with the elements of  $D$ ; we also suppose that all the other vertices are 3-valent. In every such tree we fix orientations of 3-valent edges (cyclic order of adjacent edges). These trees are called binary.

Consider the  $\mathbf{Z}$ -module  $T_D$  (in fact  $T_D$  is the homology of the graph-complex of trees), that is generated by all binary trees, and the relations are of the two following types:

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relation of  
the type  $AS$

$$\text{relation of the type } AS \quad \text{circle with 3 lines} + \text{circle with 3 lines and arrow} = 0$$

relation of  
the type  $IHX$

$$\text{relation of the type } IHX \quad \text{circle with 2 vertical and 1 horizontal line} - \text{circle with 2 vertical and 1 horizontal line} + \text{circle with 2 crossing lines and 1 horizontal line} = 0$$

(absence of an arrow that would orientate a 3-valent vertex means a clock-wise orientation; outside the circles the trees are supposed to coincide).

We define  $T_n = T_{M_n}$  similarly, where  $M_n = \{1, 2, \dots, n\}$ .

**THEOREM 3 [W].**  $T_n \simeq \mathbf{Z}^{(n-2)!}$ . As  $S_n$ -modules  $T_n \otimes \mathbf{C} \simeq \text{Ind}_{\mathbf{Z}_{n-1}}^{S_n} \chi / \text{Ind}_{\mathbf{Z}_n}^{S_n} \psi$ , where  $\mathbf{Z}_{n-1}$  (resp.  $\mathbf{Z}_n$ ) is a subgroup of the symmetric group  $S_n$  generated by a cycle of the length  $n-1$  (resp.  $n$ ),  $\chi$  and  $\psi$  are their primitive characters.  $\square$

**MAIN THEOREM.**  $T_n \simeq H^{2n-4}(\Delta_n^2, \mathbf{Z})$  as  $S_n$ -modules.

**COROLLARY.**  $H^{2n-4}(\Delta_n^2, *; \mathbf{C}) \simeq \text{Ind}_{\mathbf{Z}_{n-1}}^{S_n} \chi / \text{Ind}_{\mathbf{Z}_n}^{S_n} \psi$  as  $S_n$ -modules.

*Remark 1.* For the first time the result in the corollary was obtained in [BBLSW] by calculation of the characters of the  $S_n$ -action.

Before proving the theorem we introduce some complementary notions and prove some auxiliary results.

Consider a decomposition of the set  $M_n$  into a (not necessary disjoint) union of its subsets (of cardinality  $\geq 2$ )  $s_1, s_2, \dots, s_k$ . Using this object we construct a certain graph of inclusions, that will have  $k$  black vertices (corresponding to the subsets  $s_1, s_2, \dots, s_k$ ) and  $n$  white ones (corresponding to the elements of  $M_n$ ). Edges of this graph can join only vertices of different colours, and it is supposed that an edge joins the white vertex number  $i$  with the black vertex number  $j$  if and only if  $i \in s_j$ .

**DEFINITION 1.** A system of subsets  $s_1, s_2, \dots, s_k$  is called a *structure (of multiplicity  $k$ )*, if the obtained graph of inclusions is a tree.

**DEFINITION 2.** Two sets are called *touching*, if their intersection consists of exactly one element. We say also, that one set *touches* the other.

**LEMMA 1.**  $(s_1, s_2, \dots, s_k)$  is a structure if and only if there exists a permutation  $\sigma \in S_k$ , such that for any  $i = 1, 2, \dots, k-1$ , the set  $s_{\sigma(i+1)}$  touches  $\cup_{j=1}^i s_{\sigma(j)}$ .

*Proof of lemma 1.* Obvious.  $\square$

Let us show that any connected graph  $\Gamma$  on  $M_n$  defines naturally some structure  $\Upsilon(\Gamma)$ .

**DEFINITION 3.** We say that a subset  $s \subset M_n$  is *proper* with respect to a graph  $\Gamma$ , if the restriction of the graph  $\Gamma$  on the subset  $s$  is a 2-connected graph. A proper set is called *maximal* (with respect to the graph  $\Gamma$ ), if it is not contained in another proper set.

It is easy to see that the system  $\Upsilon(\Gamma)$  of all maximal subsets is a structure (connectedness of the resulting "black-and-white" graph follows from the connectedness of the initial graph, absence of cycles — from the property of maximality (def. 3)).

*Example 1.* If the initial graph is 2-connected then there is only one maximal set that coincides with  $M_n$ .

*Example 2.* If the initial graph is a tree then each of the maximal sets has exactly 2 points and corresponds to a certain edge of the graph.

Now let us consider the complex of connected graphs  $\Delta_n$  and a decreasing filtration on it:

$$\Delta_n = Y_1 \supset Y_2 \supset \dots \supset Y_n = *.$$

Here  $Y_i$  is generated by the graphs that have a structure of multiplicity  $\geq i$ .

Let  $\Upsilon = (s_1, s_2, \dots, s_k)$  be a structure of multiplicity  $k$ . Let us consider the subcomplex  $Y_\Upsilon$  (of the complex  $Y_k/Y_{k+1}$ ) generated by all graphs  $\Gamma$ , such that  $\Upsilon(\Gamma) = \Upsilon$ . Any such graph is a union of  $k$  2-connected graphs respectively on the sets  $s_1, s_2, \dots, s_k$  (by a union of graphs we mean a union of the sets of their edges).

LEMMA 2. The complex  $Y_\Upsilon$  is  $(k-1)$ -fold suspension (in the category of pointed sets; in other words we factorize a usual suspension by the segment over a marked point) of the tensor product of the  $k$  complexes of 2-connected graphs respectively on the sets  $s_1, s_2, \dots, s_k$ :

$$Y_\Upsilon = \Sigma^{k-1}(\#_{i=1}^k \Delta_{s_i}^2)$$

(tensor product of two spaces is  $X \# Y = (X \times Y)/(X \vee Y)$ ).

*Proof of lemma 2.* The lemma is an obvious consequence of the previous considerations and of the fact, that if  $X$  and  $Y$  are two CW-complexes,  $X_1 \subset X$ ,  $Y_1 \subset Y$  — their subcomplexes, then

$$(X * Y) / ((X_1 * Y) \cup (X * Y_1)) \simeq \Sigma((X \times Y)/(X_1 \times Y) \cup (X \times Y_1)) \simeq \Sigma((X/X_1) \# (Y/Y_1)).$$

The symbol  $*$  means a join,  $\Sigma$  means a suspension (in the category of pointed sets).  $\square$

*Remark 2.* The complex  $Y_\Upsilon$  can be described in the following way:

$$Y_\Upsilon \simeq (\Delta^{k-1} \times \#_{i=1}^k \Delta_{s_i}^2) / (\partial \Delta^{k-1} \times \#_{i=1}^k \Delta_{s_i}^2).$$

Here  $\Delta^{k-1}$  is a simplex of dimension  $k-1$ , whose  $k$  vertices correspond to the sets  $s_1, s_2, \dots, s_k$ . In particular the orientation of the simplex  $\Delta^{k-1}$  changes after renumeration of the sets depending on the sign of a permutation.

LEMMA 3.  $Y_k/Y_{k+1}$  is a wedge of the complexes  $Y_\Upsilon$  over all structures  $\Upsilon$  of multiplicity  $k$ .

*Proof of lemma 3.* Obvious.  $\square$

*Remark 3.* It follows from lemma 1 that  $\sum_{i=1}^k \#s_i = n + k - 1$  ( $\#$  is the cardinality), and hence,  $H^j(Y_\Upsilon, *) \neq 0$  if and only if  $j = \sum_{i=1}^k (2\#s_i - 4) + k - 1 = 2n + 2k - 2 - 4k + k - 1 = 2n - k - 3$ . Besides that  $H^{2n-k-3}(Y_\Upsilon) \simeq \bigotimes_{i=1}^k H^{2\#s_i-4}(\Delta_{s_i}^2, *)$ . Then any collection of elements  $\alpha_1, \alpha_2, \dots, \alpha_k$  respectively in the groups  $H^{2\#s_i-4}(\Delta_{s_i}^2, *)$ ,  $i = 1, \dots, k$  defines a cycle in  $H^{2n-k-3}(Y_\Upsilon, *)$  and therefore a cycle in  $H^{2n-k-3}(Y_k, Y_{k+1})$ . We denote this cycle by  $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k$ . This notation is relevant, because the following holds:

$$\alpha_{\sigma(1)} \wedge \alpha_{\sigma(2)} \wedge \dots \wedge \alpha_{\sigma(k)} = \text{sgn}(\sigma) \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k,$$

for all  $\sigma \in S_k$  (by remark 2 and by the fact that all the  $\alpha_i$ ,  $i = 1, \dots, k$ , belong to even dimensional homology).

Let us consider a spectral sequence associated with the filtration

$$* = Y_n \subset Y_{n-1} \subset \dots \subset Y_1 = \Delta_n,$$

and calculating the groups  $H^*(\Delta_n, *)$ .

LEMMA 4. a) The first term  $E_1$  of this spectral sequence has only one non trivial line.

b) The spectral sequence degenerates in the second term.

c) The first differential  $d_1$  in this spectral sequence is acyclic everywhere but in the term  $Y_n$  of the filtration.

*Proof of lemma 4.* a) Follows from remark 3.

b) Follows from ).

c) Follows from ) and theorem 1.  $\square$

Let  $\Upsilon = (s_1, s_2, \dots, s_k)$  be a structure,  $s_i$  and  $s_j$  be two touching sets,  $\alpha_i$  and  $\alpha_j$  be elements respectively in  $H^{2\#s_i-4}(\Delta_{s_i}^2, *)$ ,  $H^{2\#s_j-4}(\Delta_{s_j}^2, *)$ . We consider also the spectral sequence associated with the same filtration on the complex of connected graphs on the set  $s_i \cup s_j$ . Let  $d_1$  be (by abuse of the language) the first differential in it. Define  $[\alpha_i, \alpha_j] = d_1(\alpha_i \wedge \alpha_j) \in H^{2\#(s_i \cup s_j)-4}(\Delta_{s_i \cup s_j}^2, *)$ . If  $s_i \cap s_j = \emptyset$ , then for all  $\alpha_i \in H^{2\#s_i-4}(\Delta_{s_i}^2, *)$ ,  $\alpha_j \in H^{2\#s_j-4}(\Delta_{s_j}^2, *)$  we set  $[\alpha_i, \alpha_j] = 0$ .

Let us return to the initial spectral sequence. Let  $d_1$  be its first differential.

LEMMA 5.  $d_1(\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k) = \sum_{i < j} (-1)^{(i+j-1)} [\alpha_i, \alpha_j] \wedge \alpha_1 \wedge \dots \hat{\alpha}_i \dots \hat{\alpha}_j \dots \wedge \alpha_k$  (a "hat" over an element means its absence).

*Proof of lemma 5.* It follows from lemmas 2 and 3.  $\square$

Let us define a similar "commutator" on the spaces of binary trees. Let  $s_1$  and  $s_2$  be two touching sets and let  $s_1 \cap s_2 = \{i\}$ . Now we are going to construct a map

$$[\cdot, \cdot] : T_{s_1} \otimes T_{s_2} \rightarrow T_{s_1 \cup s_2}.$$

For binary trees  $T_j \in T_{s_j}$ ,  $j = 1, 2$ , we define  $[T_1, T_2]$  as follows. We glue the trees  $T_1$  and  $T_2$  at the point  $i$ . So we get there two adjacent edges. Then we add the third one. Finally we have a new 3-valent vertex (the former point  $i$ ) and a new univalent vertex (the second endpoint of the constructed edge). The new univalent vertex inherits the number  $i$ ; the new 3-valent vertex gets the following orientation (cyclic order of the adjacent edges): as the first we take the edge going to the tree  $T_1$ , then the just constructed edge and then the edge going to the tree  $T_2$ .

Obviously, this operation can be correctly defined on the whole group  $T_{s_1} \otimes T_{s_2}$  and this operation is antisymmetric, that is  $[T_1, T_2] = -[T_2, T_1]$  (by the AS-relation).

If  $s_1 \cap s_2 = \emptyset$ , then the "commutator" is, by definition, identically zero on  $T_{s_1} \otimes T_{s_2}$ .

Now let us prove the theorem.

*Proof of the main theorem.* We are going to construct isomorphisms (that agree with  $S_n$ -action)  $\lambda_n : H^{2n-4}(\Delta_n^2, *) \rightarrow T_n$ ,  $n \geq 2$ . We do it by induction and in a way compatible with the "commutator". That is, let  $s_1$  and  $s_2$  touch each other,  $\lambda_{s_1}$ ,  $\lambda_{s_2}$ ,  $\lambda_{s_1 \cup s_2}$  be the isomorphisms between the homologies of the complexes of 2-connected graphs and the spaces of binary trees on the sets  $s_1$ ,  $s_2$ ,  $s_1 \cup s_2$  respectively (these isomorphisms are uniquely defined in a natural way by the isomorphisms  $\lambda_{\#s_1}$ ,  $\lambda_{\#s_2}$ ,  $\lambda_{\#(s_1 \cup s_2)}$ , because the latters agree with the action of the symmetric group). Let  $\alpha_i \in H^{2\#s_i-4}(\Delta_{s_i}^2, *)$ ,  $i = 1, 2$ . We demand

$$\lambda_{s_1 \cup s_2}([\alpha_1, \alpha_2]) = [\lambda_{s_1}(\alpha_1), \lambda_{s_2}(\alpha_2)].$$

Let us define the isomorphism  $\lambda_2$ . There is only one (non-trivial) graph on two points. This graph is 2-connected and provides the only cycle in  $H^0(\Delta_2^2, *)$ . Let the map  $\lambda_2$  take it to the only binary tree on two points (Note that this tree is also a segment). It is easy to see that the property of the compatibility with the "commutator" defines the unique map  $\lambda_3 : H^2(\Delta_3^2, *) \rightarrow T_3$ , that turns out to be an isomorphism of one-dimensional  $S_3$ -modules that realize the sign representation.

Now let  $n \geq 4$ . We have constructed already the isomorphisms  $\lambda_2, \lambda_3, \dots, \lambda_{n-1}$ , that are compatible with the "commutator". First of all we construct a certain map

$$\rho : H^{2n-5}(Y_2, Y_3) \rightarrow T_n.$$

If  $(s_1, s_2)$  is a structure (of multiplicity 2),  $\alpha_i \in H^{2\#s_i-4}(\Delta_{s_i}^2, *)$ ,  $i = 1, 2$ , then  $\alpha_1 \wedge \alpha_2 \in H^{2n-5}(Y_2, Y_3)$ . We set

$$\rho(\alpha_1 \wedge \alpha_2) = [\lambda_{s_1}(\alpha_1), \lambda_{s_2}(\alpha_2)].$$

Evidently,  $\rho$  can be extended correctly to the whole  $H^{2n-5}(Y_2, Y_3)$ , because  $\alpha_2 \wedge \alpha_1 = -\alpha_1 \wedge \alpha_2$ , and because the "commutator" on the spaces of binary trees is antisymmetric. Now let us check that  $\rho$  takes  $\text{Im}d_1(H^{2n-6}(Y_3, Y_4))$  to zero.

We have to show that, if  $(s_1, s_2, s_3)$  is a structure (of multiplicity 3) on  $M_n$ ,  $\alpha_i \in H^{2\#s_i-4}(\Delta_{s_i}^2, *)$ ,  $i = 1, 2, 3$ , then  $\rho(d_1(\alpha_1 \wedge \alpha_2 \wedge \alpha_3)) = 0$ . We have

$$\begin{aligned} \rho(d_1(\alpha_1 \wedge \alpha_2 \wedge \alpha_3)) &= \rho([\alpha_1, \alpha_2] \wedge \alpha_3) + \rho([\alpha_2, \alpha_3] \wedge \alpha_1) + \rho([\alpha_3, \alpha_1] \wedge \alpha_2) = \\ &= [\lambda_{s_1 \cup s_2}([\alpha_1, \alpha_2]), \lambda_{s_3}(\alpha_3)] + [\lambda_{s_2 \cup s_3}([\alpha_2, \alpha_3]), \lambda_{s_1}(\alpha_1)] + [\lambda_{s_3 \cup s_1}([\alpha_3, \alpha_1]), \lambda_{s_2}(\alpha_2)] = \\ &= [[\lambda_{s_1}(\alpha_1), \lambda_{s_2}(\alpha_2)], \lambda_{s_3}(\alpha_3)] + [[\lambda_{s_2}(\alpha_2), \lambda_{s_3}(\alpha_3)], \lambda_{s_1}(\alpha_1)] + [[\lambda_{s_3}(\alpha_3), \lambda_{s_1}(\alpha_1)], \lambda_{s_2}(\alpha_2)] \end{aligned}$$

(the first equality holds by lemma 5, the second one by the definition of  $\rho$ , the third one is the expression of the compatibility with the "commutator").

Let us prove the "Jacobi's identity" for the spaces of binary trees, that is

$$[[T_1, T_2], T_3] + [[T_2, T_3], T_1] + [[T_3, T_1], T_2] = 0.$$

for any trees  $T_i \in T_{s_i}$ ,  $i = 1, 2, 3$ .

If  $s_1 \cap s_2 \cap s_3 = \emptyset$ , then it is evident, because one of the summands equals zero and the other two provide the same trees but with different signs. The case when  $s_1 \cap s_2 \cap s_3 \neq \emptyset$  (and hence consists of one point) follows from the  $IHX$ -relation.

Finally we obtain  $\rho \equiv 0$  on  $\text{Im}d_1(H^{2n-6}(Y_3, Y_4))$ . But on the other hand from lemma 4 we get (for  $n \geq 4$ )

$$H^{2n-5}(Y_2, Y_3) / \text{Im}d_1(H^{2n-6}(Y_3, Y_4)) \simeq H^{2n-5}(Y_2, *) \simeq H^{2n-4}(Y_1, Y_2) \simeq H^{2n-4}(\Delta_n^2, *).$$

Consequently, by means of the map  $\rho$  and the above isomorphisms we have constructed a map

$$\lambda_n : H^{2n-4}(\Delta_n^2, *) \rightarrow T_n.$$

Note that  $\rho$  is a surjection, therefore so is  $\lambda_n$ , but  $T_n \simeq \mathbf{Z}^{(n-2)!}$  (theorem 3),  $H^{2n-4}(\Delta_n^2, *) \simeq \mathbf{Z}^{(n-2)!}$  (theorem 2). It follows that  $\lambda_n$  is an isomorphism. By construction  $\lambda_n$  agrees with  $S_n$ -action and is compatible with the "commutator". Thus the theorem is proved.  $\square$

Finally I would like to explain some notations used above ("exterior product" and "commutator").

The homology of the complex of connected graphs is isomorphic to some component of the cohomology algebra of the group of coloured braids (see [V2, V3]). On the other hand the space of binary trees can be considered as some component of the infinitesimal algebra of coloured braids. These

algebras are Koszul dual. It means, for example, that each of them is the cohomology algebra of the other. As soon as the second is the universal enveloping algebra of some Lie algebra, its cohomology can be calculated as the cohomology of the Lie algebra. In fact, the non-trivial line in the term  $E_1$  of our spectral sequence together with the first differential  $d_1$  is isomorphic to a certain part of the chain complex of this Lie algebra.

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