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On Finite Order Invariants of Triple Point Free Plane Curves

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To D. B. Fuchs with love.

ABSTRACT. We describe some regular techniques of calculating finite-order invariants of triple point free smooth plane curves $S^1 \to \mathbf{R}^2$. They are a direct analog of similar techniques for knot invariants and are based on the calculus of *triangular diagrams* and *connected hypergraphs* in the same way that the calculation of knot invariants is based on the study of chord diagrams and connected graphs.

For example, the simplest of these invariants is of order 4 and corresponds to the triangular diagram  in the same way as the simplest knot invariant (of order 2) corresponds to the 2-chord diagram $\oplus$. Also, following V. I. Arnold and other authors, we consider invariants of *immersed* triple point free curves and describe similar techniques also for this problem, and, more generally, for the calculation of homology groups of the space of immersed plane curves without points of multiplicity $\geq k$ for any $k \geq 3$.

Introduction

The intensive study of invariants of generic immersions $S^1 \to \mathbf{R}^2$ was started by Arnold in [**5**] and continued in [**6**], [**7**], [**38**], [**1**], [**30**], [**27**], [**25**], [**24**], etc.

The most interesting invariant of such objects, the *strangeness*, is in fact an invariant of triple point free immersions $S^1 \to \mathbf{R}^2$ (with allowed self-tangencies).

Almost simultaneously ([**35**], [**33**]), I considered the *ornaments*, i.e., collections of plane curves (maybe with singularities), without intersections of three different components, and developed regular techniques for calculating their invariants. The present work is the implantation (promised in [**35**]) of these methods in the theory of triple point free plane curves.

Below we describe a natural filtration of invariants by their *orders*, and a regular method of calculating all invariants of all finite orders for triple point free plane curves. Following the idea of [**3**], we reduce the study of invariants (and other cohomology classes of the space of generic objects) to that of the homology groups of the complementary *discriminant set* of objects with forbidden singularities (i.e.,

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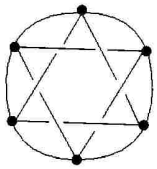

FIGURE 1. Unique invariant of order 4 for doodles

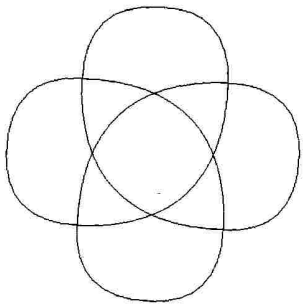

FIGURE 2. A plane curve not equivalent to the circle in the space of singular triple point free curves (after A. B. Merkov)

in our case, of curves with triple points). The more technical tools of this method are the simplicial resolutions of discriminants (see [**31**]) and the calculus (arising from them) of *triangular diagrams* and *connected hypergraphs*, which are analogs of chord diagrams and connected graphs arising in the theory of finite-order knot invariants.

The simplest such invariants are described in following two theorems.

First, as was proposed in [**35**, §9, Problem 2], we consider the space of all plane curves $\phi : S^1 \to \mathbf{R}^2$ having no triple points and no singularities obtained as degenerations of triple points (i.e., either the double points at which one of two local branches has a singular point with $\phi' = 0$, or the points at which $\phi' = \phi'' = 0$). The problem of classifying such objects (called the *doodles*) is in the same relation with the classification of ornaments, in which the isotopy classification of links is with the homotopy classification.

In this setting, the curves "0" and "8" become equivalent, and the strangeness fails to be an invariant of such objects; this is an analog of the fact that the trivial chord diagram $\ominus$ does not define a knot invariant.

THEOREM 1. *There are no invariants of doodles of orders* 1, 2 *or* 3, *and there is exactly one invariant of order* 4.

This invariant is depicted by the triangular diagram shown in Figure 1. (This diagram is an adequate analog of the chord diagram $\bigoplus$ defining the first nontrivial knot invariant: they both are simplest diagrams of corresponding kinds, not containing elements with neighboring vertices.)

This invariant proves, in particular, that the curve of Figure 2 (discovered previously by A. B. Merkov) is not equivalent to a circle.

Following [**5**], let us now consider the immersed curves in $\mathbf{R}^2$.

FIGURE 3. Notation for invariants of order 2 and 3 of immersed curves

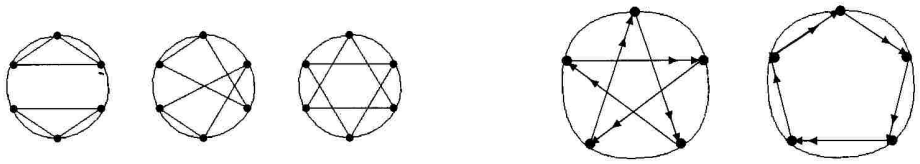

FIGURE 4. Notation for invariants of order 4

THEOREM 2. *There are only the following invariants of orders* ≤ 4 *of triple point free plane* immersed *curves* $S^1 \to \mathbf{R}^2$*:*

1) *no invariants of order* 1;

2) *one invariant of order* 2 (*the Arnold* strangeness; *for a certain reason to be explained later we denote this invariant by the simplest "triangular diagram" of Figure* 3a);

3) *one more invariant of order* 3 (*its natural notation given in Figure* 3b);

4) five more invariants of order 4 (*they are described in Figure* 4).

Our methods allow us to calculate also some higher-dimensional cohomology classes of spaces of k-points free plane curves (both immersed or just C^∞-parametrized) for any $k \geq 3$.

For example, let $k = 4$. The set $\Sigma 4$ of all curves with 4-fold self-intersections has codimension 2 in the space of all plane curves, thus the first interesting problem is the calculation of the 1-dimensional cohomology group of the complementary space of immersed plane curves without such points.[1]

This cohomology group $H^1(\mathrm{Imm}(S^1, \mathbf{R}^2) \setminus \Sigma 4)$ also has a natural filtration, so that the *orders* of (some) its elements are well defined.

THEOREM 3. *For any connected component of the space of immersions* $S^1 \to \mathbf{R}^2$*, the first few groups* $\mathcal{F}_d$ *of* 1*-dimensional integer cohomology classes of order* d *of the space of four-point free immersions lying in this component are as follows:*

$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = 0, \qquad \mathcal{F}_5 \simeq \mathbf{Z}^2.$$

However, if we calculate the $\mathbf{Z}_2$*-cohomology, then we have* $\mathcal{F}_3 \simeq \mathbf{Z}_2$, $\mathcal{F}_4/\mathcal{F}_3 = 0$, $\mathcal{F}_5/\mathcal{F}_4 \simeq \mathbf{Z}_2^2$, *and if we calculate the* $\mathbf{Z}_5$*-cohomology, then* $\mathcal{F}_3 = 0, \mathcal{F}_4 \simeq \mathbf{Z}_5, \mathcal{F}_5/\mathcal{F}_4 \simeq \mathbf{Z}_5^2$.

[1]The problem, posed by Arnold (see [**8**, problem 1996-2]), of calculating such homology groups stimulated me to write this paper.

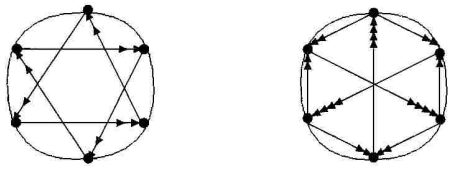

FIGURE 5. Two generators of order 5 of the 1-cohomology group of the space of plane immersed curves without 4-fold points

Two generators of the group $\mathcal{F}_5/\mathcal{F}_4$ (with integer coefficients) are naturally depicted by two chains shown in Figure 5.

Many invariants of immersions from Theorem 1 have an elementary description. Namely, the sum of three generators from Figure 4a is equal to the square of the strangeness. Moreover, strangeness itself, the unique invariant of order 3, and the sum of two invariants of order 4, shown in Figure 4b, are "index-type invariants," see subsection 1.3 below, thus initiating an infinite series of finite-order invariants (one in each order) of this sort.

IMPORTANT NOTE. Our notion of the order of invariants differs from the one used in [**5**]–[**7**], [**27**], [**30**], etc. Any invariant of finite order k in the sense of our work is also of order $\leq [k/2]$ in the sense of these works; the converse is very much false.

There are (among others) three equivalent definitions of finite order invariants of knots in $\mathbf{R}^3$: 1) the "geometrical", in terms of resolved discriminants and their filtrations, 2) the "axiomatic", in terms of finite differences of knot diagrams, and 3) the "combinatorial" (developed in [**26**]) in terms of homomorphisms of chord diagrams. The equivalence of two first definitions was clear from the very beginning, their equivalence to the third one is a nontrivial fact, conjectured by M. Polyak and O. Viro and proved by M. Goussarov.

There is a wide class of objects (including knots, ornaments, and doodles), whose invariants can be calculated by the methods, developed in [**32**] and [**35**], i.e., in the terms of resolved discriminants, thus leading to the "geometrical" definition of finite-order invariants. An "axiomatic" elementary reformulation of the resulting notion in our present situation also exists, but it is not a straightforward translation of that from [**32**]; see §2 below. I believe that it will lead to the most interesting algebraic structures, reflecting the rich geometric structures behind it.

The "combinatorial" definition and related aspects of the same invariants of ornaments and doodles are introduced and investigated by A. B. Merkov, [**21**]–[**24**] as a far generalization of the index-type invariants from [**35**].

In particular, he proved that these invariants distinguish any two nonequivalent collections of (arbitrarily many) plane curves without triple intersections or self-intersections. However I believe that the techniques of the present work allow us to calculate all such invariants in the most direct and regular way.

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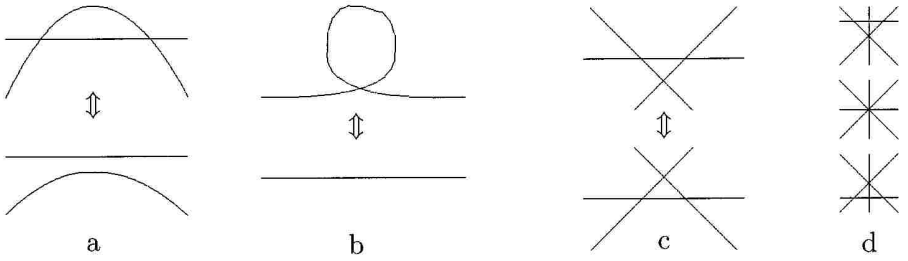

FIGURE 6. Standard moves of quasidoodles

1. Elementary theory

This and the next sections are almost exact analogues of [**35**, §§1,2].

1.1. First definitions and Reidemeister moves.

DEFINITION 1. A *doodle* is a C^∞-map $\phi : S^1 \to \mathbf{R}^2$ such that for no three different points $x, y, z \in S^1$ one of the following conditions holds:[2]

$$\phi(x) = \phi(y) = \phi(z), \tag{1.1}$$

$$\phi'(x) = 0, \phi(x) = \phi(y), \tag{1.2}$$

$$\phi'(x) = \phi''(x) = 0. \tag{1.3}$$

An *I-doodle* (i.e., immersed doodle) is a doodle which is an immersion (i.e., a map ϕ without degenerations of two types (1.1) and

$$\phi'(x) = 0\ ). \tag{1.4}$$

Two doodles (respectively, I-doodles) are *equivalent* if there is a continuous family of doodles (I-doodles) connecting them. An *invariant* of doodles or I-doodles is any function on the space of these objects, taking equal values at equivalent objects.

A doodle is *regular* if it is an immersion having only transverse double points.

PROPOSITION 1. *Any equivalence class of doodles or I-doodles contains regular doodles. Two regular doodles define equivalent doodles (respectively, I-doodles) if and only if they can be transformed one into the other by a finite sequence of isotopies of* $\mathbf{R}^2$ *(which do not change the topological picture of the image of the doodle), and of local moves shown in Figure* 6a, b *(respectively,* 6a *only).*

In other words, the move of Figure 6c is prohibited in the classification of doodles, and both 6b, 6c in the case of I-doodles.

The proof of this proposition is trivial.

DEFINITION 2. A *quasidoodle* is any C^∞-map $S^1 \to \mathbf{R}^2$. The space of all such maps is denoted by $\mathcal{K}$. The space $\mathrm{Imm}(S^1, \mathbf{R}^2)$ of all immersions $S^1 \to \mathbf{R}^2$ will be

[2]In a more general theory (see [**17**], [**24**]) this object is called a 1-doodle. We consider here only such one-component doodles and call them simply *doodles*.

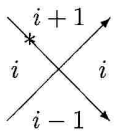

FIGURE 7

denoted by $I\mathcal{K}$. The *discriminant* (respectively, I-discriminant) $\Sigma \subset \mathcal{K}$ (respectively, $I\Sigma \subset I\mathcal{K}$) is the set of all maps from this space for which one of prohibited conditions (1.1)–(1.3) (respectively, (1.1)) holds.

PROPOSITION 2. *The set Σ is a closed subvariety of codimension 1 in $\mathcal{K}$. The set of all maps, for which only* (1.1) *holds, is dense in Σ, and the maps satisfying* (1.2) *or* (1.3) *lie in its closure. The set of quasidoodles having no degenerations of types* (1.1) *and* (1.2) *but with allowed degenerations of type* (1.3) *is path-connected in $\mathcal{K}$.*

The proof of the last assertion essentially coincides with that of the fact that *all* embeddings $S^1 \to \mathbf{R}^3$ (maybe not regular) form a path-connected subset in $C^\infty(S^1, \mathbf{R}^3)$; see e.g., [**13**]. All other statements of the proposition are elementary.

1.2. On Arnold's invariants of immersed plane curves. In [**5**] Arnold introduced three invariants of generic immersed plane curves. One of them is the *strangeness*, defined as the linking number in $I\mathcal{K}$ with the suitably (co)oriented variety $I\Sigma \subset I\mathcal{K}$. The coorientation of this variety, participating in this construction, will be specified in subsection 1.4.

1.3. Index-type invariants of I-doodles. Let us fix an orientation of the plane $\mathbf{R}^2$.

Recall that any closed oriented immersed curve c in $\mathbf{R}^2$ defines an integer-valued function ind_c on its complement: for any point t of the complement, $ind_c(t)$ equals the (counterclockwise) rotation number of the vector (t, x) when x runs one time along c.

Consider a regular I-doodle $\phi : S^1 \to \mathbf{R}^2$. To any self-intersection point x of the curve $c = \phi(S^1)$ assign its index $i(x)$ equal to the arithmetical mean of four values of ind_c in four neighboring components of $\mathbf{R}^2 \setminus c$; see Figure 7. Let us fix a regular (not intersection) point $*$ in c and define its index $i(*)$ as the greatest value of ind_c in two neighboring domains of the complement of c. For any self-intersection point x consider the frame in it, formed by (oriented) tangent vectors to c. These vectors are ordered according to the order of visits of c after leaving the point $*$. Define the *sign* $\sigma(x)$ of the point x as the sign of the orientation of this ordered frame.

For any integer i and natural β, denote by $\binom{i}{\beta}$ the number

$$i(i-1)\cdots(i-\beta+1)/\beta!;$$

cf. [**22**], [**35**]. It is easy to see that this number is always an integer.

For any natural β, define the *βth moment* $M(\beta)$ of the regular doodle by the equality

$$M(\beta) = \sum_x \sigma(x) \binom{i(x)}{\beta} + 2\binom{i(*)}{\beta+1}. \tag{1.5}$$

PROPOSITION 3 (cf. [28], [35]). *All numbers $M(\beta)$, $\beta = 1, 2, \ldots$, are invariants of I-doodles, in particular they do not depend on the choice of the distinguished point $*$. The first of them, $M(1)$, is the Arnold strangeness.*

REMARK 1. In a very similar case of ornaments such invariants were introduced in [35, §1.4], as the first nontrivial examples of finite-order invariants; see also [22]. In [30] similar expressions appeared as invariants of one-component *long curves*; i.e., essentially of curves with a fixed nonsingular point $*$. The formulae for all these invariants contained only the terms similar to the first term of the right-hand part of (1.5). Finally, Shumakovich [28] introduced a correcting second term and obtained invariants independent of the choice of this point: his invariants coincide with (1.5) up to a linear transformation with rational coefficients. (The simplest version of this correcting term, corresponding to the case $\beta = 1$ and providing a combinatorial expression for the Arnold strangeness, appeared previously in [27].) Numerous more general combinatorial expressions for invariants of doodles, ornaments, I-doodles, etc., were given in [21]–[24]. It seems likely that the method described below (see also [35], [22]) is the most universal algorithm for guessing such expressions: first one calculates several first elements of a spectral sequence converging to the group of all invariants of finite degree, and then finds an elementary interpretation for them; cf. also [32], [33].

1.4. Coorientation of the discriminant. The discriminant set Σ has a natural (co)orientation in its regular points: if we go along a generic path in the space $\mathcal{K}$ and traverse the discriminant, doing the local surgery shown in Figure 6c, then there is an invariant way to say which one of these two resolved pictures lies on the positive side of the discriminant, and which on the negative. There are numerous equivalent definitions of this coorientation; see, e.g., [5], [27], [22]. One of them can be formulated as follows. We consider the sum like (1.5) with $\beta = 1$, but with summation only over 3 points participating in the surgery. The positive (negative) side of discriminant is that with the greater (smaller) value of this sum.

This coorientation is well defined even if the curve has forbidden multiple (or, in the theory of I-doodles, forbidden singular) points far away from the location of the surgery: in fact, this is a coorientation of the locally irreducible branch of the discriminant set.

2. Elementary definition of the order of invariants

2.1. Degeneration modes and characteristic numbers. The orders of invariants of doodles and I-doodles will be defined in the same way.

DEFINITION 3. Suppose that j is a natural number, $j \geq 2$. A *degree j standard singularity* of doodles is a pair of the form {a quasidoodle $\phi : S^1 \to \mathbf{R}^2$; a point $x \in \mathbf{R}^2$} such that $\phi^{-1}(x)$ consists of exactly $j+1$ points $z_1, \ldots, z_{j+1}$, the map ϕ in a neighborhood of all these points is an immersion, and the corresponding $j+1$ local branches of the curve $\phi(S^1)$ are pairwise nontangent at x. A quasidoodle is called a *regular quasidoodle of complexity i*, if it is an immersion, all its forbidden points

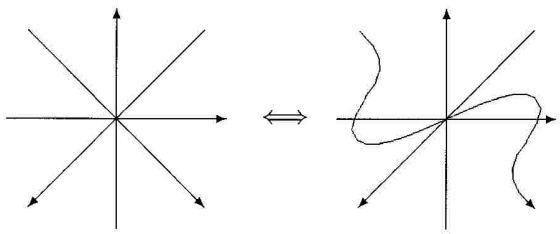

FIGURE 8. Nonessential move of a complicated multiple point

(i.e., the points, at which at least three different components meet) are standard singular points, and the sum of degrees of these singularities is equal to i.

Any regular quasidoodle can be obtained from regular doodles by a sequence of elementary degenerations. Namely, first we move along a generic path in the space $\mathcal{K}$, up to the first instant when some three points of $\phi(S^1)$ meet at the same point, forming a regular singularity of degree 2 (we do not pay attention to the surgeries shown in Figures 6a and 6b). Then we consider the vector subspace in $\mathcal{K}$, consisting of maps gluing together these three points of S^1, and go along a generic path in it; at some instant either another triple point occurs or a fourth branch joins these three. Again, we fix the smaller subspace consisting of maps gluing together all the same points, and move inside it. On the third step a point of multiplicity 5 can occur, or two points of multiplicities 4 and 3, of 3 points of multiplicity 3, etc. (In this case we do not also pay attention to the nonessential local moves like the one shown in Figure 8.) At the last step we get our quasidoodle.

Any such sequence of paths is called the *degeneration process* of our regular quasidoodle.

EXAMPLES. If our quasidoodle has only one triple point, then it can be obtained by two essentially different processes, corresponding to two of its resolutions shown in Figure 6c.

A quasidoodle with one singular point of multiplicity 4 has $16 = 4 \times 2 \times 2$ essentially different degeneration processes: at the first step any 3 of 4 points can meet at the same point of $\mathbf{R}^2$ (and this can be done in two different ways; see Figure 6c), and on the second, the fourth point joins them in one of two different ways, as shown in Figure 6d.

If our quasidoodle has one more point of multiplicity 3, then there are $96 = 4 \times 3 \times 2^3$ different degeneration processes: the points of the second group can meet before, after, or between two steps of degeneration of the first group.

Any degeneration process of a regular quasidoodle and any invariant of doodles (or I-doodles) defines a *characteristic number*; cf. [**35**, §2]. Namely, if we have a quasidoodle of some complexity j and a degeneration process DP of it, then there are exactly two quasidoodles of smaller complexity, whose degeneration processes coincide with DP without its last step. If at this last step some new group of multiplicity 3 occurs, then these are two resolutions of this group shown in Figure 6c; if at this step one branch of $\phi(S^1)$ joined an existing group of multiplicity ≥ 3, then we can move this branch to exactly two sides from this point; see Figure 6d.

In all cases these two resolutions are ordered, i.e., one of them can be called positive and the other negative. In the first case this order is described in subsection 1.4.

In the second case we define the index of a multiple point as the arithmetical mean of indices of points from all neighboring components of the complement of $\phi(S^1)$, and call *positive* the side for which the close point of multiplicity ≥ 3 has greater index with respect to the curve.

The characteristic number, which is an invariant and is defined at the pair {a regular quasidoodle, some its degeneration process}, is equal to the difference of similar numbers at two corresponding one-step resolutions (at the positive minus, at the negative one) supplied with the same degeneration process less its last step. (For these simpler singularities such characteristic numbers are well defined by the inductive conjecture.)

DEFINITION 4. An invariant is *of order* j if for any quasidoodle, whose complexity is greater than j, and any degeneration process, the corresponding characteristic number is equal to 0.

PROPOSITION 4 (cf. Theorem 4 in [**35**]). *Any index-type invariant* $M(\beta)$ *from subsection* 1.3 *is of order* $\beta + 1$. □

A wide class of invariants, generalizing these from subsection 1.3, was constructed by A. B. Merkov (see [**22**]). A further generalization of these invariants [**24**] classifies all doodles up to equivalence; in particular it is as strong as the entire space of all finite-order invariants.

2.2. Coding and calculation of finite-order invariants. Of course, the characteristic numbers of a degeneration process (and hence also the notion of the order) depend not on its geometrical realization, but only on some discrete data related to it, such as the combinatorial type of the set of points in S^1 pasted together at its different steps. Let us describe these data.

Let A be a finite set of integer numbers $A = (a_1, a_2, \ldots, a_m)$, all of which are ≥ 3. Denote by $|A|$ the number $a_1 + \cdots + a_m$, and by $\#A$ the number of elements a_l of the set A (denoted in the previous line by m).

DEFINITION 5. An *A-configuration* is a collection of $|A|$ pairwise different points in S^1 divided into groups of cardinalities $a_1, \ldots, a_{\#A}$. Two A-configurations are *equivalent* if they can be transformed one into the other by an orientation-preserving diffeomorphism of S^1. A quasidoodle $\phi : S^1 \to \mathbf{R}^2$ *respects* an A-configuration if it sends any of corresponding $\#A$ groups of points into one point in $\mathbf{R}^2$. We say ϕ *strictly respects* the A-configuration if, moreover, all these $\#A$ points in $\mathbf{R}^2$ are distinct, have no extra preimages than these $|A|$ points, and ϕ has no extra points in $\mathbf{R}^2$ at which images of three of more different points of S^1 meet.

Obviously, the space of all quasidoodles respecting a given A-configuration J is a linear subspace of codimension $2(|A| - \#A)$ in the space $\mathcal{K}$ of all quasidoodles. The number $|A| - \#A$ is thus called the *complexity* of the configuration. We shall denote this subspace by $\chi(J)$. The set of all quasidoodles, which *strictly* respects this configuration, is an open dense subset in this subspace.

DEFINITION 6. A *degeneration mode* of an A-configuration is some arbitrary order of marking all the points of the configuration, satisfying the following conditions. On any step we mark either some three points of some of $\#A$ groups (if no

point of the same group is already marked) or one point of a group, some three or more points of which are already marked.

Any degeneration process of a regular quasidoodle ϕ defines in the obvious way a degeneration mode of the A-configuration strictly respected by ϕ.

Let M be an invariant of doodles, and J an A-configuration.

PROPOSITION 5 (cf. Theorem 5 in [**35**]). *If M is an invariant of order i, and $|A| - \#A = i$, then for any regular quasidoodle ϕ, which strictly respects the A-configuration J, any characteristic number defined by the triple consisting of M, ϕ and a degeneration process of ϕ, depends only on the pair consisting of the configuration J and its degeneration mode defined by this degeneration process.* □

COROLLARY. *For any j the group of order j invariants of doodles or I-doodles is finitely generated.)*

Indeed, the number of its generators does not exceed the sum (over all equivalence classes of A-configurations of complexity $\leq i$) of numbers of their degeneration modes. □

Any invariant of order i can be encoded by its *characteristic table*, which we now describe.

This table has $i+1$ levels numbered by $0, 1, \ldots, i$. The lth level consists of several cells, which are in one-to-one correspondence with all possible pairs consisting of:

a) an equivalence class of A-configurations of complexity l in S^1;

b) a degeneration mode of this A-configuration.

In each cell we indicate

a) a picture (or a code) representing a "model" regular quasidoodle, which strictly respects some A-configuration from the corresponding equivalence class (this picture is the same for all invariants),

b) a degeneration process of this quasidoodle, defining this degeneration mode (also not depending on the invariant), and

c) the characteristic number, which our invariant and the degeneration process, corresponding to the cell, assign to this quasidoodle.

By Proposition 5, we may not specify the pictures and degeneration processes in the cells of the highest (ith) level of the table; indeed, the corresponding characteristic numbers depend only on the data indexing the cell.

For instance, the 0th level consists of the trivial ◯-like doodle, and the corresponding characteristic number equals 0 (we can normalize all invariants so that they take zero value on the trivial doodle). The first level is empty, because there are no configurations of complexity 1.

Having these data, we can calculate our invariant by the inductive process, coinciding identically with the one described in [**35**, §3] or [**32**, §4.2] (where, however, the characteristic numbers are sometimes called "actuality indices", and all (quasi)doodles should be replaced by (quasi)ornaments or (singular) knots).

REMARK 2. Of course, the characteristic numbers corresponding to different degeneration modes of the same regular quasidoodle satisfy some natural relations. For instance, if two degeneration modes differ only by a reordering of markings, preserving their order inside any group of the A-configuration, then the corresponding characteristic numbers coincide.

Less trivial identities, relating different degeneration modes inside the same group, follow from the differentials in the chain complex of *connected hypergraphs*; see [**34**], [**12**].

3. Invariants of doodles in terms of the resolved discriminant

We shall work with the space $\mathcal{K} \equiv C^\infty(S^1, \mathbf{R}^2)$ as with a Euclidean space of a very large but finite dimension Δ. The justification of this assumption uses the finite-dimensional approximations of this space and is similar to that given in [**35**] and [**32**]. Everywhere below, a cautious reader can consider $\mathcal{K}$ to be a generic finite-dimensional subspace in $C^\infty(S^1, \mathbf{R}^2)$.

In particular, we shall use the Alexander duality formula

$$\tilde{H}^i(\mathcal{K} \setminus \Sigma) \simeq \bar{H}_{\Delta-1-i}(\Sigma), \tag{3.1}$$

where $\tilde{H}^*$ is the usual reduced cohomology group (we are especially interested in the group $\tilde{H}^0(\mathcal{K} \setminus \Sigma)$ of invariants taking zero value on the trivial doodle), and $\bar{H}_*$ is the *Borel–Moore homology group*, i.e., the homology group of the one-point compactification reduced modulo the added point.

3.1. Simplicial resolution of the discriminant variety. Denote by Ψ the *configuration space* of all unordered collections of three points in S^1, so that $\Psi = (S^1)^3/S(3)$. It is a smooth 3-dimensional manifold with corners, homotopy equivalent to S^1. More precisely, it is the space of an orientable fiber bundle over S^1, whose projection p sends a triple of points to their sum in the Lie group S^1, the fiber is a closed filled triangle, and the monodromy over the entire base S^1 provides the cyclic permutation of sides and vertices of the triangles. The "zero section", consisting of centers of these fibers, consists of configurations, all of whose points are at the distance $2\pi/3$ one from the other.

Let us fix a space $\mathbf{R}^N$ of a huge dimension (much greater than that of the space $\mathcal{K}$) and fix a generic embedding $\lambda : \Psi \to \mathbf{R}^N$. For any point $\phi \in \Sigma$ consider all such points $\{x, y, z\} \in \Psi$ that one of three conditions holds:

a) $x \neq y \neq z \neq x$ and $\phi(x) = \phi(y) = \phi(z)$;

b) $x = z \neq y$ and $\phi'(x) = 0, \phi(x) = \phi(y)$;

c) $x = y = z$ and $\phi'(x) = \phi''(x) = 0$.

Then consider all points $\lambda(\{x, y, z\}) \in \mathbf{R}^N$ for all such triples $\{x, y, z\}$. Since our embedding is generic (and N is sufficiently large), then for any $\phi \in \Sigma$ the convex hull of all such points is a simplex with vertices at all these points. (Using the *generic* finite-dimensional approximations of the space $\mathcal{K}$, we can ignore the situation when the number of such triples is infinite; moreover, we can assume that the number of such triples has a finite upper estimate, uniform over all $\phi \in \Sigma$.)

Denote this simplex by $\sigma(\phi)$.

Finally, define the *resolution set* $\sigma \subset \mathcal{K} \times \mathbf{R}^N$ as the union of all simplices of the form $\phi \times \sigma(\phi)$ over all $\phi \in \Sigma$.

The obvious projection $\mathcal{K} \times \mathbf{R}^N \to \mathcal{K}$ provides the map $\pi : \sigma \to \Sigma$. By definition, this map is surjective, and by the previous "finiteness assumption" it is also proper.

PROPOSITION 6 (cf. [**32**], [**35**]). *The map π provides the homotopy equivalence of one-point compactifications of spaces σ and Σ.* □

In particular, the Borel–Moore homology groups (see (3.1)) of these spaces are canonically isomorphic.

3.2. A-cliques and the main filtration of the resolved discriminant. The space σ admits a natural filtration, which can be defined in two equivalent ways. To do it, we need to extend the notion of the A-configuration used in subsection 2.2. Again, let A be a finite set of $\#A$ integer numbers $A = (a_1, a_2, \dots, a_{\#A})$, all of which are ≥ 3.

DEFINITION 7. An *A-clique* is an unordered collection of $a_1 + \cdots + a_{\#A}$ points in S^1, divided into groups of cardinalities $a_1, \dots, a_{\#A}$, such that a) points of different groups do not coincide geometrically; b) points inside a group can coincide, but with multiplicity at most 3. Again, the *complexity* of a clique J is the number $|A| - \#A$; another important characteristic, $\rho(J)$, is the number of geometrically distinct points in it, i.e., the dimension of the space of cliques equivalent to it.

The map $\phi : S^1 \to \mathbf{R}^2$ *respects* an A-clique if it glues together all geometrically distinct points inside any of its groups, satisfies the condition $\phi' = 0$ at all points of multiplicity 2, and satisfies the condition $\phi' = \phi'' = 0$ at all points of multiplicity 3. ϕ *strictly respects* it, if additionally it does not respect any cliques of larger complexity.

For any A-clique J, consider all triples of points in S^1 belonging to the same group. All such triples are the points of the configuration space Ψ. Consider the images in $\mathbf{R}^N$ of all these points under the embedding λ and define the simplex $\sigma(J) \subset \mathcal{K} \times \mathbf{R}^N$ as the convex hull of all such points.

EXAMPLE 1. Suppose that $\#A = 1$, $a_1 = 4$, and the unique group of our A-clique is a quadruple of points (x, x, y, z), exactly two of which coincide. Then the simplex $\sigma(J)$ is a triangle with 3 vertices (x, x, y), (x, x, z), (x, y, z).

Now, for any natural i we take all quasidoodles, *strictly* respecting all possible A-cliques of complexities $\leq i$, then consider the union of their complete preimages in σ and, finally, define the term σ_i of our *main filtration* as the closure of this union. It also contains some points of the form $\phi \times \theta \in \mathcal{K} \times \mathbf{R}^N$, where ϕ is a quasidoodle of complexity $> i$, and θ is some boundary point of the corresponding simplex $\sigma(\phi)$.

Equivalently, for any A-clique J we can define the linear subspace $\chi(J) \subset \mathcal{K}$ consisting of all quasidoodles respecting (strictly or not) this clique. The term $\sigma_i \subset \sigma$ of the main filtration is then defined as the union of all subsets $\chi(J) \times \sigma(J) \subset \mathcal{K} \times \mathbf{R}^N$ over all cliques J of complexity $\leq i$. It is easy to see that these two definitions of the main filtration are equivalent.

DEFINITION 8. An element of the group $\bar{H}_*(\Sigma) \equiv \bar{H}_*(\sigma)$ is *of order i* if it can be realized by a locally finite cycle lying in the term σ_i of this filtration. In particular, an invariant of doodles is of order i if its class in the group (3.1) can be realized as a linking number with the direct image of a cycle lying in σ_i.

PROPOSITION 7 (cf. [**35**], Theorem 7). *The latter definition of the order of invariants is equivalent to that given in Section* 2. □

4. Calculation of invariants of doodles

Consider the spectral sequence $E^r_{p,q}$ calculating the Borel–Moore homology group of the space σ and generated by our filtration. Its term $E^1_{p,q}$ is isomorphic to $\bar{H}_{p+q}(\sigma_p \setminus \sigma_{p-1})$.

PROPOSITION 8 (cf. [35]). *All groups $E^1_{p,q}$ with $p+q \geq \Delta$ vanish.*

The proof of this proposition will be given in subsection 4.4.

Hence, for the calculation of the group of invariants (by (3.1) coinciding with $\bar{H}_{\Delta-1}(\sigma)$) only the $(\Delta-1)$- and $(\Delta-2)$-dimensional Borel–Moore homology groups of these spaces $\sigma_i \setminus \sigma_{i-1}$ are interesting.

In this section we calculate these groups for $i \leq 4$.

4.1. Stratification of the resolved discriminant. By construction, any space $\sigma_i \setminus \sigma_{i-1}$ consists of several $\mathcal{J}$-*blocks*, numbered by all equivalence classes $\mathcal{J}$ of A-cliques of complexity exactly i. Given such an equivalence class $\mathcal{J}$, the corresponding block $B(\mathcal{J})$ is the space of a fiber bundle, whose base is the space of all A-cliques J of this class, and the fiber over a clique J is the direct product of:

a) a linear subspace of codimension $2i$ in $\mathcal{K}$, consisting of all quasidoodles respecting J (the vector bundle of such subspaces is always orientable); and

b) a dense subset of the simplex $\sigma(J)$; namely, this simplex minus some its faces, which may belong to σ_{i-1}.[3]

By the Thom isomorphism, the Borel–Moore homology group $\bar{H}_*$ of any such block is canonically isomorphic to the group $\bar{H}_{*-(\Delta-2i)}$ of the space of only the second bundle of complexes b). Such spaces will be called the *reduced* $\mathcal{J}$-blocks.

The bases of these fiber bundles are $\rho(\mathcal{J})$-dimensional manifolds, where $\rho(\mathcal{J})$ is the number of geometrically distinct points in any clique J of the class $\mathcal{J}$.

4.2. The auxiliary filtration.

DEFINITION 9. We introduce the *auxiliary filtration* in the space $\sigma_i \setminus \sigma_{i-1}$, defining its term F_α as the union of all above-described blocks over all classes $\mathcal{J}$ of A-cliques such that $\rho(\mathcal{J}) \leq \alpha$.

EXAMPLE 2. The term σ_2 of the main filtration consists of exactly 3 terms $F_1 \subset F_2 \subset F_3 \equiv \sigma_2$ of the auxiliary filtration, because the (3)-cliques $\{x, y, z\}$ can consist of 1, 2 or 3 geometrically different points.

DEFINITION 10. The spectral sequence, calculating the group $\bar{H}_*(\sigma_i \setminus \sigma_{i-1})$ and generated by this auxiliary filtration, is called the *auxiliary spectral sequence* in contrast to the *main* spectral sequence generated by the main filtration and calculating the homology groups of entire σ.

4.3. Columns $p = 1$ and $p = 2$ of the main spectral sequence. The term σ_1 is empty, because there are no cliques of complexity 1.

PROPOSITION 9. *The column $\{p = 2\}$ of the term E^1 of the main spectral sequence has only the following nontrivial elements: $E^1_{2,\Delta-5} \simeq \mathbf{Z}$ and $E^1_{2,\Delta-6} \simeq \mathbf{Z}$.*

PROOF. The term σ_2 is the space of a fiber bundle, whose base is the configuration space Ψ, and the fiber over its point $\{x, y, z\}$ is the linear subspace of codimension 4 in $\mathcal{K}$ consisting of all quasidoodles respecting the corresponding (3)-clique. This bundle is orientable; therefore $\bar{H}_*(\sigma_2) = \bar{H}_{*-\Delta+4}(\Psi) \simeq H_{*-\Delta+4}(S^1)$. □

[3] If J is an A-configuration, i.e., has no multiple points, then these are exactly the faces corresponding to *not connected* 3-hypergraphs; see [34], [12].

4.4. Revised resolution. Before continuing, we improve slightly the construction of our $\mathcal{J}$-blocks, described in the subsection 4.1. Namely, any of these blocks contains a deformation retract having the same Borel–Moore homology group; it will be convenient to consider these new blocks $VB(\mathcal{J})$, which we (in accordance with the terminology of [**35**]) shall call the *visible blocks.*

Again, any such block, corresponding to an equivalence class $\mathcal{J}$ of A-cliques, is the space of a fibered product of two bundles, whose base and the first factor of the fiber are the same as described previously (i.e., respectively the space of all cliques $J \in \mathcal{J}$ and an oriented vector space of dimension $\Delta - 2(|A| - \#A)$). However, the second factors of the fibers of this new bundle are some subcomplexes of the barycentric subdivision of the corresponding fibers of the former bundle of simplices.

Namely, consider any A-clique J and the corresponding simplex $\sigma(J)$. Any vertex of this simplex, i.e., a triple of points of our A-clique lying inside one of its groups, defines a vector subspace of codimension 4 in $\mathcal{K}$. Consider all these subspaces corresponding to our clique (e.g., if the clique is an A-configuration, then there are exactly $\sum_{i=1}^{\#A} \binom{a_i}{3}$ such different subspaces). These subspaces together with all their possible intersections form a partially ordered set $\Pi(J)$ with respect to the relation of the (inverse) inclusion. This poset has unique maximal element: the subspace $\chi(J)$, i.e., the intersection of all our subspaces.

Having such a partially ordered set, we can define its *order complex* $\diamond(J)$ (see e.g., [**11**]): this is a formal simplicial complex, whose simplices are all the strictly monotone sequences of elements of our poset. For example, if the A-clique J is an A-configuration, then such simplices of the maximal dimension in $\diamond(J)$ are nothing but the degeneration modes of J described in subsection 2.2.

This order complex $\diamond(J)$ can be naturally considered as a subcomplex of the barycentric subdivision of the simplex $\sigma(J)$ (while its maximal element $\{\chi(J)\}$ corresponds to the center of $\sigma(J)$).

If the complexity $|A| - \#A$ of J is equal to j, then the set $\diamond(J) \cap \sigma_{j-1}$ of "marginal" faces of this complex consists of all its faces not containing the main vertex $\{\chi(J)\}$.

Define the *visible block* $VB(\mathcal{J})$ corresponding to our equivalence class $\mathcal{J}$ of A-cliques as the subset of the previously defined block $B(\mathcal{J})$, described in subsection 4.1, in which elements of the fibers $\sigma(J)$ should belong to the subcomplex $\diamond(J) \subset \sigma(J)$.

Define the revised resolved discriminant $\diamond \subset \sigma$ as the union of all such revised blocks $VB(\mathcal{J})$. It has a natural *main* filtration $\{\diamond_i\}$ induced by the identical embedding from the main filtration in σ. Similarly, in any space $\diamond_i \setminus \diamond_{i-1}$ the auxiliary filtration is induced from that in $\sigma_i \setminus \sigma_{i-1}$.

PROPOSITION 10 (see [**35**], [**37**]). *The inclusion $\diamond \hookrightarrow \sigma$ induces a homotopy equivalence of one-point compactifications of these spaces. The same is true for the inclusion $\diamond_j \hookrightarrow \sigma_j$ of any terms of the main filtrations and also for the inclusions $VB(\mathcal{J}) \hookrightarrow B(\mathcal{J})$ of their blocks corresponding to the same classes of equivalent A-cliques of complexity j. In particular this inclusion induces an isomorphism of both the main and auxiliary spectral sequences calculating the Borel–Moore homology groups of all these objects.* □

Proposition 8 follows immediately from Proposition 10 and Lemma 1.

LEMMA 1. *The dimension of any block $VB(\mathcal{J})$ of $\diamond$ does not exceed $\Delta - 1$.*

PROOF. This dimension consists of three numbers:

a) the dimension $\rho(\mathcal{J})$ of the space of cliques of our class (which does not exceed the number $|A|$);
b) the dimension $\Delta - 2(|A| - \#A)$ of the standard fiber $\chi(J)$ of the first (vector) bundle;
c) the dimension of the order complex of subspaces associated with the clique J.

The last dimension is equal to the length of the maximal monotone chain of subspaces constituting our poset $\Pi(J)$ minus 1. This length is equal to $|A| - 2\#A$. Indeed, all our subspaces are of even codimension in $\mathcal{K}$, their maximal element is the space $\chi(J)$ of codimension $2(|A| - \#A)$, and there are exactly $\#A$ steps in the chain, when the dimension jumps by 4 (when we start a new group of points; the very first step is one of them).

Finally, we obtain that the dimension of our block is not greater than $|A| + \Delta - 2(|A| - \#A) + |A| - 2\#A - 1$. □

COROLLARY. Calculating the invariants of doodles, we can consider only the $\mathcal{J}$-blocks corresponding to such A-cliques J, that $|A| - \rho(A) \leq 1$, i.e., either all their points are geometrically distinct or there is at most one point of multiplicity 2. Moreover, the blocks of the latter kind can only provide relations in the group of invariants $\bar{H}_{\Delta-1}(\sigma)$, but not its generators.

4.5. The third column of the spectral sequence.

THEOREM 4. *The group $E^1_{3,q} = \bar{H}_{3+q}(\diamond_3 \setminus \diamond_2)$ of the main spectral sequence is trivial for all numbers q not equal to $\Delta - 6$ and $\Delta - 7$, while $E^1_{3,\Delta-6} \simeq \mathbf{Z} \simeq E^1_{3,\Delta-7}$.*

The proof of this theorem occupies the rest of this subsection.

The term $\diamond_3 \setminus \diamond_2$ consists of exactly four $\mathcal{J}$-blocks corresponding to different equivalence classes $\mathcal{J}$ of cliques of complexity 3: the main block A (all four points in the cliques are distinct), the block B of auxiliary filtration 3 (exactly two points coincide), and two blocks of auxiliary filtration 2: C (one simple point and one point of multiplicity 3) and D (two double points).

Consider the corresponding reduced blocks (see subsection 4.1) $[A], [B], [C]$ and $[D]$.

The main reduced block $[A]$ is the space of a fiber bundle, whose base $\mathcal{J}$ is the configuration space $B(S^1, 4)$ of subsets of cardinality 4 in S^1, and the fiber is a cross with its four endpoints removed. The vertices of such a cross correspond to all possible choices of some three points of the 4-point configuration; this notation is transparent in Figure 9a.

The base $B(S^1, 4)$, in its turn, is the space of the fiber bundle

$$p : B(S^1, 4) \to S^1, \tag{4.1}$$

where the projection sends a quadruple of points in $S^1 = \mathbf{R}/\mathbf{Z}$ to their sum (mod $\mathbf{Z}$), and the fiber is diffeomorphic to an open 3-dimensional disc. The monodromy over the basis circle of (4.1) violates the orientation of the bundle of 3-dimensional discs and acts on the bundle of crosses as a cyclic permutation of their edges.

Thus the Wang exact sequence of this bundle gives us the following assertion.

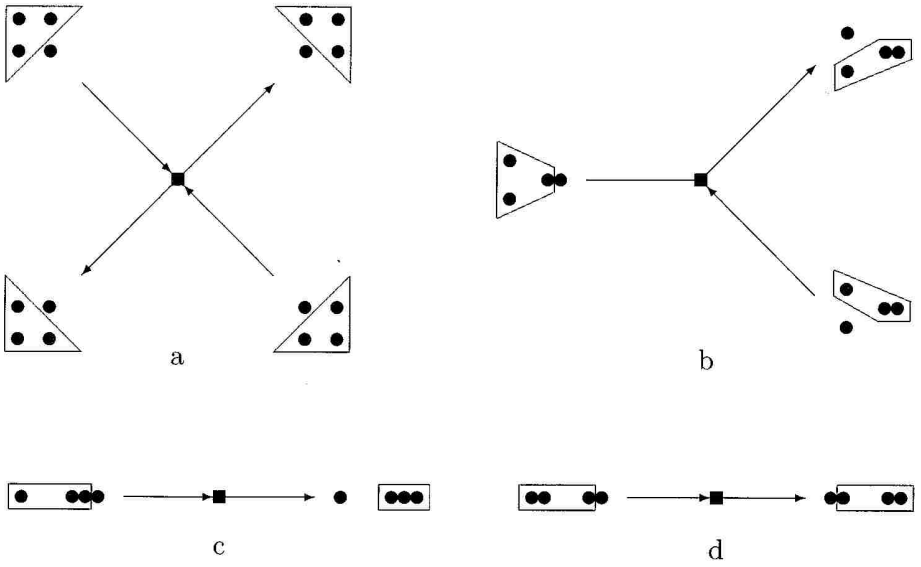

FIGURE 9. Order complexes for blocks in $\diamond_3$

PROPOSITION 11. *The Borel–Moore homology group of the reduced main block* $[A]$ *of* $\diamond_3 \setminus \diamond_2$ *is nontrivial only in dimensions* 5 *and* 4 *and is isomorphic to* $\mathbf{Z}$ *in these dimensions. The generator of the* 5*-dimensional group is swept out by the bundle over* $B(S^1, 4)$ *of* 1*-chains shown in Figure* 3b $\equiv$ *Figure* 9a *(i.e., the unique chains anti-invariant under the monodromy action). The generator of the* 4*-dimensional group is swept out by the fiber bundle, whose base is the fiber* $p^{-1}(0)$ *of the bundle* (4.1)*, and fibers are* 1*-chains shown in these pictures by the sum of two right-hand arrows.* □

The block $[B]$ is also the space of a (trivial) fiber bundle, whose base is the space of all configurations of three points in S^1, one of which (the double point) is distinguished, and the fiber is a star with three rays without endpoints. These fibers and their endpoints are shown in Figure 9b: two right-hand endpoints correspond to the subspaces in $\mathcal{K}$ given by the conditions of the form $\phi(x) = \phi(y), \phi'(x) = 0$ and $\phi(x) = \phi(z), \phi'(x) = 0$, while the left endpoint corresponds to the equation $\phi(x) = \phi(y) = \phi(z)$; cf. Example in subsection 3.2.

The base of this bundle is diffeomorphic to the direct product $S^1 \times B^2$, where B^2 is an open 2-dimensional disc. Monodromy along the basic circle S^1 acts trivially on the bundle of 3-stars. Therefore we have the following statement.

PROPOSITION 12. *The Borel–Moore homology group of the reduced block* $[B]$ *is nontrivial only in dimensions* 4 *and* 3 *and is isomorphic to* $\mathbf{Z}^2$ *in both these dimensions.* □

In a similar way we get the following statements.

PROPOSITION 13. *The Borel–Moore homology group of the reduced block* $[C]$ *is nontrivial only in dimensions* 3 *and* 2 *and is isomorphic to* $\mathbf{Z}$ *in both these dimensions.* □

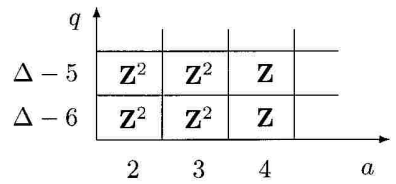

FIGURE 10. Auxiliary spectral sequence for the column $p = 3$

PROPOSITION 14. *The Borel–Moore homology group of the reduced block* $[D]$ *is nontrivial only in dimensions* 3 *and* 2 *and is isomorphic to* $\mathbf{Z}$ *in both these dimensions.* □

The corresponding order complexes are shown in Figures 9c and 9d, respectively.

COROLLARY. *The term* E^1 *of the spectral sequence, calculating the Borel–Moore homology group of the (not reduced) term* $\diamond_3 \setminus \diamond_2$ *of the main filtration of our resolved discriminant and generated by the auxiliary filtration in this term, is as shown in Figure* 10 *(i.e., all its cells* $E^1_{p,q}$ *other than the six indicated there are trivial).*

PROPOSITION 15. *The Borel–Moore homology group of the subspace in* $\diamond_3 \setminus \diamond_2$ *formed only by the blocks A, B and C, is acyclic in all dimensions.*

This follows immediately from the shape of all these blocks and their generators, and from accounting the limit positions of subspaces in $\mathcal{K}$ corresponding to the cliques of types A and B when they degenerate and form configurations of types B and C, respectively.

For example, let us consider the (4)-clique :: drawn at any endpoint of Figure 9a and let the two right-hand points of it move one towards the other, forming at the last instant a clique of type B. The subspaces of codimension 2 in $\mathcal{K}$, corresponding to all endpoints of the cross, tend to similar subspaces for endpoints of the star of Figure 9b. Namely, to both left-hand endpoints of the cross $\times$ there corresponds the unique left-hand endpoint of the star, and the other two endpoints "remain unmoved". Therefore the boundary under this degeneration of the basic cycle shown in Figure 9a is equal to the basic cycle in $\bar{H}_{\Delta-3}$ (the block B) depicted by two arrows in Figure 9b (i.e., swept out by the fiber bundle of such 1-chains over the entire base $S^1 \times B^2$ of this block).

Other boundary operators $\bar{H}_*(A) \to \bar{H}_*(B)$ and $\bar{H}_*(B) \to \bar{H}_*(C)$ can be considered in a similar way and give the assertion of Proposition 15. □

COROLLARY. *The Borel–Moore homology group of the space* $\diamond_3 \setminus \diamond_2$ *coincides with that of its unique block D (described in Proposition* 14).

This concludes the proof of Theorem 4. □

4.6. Invariants of order 4. In this subsection, we shall be interested only in the $(\Delta - 1)$-dimensional homology group of the space $\diamond_4 \setminus \diamond_3$, which can provide invariants of doodles.

In accordance with the Corollary to Proposition 10, we shall consider only those blocks of complexity 4 having at most one double point.

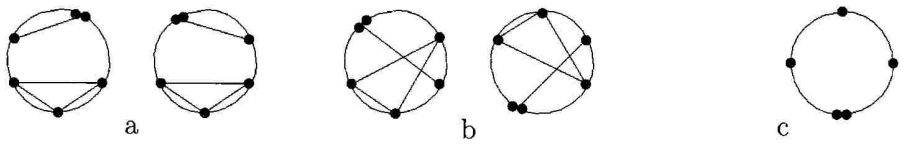

FIGURE 11. Degenerated blocks of complexity 4

PROPOSITION 16. *There are exactly four $\mathcal{J}$-blocks of complexity* 4 *corresponding to classes $\mathcal{J}$ of A-configurations (i.e., of A-cliques, all whose points are geometrically distinct). Three of them correspond to* $(3,3)$*-cliques consisting of six points in S^1 separated into two triples in one of ways shown in Figure* 4a. *The fourth corresponds to the unique class of* (5)*-configurations.*

Also, there are exactly five blocks of complexity 4 *corresponding to cliques with exactly one point of multiplicity* 2. *Four of them also are of type $A = (3,3)$ and can be obtained from the first two pictures in Figure* 4a *by some degenerations; see Figures* 11a *and* 11b, *respectively. The fifth $\mathcal{J}$-block corresponds to* (5)*-cliques with exactly one double point; see Figure* 11c. □

Let us study all these $\mathcal{J}$-blocks and their homology groups.

PROPOSITION 17. *All three $\mathcal{J}$-blocks corresponding to three pictures of Figure 4a are smooth orientable manifolds diffeomorphic to $S^1 \times \mathbf{R}^{\Delta-2}$, in particular any of them has only two nontrivial Borel–Moore homology groups $\bar{H}_{\Delta-1} \simeq \mathbf{Z} \simeq \bar{H}_{\Delta-2}$.* □

More precisely, in the first and the third (respectively, in the second) case the corresponding space $\mathcal{J}$ of equivalent cliques is diffeomorphic to a nonorientable (respectively, orientable) fiber bundle over S^1 with fiber B^5. The order complexes $\diamond(J)$ in all cases are homeomorphic to open intervals, whose endpoints correspond to subspaces of codimension 2 in $\mathcal{K}$ defined by the (3)-cliques forming the triangles, and the center corresponds to the subspace of codimension 4, defined by their intersection. The bundle of these intervals is (non)orientable exactly in the same cases when the corresponding configuration space is.

PROPOSITION 18. *The $(\Delta-1)$-dimensional Borel–Moore homology group of the $\mathcal{J}$-block in $\diamond_4 \setminus \diamond_3$, corresponding to the unique class $\mathcal{J}$ of* (5)-configurations, *is equal to $\mathbf{Z}^2$.*

(I thank A. B. Merkov very much who proved this proposition, and also suggested the following notation, convenient for the homological study of such blocks.)

PROOF. Let J be a configuration of five different points in S^1. The corresponding order complex $\diamond(J)$ is two-dimensional. Its twenty simplices of dimension 2 are in a natural one-to-one correspondence with the triples of the form {some three points of J; some four points containing these three; all five points}. Denote such a simplex by the arrow, connecting two points *not participating* in the first triple and directed towards the point not participating in the quadruple. The one-dimensional simplices of the same order complex $\diamond(J)$ are the segments of the following three kinds.

A) Connecting (the vertex corresponding to the subspace in $\mathcal{K}$ defined by) a triple of our points and (the vertex corresponding to its subspace defined by) a

quadruple containing this triple. Such edges do not belong to $\diamond_4$ and are not interesting for us.

B) Connecting (the vertex corresponding to) a triple and (that corresponding to) the maximal element $\chi(J) \in \diamond(J)$ (defined by the entire (5)-clique J). These ten edges are denoted by nonoriented edges connecting two points not participating in the triple.

C) Connecting (the vertices corresponding to) a quadruple and $\chi(J)$. These five edges are denoted by marking the point not participating in the quadruple.

In this notation, the boundary of an arrow (i.e., a 2-simplex of $\diamond(J)$) is equal to the edge, obtained from this arrow by forgetting the orientation, minus the endpoint of the arrow.

LEMMA 2. *The Borel–Moore homology group of this complex is concentrated in dimension* 2 *and is isomorphic to* $\mathbf{Z}^6$.

PROOF. Indeed, the generating 2-cycles look as follows.

First of all, any arrow can appear in such a cycle only together with its opposite, taken with opposite coefficient; otherwise the boundary of this cycle will contain an edge of type B) with a nonzero coefficient. Thus it is sufficient to consider the 10-dimensional group, generated by the linear combinations of the form {an arrow minus its opposite}. Such an element will be depicted by a double arrow directed as the first arrow in this combination; see Figure 4b. A linear combination of such double arrows is a cycle of the complex of closed chains of $\diamond(J) \setminus \diamond_3$ if and only if the correspondingly oriented segments form a cycle of the complete graph with five vertices. The group H_1 of this complete graph is isomorphic to $\mathbf{Z}^6$, and Lemma 2 is proved. □

Further, our $\mathcal{J}$-block is the space of a fiber bundle, whose base is the configuration space $B(S^1, 5) \cong S^1 \times B^4$, and the fiber over the configuration J is the direct product of the oriented $(\Delta - 8)$-dimensional subspace $\chi(J) \subset \mathcal{K}$ and the complex $\diamond(J) \setminus \diamond_3$. The monodromy over the generator S^1 of the fundamental group of the base acts on this complex (and its homology) as a cyclic permutation of five vertices. Thus by the Wang exact sequence, the group considered in Proposition 18 is generated exactly by all cycles of a complete 5-graph which are invariant under this action. This group is two-dimensional; its generators are shown in Figure 4b. □

We have found all possible generators of the group

$$\bar{H}_{\Delta-1}(\diamond_4 \setminus \diamond_3); \tag{4.2}$$

namely, the following statement holds.

PROPOSITION 19. *Each element of the group* (4.2) *is a linear combination of the five chains shown in Figure* 4. □

Now let us study the boundaries of these chains in other blocks.

PROPOSITION 20. *The two chains corresponding to the two left pictures in Figure* 4a *cannot participate in an element of the group* (4.2) *with nonzero coefficients.*

Indeed, the boundary of the first (respectively, the second) of them contains the sum of generators of $(\Delta - 2)$-dimensional homology groups of two blocks shown in Figure 11a (respectively, 11b). These generators do not appear in the boundaries of any other of our five chains. □

FIGURE 12. Boundaries of basic cycles of Figure 4b and Figure 1

PROPOSITION 21. *The two chains corresponding to the two pictures in Figure* 4b *cannot participate in an element of the group* (4.2) *with nonzero coefficients.*

To prove this proposition, let us consider the homology group of the $\mathcal{J}$-block, corresponding to (5)-cliques $J = (x, x, y, z, w)$, as shown in Figure 11c. The corresponding order complex $\diamond(J)$ again is two-dimensional.

PROPOSITION 22. *For any* (5)*-clique* J *consisting of exactly four geometrically different points in* S^1, *the group* $\tilde{H}_*(\diamond(J) \setminus \diamond_3)$ *is concentrated in dimension* 2 *and is isomorphic to* $\mathbf{Z}^3$. *The Borel–Moore homology group of the corresponding block in* $\diamond_4 \setminus \diamond_3$ *is concentrated in dimension* $\Delta - 2$ *and also is isomorphic to* $\mathbf{Z}^3$.

PROOF. Exactly as in the proof of Proposition 18, almost all two-dimensional simplices of the complex $\diamond(J)$ are naturally depicted by arrows connecting some of 4 geometrically distinct points of the clique J. (The unique extra triangle is the triple of subspaces, whose first element is defined by three points of multiplicity 1: it should be denoted by a loop edge, connecting the double point with itself. This triangle cannot participate with nonzero coefficient in any cycle of the complex $\diamond(J) \setminus \diamond_3$.)

Again, the cycles of this complex are depicted by double arrows, forming cycles (in the usual sense) of the complete graph on our 4 vertices. This proves the first assertion of Proposition 22.

The entire J-block is the space of a fiber bundle over the space of all cliques of this type (which is diffeomorphic to $S^1 \times B^3$), namely, of a fibered product of an orientable $(\Delta - 8)$-dimensional vector bundle and the (trivial) bundle of complexes $\diamond(J) \setminus \diamond_3$. This proves the last assertion of Proposition 22. Moreover, the $(\Delta - 2)$-dimensional cycles of the block are in a one-to-one (Künneth) correspondence with 2-dimensional cycles of $\diamond(J) \setminus \diamond_3$, where J is any clique of this class. □

PROPOSITION 23. *The boundaries in this block of two basic* $(\Delta-1)$*-dimensional cycles, shown in Figure* 4b, *are the two cycles shown in Figure* 12a. □

PROPOSITION 24. *The chain in* $\diamond_4$, *shown in Figure* 1 (≡ *the right-hand picture in Figure* 4a), *defines a cycle in* $\diamond_4 \setminus \diamond_3$.

PROOF. The $(3, 3)$-cliques of the type $*$ can degenerate only in the following way: some two neighboring points of *different* groups coincide, thus forming a (5)-configuration, see the left picture of Figure 12b. Given a (5)-configuration J, the $\mathcal{J}'$-block of the type $*$ adjoins the corresponding complex $\diamond(J)$ exactly five times, because such coincidence can happen at any of its five points. The boundary position of complexes $\diamond(J')$, when $J' \in *$ tends to J in the way shown in Figure 12b left, is (in the notation used in the proof of Proposition 18) equal to the difference of two 1-dimensional simplices in $\diamond(J)$ denoted by two edges in Figure 12b right.

The sum of such differences over all five vertices of the configuration J is equal to 0, and Proposition 24 is proved. □

REMARK 3. A much more general fact was proved in [**22**]: any "horizontal" boundary operator d^1 of the auxiliary spectral sequence, corresponding to the collision of two points of two different groups, is always trivial.

Summarizing Propositions 16–24, we get the following statement.

THEOREM 5. *The group* $\bar{H}_{\Delta-1}(\diamond_4 \setminus \diamond_3)$ *is isomorphic to* $\mathbf{Z}$ *and is generated by the fundamental cycle of the* $\mathcal{J}$*-block corresponding to the* $(3,3)$*-cliques shown in Figure* 1.

As the $(\Delta-1)$- and $(\Delta-2)$-dimensional Borel–Moore homology groups of both spaces $\diamond_2$ and $\diamond_3 \setminus \diamond_2$ are trivial (see Proposition 9 and Theorem 4), this implies Theorem 1 of the Introduction.

4.7. A nontrivial doodle. The theory of finite-order invariants provides a method of constructing a priori nontrivial (and nonequivalent) objects. For example, imagine that we do not know any nontrivial knot in $\mathbf{R}^3$ and wish to construct it. To do it, we can calculate the simplest finite-order knot invariant (given by the chord diagram $\bigoplus$), then draw the simplest singular knot respecting this diagram (i.e., having two transverse self-intersections), and then consider four knots obtained from it by all possible local resolutions of both these points. At least one of the knots obtained will surely be nonequivalent to the others (and indeed, if we do all this in the simplest possible way, we get three trivial knots and one trefoil).

In exactly the same way, we can construct the simplest quasidoodle with two generic triple points, respecting the triangular diagram of Figure 1. Perturbing it in four different ways, we shall obtain three trivial (equivalent to a circle) doodles, and one equivalent to Figure 2a.

(However, A. B. Merkov, who discovered this doodle, arrived at it from very different considerations.)

5. Invariants of I-doodles

In this and the next section we consider only the immersed curves in $\mathbf{R}^2$. To any immersion $\phi : S^1 \to \mathbf{R}^2$ there corresponds a map $S^1 \to S^1$: any point $x \in S^1$ goes to the direction of the tangent vector $\phi'(x)$. According to Smale [**29**], this correspondence is a homotopy equivalence between spaces $I\mathcal{K} \equiv \mathrm{Imm}(S^1, \mathbf{R}^2)$ and $C(S^1, S^1)$. These spaces split into countably many components labeled by the "winding numbers" (i.e., the indices of corresponding maps $S^1 \to S^1$). Any of these components is homotopy equivalent to S^1, the homotopy equivalence being provided by the image of (the tangent direction of ϕ at) the distinguished point of S^1.

The discriminant $I\Sigma$ in the space $I\mathcal{K}$ is just the intersection of this space $I\mathcal{K}$ with the discriminant set $\Sigma \subset \mathcal{K}$ considered in the previous sections. Its resolution $I\diamond$ is a subset in $\diamond$, namely the complete preimage of $I\Sigma$.

In its decomposition into $\mathcal{J}$-blocks only the *J-configurations*, i.e., the $\mathcal{J}$-cliques without multiple points, can take part.

PROPOSITION 25. *For any connected component* $\mathcal{C}$ *of the space* $I\mathcal{K}$ *and for any* A*-configuration* J *in* S^1, *the space of immersions* $S^1 \to \mathbf{R}^2$ *respecting this*

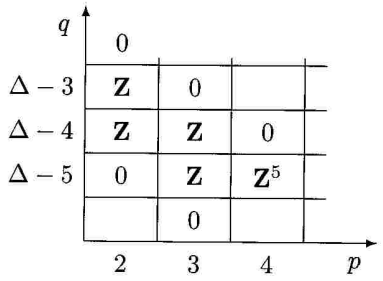

FIGURE 13. Spectral sequence for invariants of orders 2, 3 and 4

configuration and lying in this component is a path-connected *open submanifold of the space* $\chi(J) \subset \mathcal{K}$.

This follows easily from the Smale's theorem. □

For any component $\mathcal{C}$, denote by $\mathcal{C}I\diamond$ and $\mathcal{C}I\diamond_i$ the intersection of the space $I\diamond$ (respectively, $I\diamond_i$) with the preimage of $\mathcal{C}$ under the projection $\diamond \to \mathcal{K}$.

EXAMPLE 3. The stratum $I\diamond_2$ is an open subset in the space of a fiber bundle, almost coinciding with that considered in Proposition 9, with the unique difference that its base is not the entire space $\Psi = (S^1)^3/S(3)$, but its open part $B(S^1, 3)$. The *strangeness* is the linking number in $\mathcal{C}$ with the direct image of the fundamental cycle of this subset. The existence of the strangeness as an integer-valued invariant is due to the fact that this configuration space $B(S^1, 3)$ is orientable.

All other $(\Delta-1)$-dimensional blocks in all spaces $I\diamond_3 \setminus I\diamond_2$ and $I\diamond_4 \setminus I\diamond_3$ are the open subsets of similar blocks considered in the previous section; the corresponding virtual generators of the group of invariants of I-doodles are shown in Figures 3b, 4a and 4b.

All these generators define elements of corresponding groups $\bar{H}_{\Delta-1}(\diamond_i \setminus \diamond_{i-1}) \equiv E^1_{i,\Delta-i-1}$. For two generators shown in Figure 4b, this follows from the fact that the entire boundaries of corresponding blocks in $I\diamond_4 \setminus I\diamond_3$ are empty. For two remaining blocks of Figure 4a we should additionally check that their boundaries in the block corresponding to the (5)-configurations are trivial; the proof of this fact essentially coincides with that of Proposition 24.

So, for any fixed component of the space $I\mathcal{K}$ the domain in the table $\{E^1_{p,q}\}$ responsible for the calculation of invariants of orders 2, 3 and 4 of I-doodles from this component is as shown in Figure 13.

PROPOSITION 26. *For any connected component $\mathcal{C}$ of the space $I\mathcal{K}$, the fragment of the spectral sequence shown in Figure* 13 *degenerates at the term E^1, i.e., all its elements extend to well-defined Borel–Moore homology classes of the space $\mathcal{C}I\diamond$.*

PROOF. For the group $E_{2,\Delta-3}$ this is obvious.

The group $E^1_{2,\Delta-4}$ is generated by the fundamental cycle of the submanifold in $\mathcal{C}I\diamond_2$, consisting of such pairs of the form {a 3-configuration $(x, y, z) \in B(S^1, 3)$; a map $\phi : S^1 \to \mathbf{R}^2$} that $x + y + z \equiv 0 \pmod{2\pi}$. It is obviously a cycle in entire $\mathcal{C}I\diamond$. Let us prove that it is not homologous to zero. As $H^2(\mathcal{C}) \simeq 0$, it is sufficient

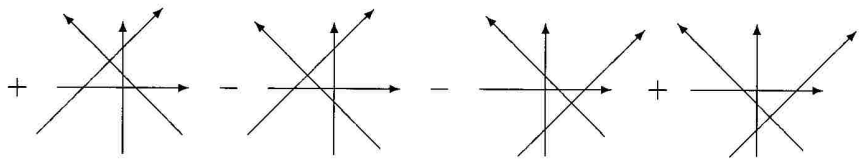

FIGURE 14. The chain of I-doodles nontrivial by the invariant of order 3

to construct two 1-dimensional cycles in $\mathcal{C} \setminus I\Sigma$, defining the same element in $H^1(\mathcal{C})$ but such that some (and then any) 2-chain realizing the homology between these cycles has nonzero intersection number with this fundamental cycle.

Consider a map $\phi \in I\Sigma \cap \mathcal{C}$ with unique generic triple point, and let ϕ_1, ϕ_2 be its two small nondiscriminant perturbations resolving this triple point in two different ways; see Figure 6c.

For any $i = 1, 2$, denote by $[\phi_i]$ the 1-cycle in $\mathcal{C} \setminus \Sigma$ swept out by all maps obtained from ϕ_i by all cyclic reparametrizations of the source circle S^1. These two cycles are obviously homologous in $\mathcal{C}$, and the intersection index of such a homology with the above manifold is equal to ± 3.

This proves the assertion of Proposition 26 for cells $E_{2,\Delta-4}$ and $E_{3,\Delta-4}$.

In particular, there exist two elements of the group $H_0(\mathcal{C} \setminus \Sigma)$, i.e., two linear combinations of doodles in $\mathcal{C}$, which cannot be distinguished by the "strangeness" (generating the group $E^{\infty}_{2,\Delta-3}$ of second order 0-cohomology classes), but can be distinguished by the invariant generating the group $E^{\infty}_{3,\Delta-4}$. (In accordance with subsection 4.7, we can find these combinations by resolving a unique point of multiplicity 4. Indeed, the linear combination of four doodles, shown locally in Figure 14 and coinciding outside it, provides such a chain.)

Exactly as above we produce from them a 1-cycle in $\mathcal{C} \setminus \Sigma$, homologous to zero in $\mathcal{C}$, and having a nonzero intersection index with the chain generating the group $E^1_{3,\Delta-5}$. This proves our proposition. □

In particular, we have proved that for any component $\mathcal{C}$ of the space $I\mathcal{K}$ all seven generators mentioned in Theorem 2 and shown in Figures 3 and 4 define independent elements of the group $\bar{H}_{\Delta-1}(\Sigma \cap \mathcal{C})$.

Finally, it is obvious that the intersection indices with all these $(\Delta - 1)$-dimensional Borel–Moore homology classes of the discriminant define zero elements in the 1-dimensional cohomology group of the component $\mathcal{C}$, and hence the linking numbers with them are well-defined invariants of I-doodles.

Theorem 2 is thus completely proved. □

6. One-dimensional cohomology of the space of immersions $S^1 \to \mathbf{R}^2$ without points of multiplicity 4

Define the discriminant $I\Sigma 4 \subset I\mathcal{K}$ as the set of immersions $\phi : S^1 \to \mathbf{R}^2$ such that images of some four different points coincide.

Its resolution $I\diamond 4$ is constructed in essentially the same way as was done above for the set $I\Sigma$. In this section we are interested in the 1-dimensional cohomology classes of the space $I\mathcal{K} \setminus I\Sigma 4$ or, which is the same, in the $(\Delta - 2)$-dimensional Borel–Moore homology classes of spaces $I\Sigma 4$ or $I\diamond 4$.

The first nonempty term of this resolution is of filtration 3. This is the space of an orientable $(\Delta - 6)$-dimensional vector bundle over the configuration space $B(S^1, 4)$. This configuration space is nonorientable. Therefore the fundamental cycle of this term defines a class only in the group $\bar{H}_{\Delta-2}(I\Sigma_4, \mathbf{Z}_2)$, but not in the integer homology group; see the statement "$\mathcal{F}_3 \sim \mathbf{Z}_2$ over $\mathbf{Z}_2$" in Theorem 3.

The next term $I\diamond 4_4 \setminus I\diamond 4_3$ of our filtration is also the space of a fiber bundle, whose base is the configuration space $B(S^1, 5)$, and the fibers are direct products of stars $\star$ with five rays (without endpoints, which belong to the smaller term of the filtration) and some (canonically oriented) $(\Delta - 8)$-dimensional vector subspaces in $\mathcal{K}$.

The base $B(S^1, 5)$ of this bundle is orientable (and diffeomorphic to $S^1 \times \mathbf{R}^4$), and the monodromy over the circle generating the group $\pi_1(B(S^1, 5)) \sim \mathbf{Z}$ acts on the fibration of 5-stars $\star$ by cyclic permutations of their rays.

Therefore the $(\Delta - 2)$-dimensional Borel–Moore homology group of this term coincides with the subgroup of the group $\bar{H}_1(\star)$ consisting of elements invariant under the rotations of these stars. For any coefficient group G this group is isomorphic to G^4. If in G the condition $5a = 0$ implies $a = 0$, then its invariant subgroup is trivial; in the case $G = \mathbf{Z}_5$ this group is isomorphic to $\mathbf{Z}_5$; see statement "$\mathcal{F}_4 \simeq \mathbf{Z}_5$" over $\mathbf{Z}_5$ of Theorem 3.

Finally, consider the term $I\diamond 4_5 \setminus I\diamond 4_4$ of our filtration. It is the space of a fiber bundle over $B(S^1, 6)$, whose fiber is the product of $\mathbf{R}^{\Delta-10}$ and some two-dimensional order complex. This complex is similar to the one considered in the proof of Proposition 18, with a single difference: its vertices correspond to choices of some 4, 5 or 6 points of our 6, and not of 3, 4 or 5 points of 5. In absolutely the same way as above, the two-dimensional cycles of this complex are the linear combinations of (double) arrows with starts and ends at these six points, forming the cycles (in the usual sense) of the complete graph on these six vertices.

However, unlike the case of 5-configurations, the base space $B(S^1, 6)$ is nonorientable. Therefore the $(\Delta - 2)$-dimensional Borel–Moore homology classes of our block are in a one-to-one correspondence with such cycles of the complete 6-graph, which are *anti-invariant* under the cyclic permutations of its six vertices.

The group of such cycles can be easily calculated and is isomorphic to $\mathbf{Z}^2$; the pictures of its generators are given in Figure 5.

By dimensional reasons, these homology classes of the space $I\diamond 4_5 \setminus I\diamond 4_4$ can be extended to these of the space $I\diamond 4_5$, and hence to the 1-dimensional cohomology classes (of order 5) of the entire space of immersions $S^1 \to \mathbf{R}^2$ without 4-fold points.

This proves Theorem 3 of the Introduction.

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