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Invariants of Ornaments

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ABSTRACT. An *ornament* is a collection of oriented closed curves in a plane, no three of which intersect at the same point. The classification of ornaments up to natural equivalence relation is parallel to the homotopy classification of links. We construct a series of invariants of ornaments, many of which have a very classical interpretation.
The general construction leads naturally to problems in modern homological combinatorics (see [BW, Bj]) and seems to be closely related to the higher-dimensional analogues of the Chern-Simons theory and Yang-Baxter equation (see [MS, FNRS]).

§0. Introduction

Denote by C_k the disjoint union of k circles.

DEFINITION. A *k-ornament* (or simply an *ornament*) is a C^∞-smooth map $C_k \to \mathbb{R}^2$ such that the images of no three different circles intersect at the same point in $\mathbb{R}^2$. Two ornaments are *equivalent* if the corresponding maps $C_k \to \mathbb{R}^2$ can be connected by a homotopy $C_k \times [0, 1] \to \mathbb{R}^2$ such that for any $t \in [0, 1]$ the corresponding map $C_k \times t \to \mathbb{R}^2$ is an ornament.

(Similar objects were considered in [FT] under the name *doodles*: the only difference is that doodles are collections of Jordan curves (without selfintersections). Of course, invariants of ornaments are also invariants of doodles; conversely, the invariants introduced in [FT] can be easily generalized to invariants of ornaments, see 1.2 in [Mx].)

In this paper we construct a series of numerical invariants of equivalence classes of ornaments. Like the knot invariants in [V1, V2], these invariants appear from the study of the *discriminant*, i.e., the space of all maps $C_k \to \mathbb{R}^2$ which necessaraly have triple intersections.

(We follow again the general strategy from [A2]: to replace the study of the soft, homogeneous space of nonsingular objects by the study of the complementary space of singular objects, which usually has a rich geometrical structure.)

Using the geometry of the discriminant, we construct a spectral sequence $E_r^{p,q}$ which calculates the cohomology groups of the space of ornaments; in particular the groups $E_\infty^{-i,i}$ of this spectral sequence provide invariants of ornaments.

A lot of our invariants can be interpreted in absolutely classical terms; these invariants are described in 1.4 below.

As in [V1, V2], the invariants coming from the cell $E^{-i,i}$ of this spectral sequence are called *invariants of order* i, and all such invariants corresponding to different i are called *finite-order invariants.*

This theory is a model version of a wide class of problems (stated in [FNRS] in connection with the higher-dimensional generalizations of Chern-Simons theory) where a similar technique works: for example, the next problem of this class is the classification of all maps of k two-spheres (or arbitrary fixed Riemannian surfaces) in $\mathbb{R}^3$ in such way that no four of them intersect at the same point. The construction of our invariants can be immediately generalized to these problems.

There are also many invariants of ornaments (due to Fenn, Taylor, and Merkov) which seem to be specifically "one-dimensional", see 1.2, 1.3 of the present paper and §§1, 2 in the article of A. Merkov in this volume.

In §§2, 3 we describe the elementary characterization of finite-order invariants and show how to calculate the values of these invariants on an ornament. In §§4–7 we construct and investigate the principal spectral sequence which provides such invariants. The first calculations are presented in §8. A large list of unsolved problems is given in §9.

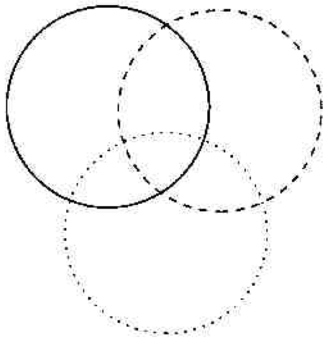

FIGURE 1. The simplest nontrivial ornament

About the pictures. In almost all the pictures and examples no more than three components of C_k participate. We depict these components by ordinary, dotted, and dashed lines, see Figure 1.

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§1. Elementary theory

1.1. Reidemeister moves.

DEFINITION. An ornament $\varphi: C_k \to \mathbb{R}^2$ is *regular* if it is an immersion of C_k, and all the multiple points of the image of C_k in $\mathbb{R}^2$ are double transversal intersection points.

THEOREM 1. *Any ornament is equivalent to a regular ornament. Two regular ornaments are equivalent if an only if they can be transformed one into the other by a finite sequence of isotopies of $\mathbb{R}^2$ (which do not change the topological picture of the image of the ornament), and of local moves shown on Figure 2 (or obtained from these moves by recoloring the strands).*

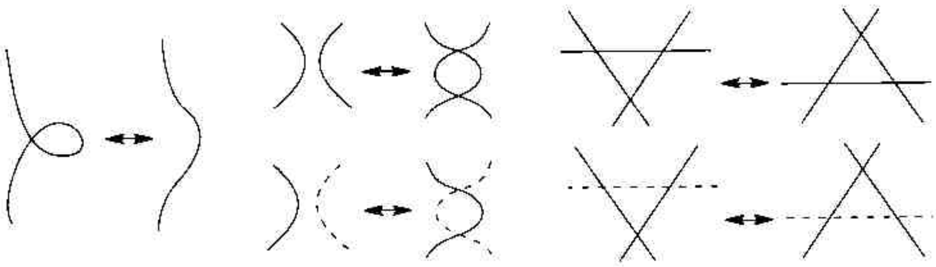

FIGURE 2. Reidemeister moves

(In other words, only the local move shown on Figure 3 is forbidden among the moves which can appear in a generic homotopy of a generic smooth map $C_k \to \mathbb{R}^2$).

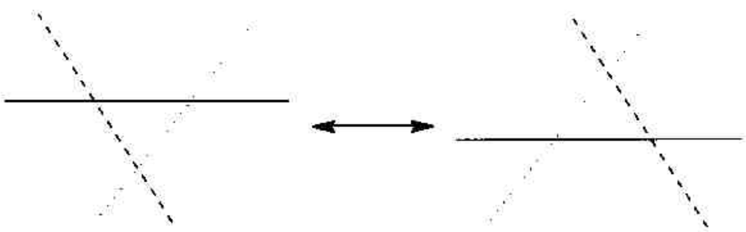

FIGURE 3. The forbidden move

PROOF. This theorem follows immediately from the Thom multijet transversality theorem, see [GG].

1.2. On Fenn-Taylor invariants. In [FT], Fenn and Taylor introduced an invariant of *doodles*, i.e., of collections of Jordan curves without triple intersections on a two-sphere. This invariant can be easily generalized to an invariant of ornaments on two-spheres. The values of this invariant are collections of k elements of the free group with $k-1$ generator considered up to cyclic permutations of symbols.

For a description and a generalization of these invariants, see [Mx].

1.3. Reduction to the homotopy classification of links. The classification of k-ornaments can be partially reduced in several ways to the homotopy classification of links. These reductions (which essentially also were introduced in [FT]) are numbered by the orientations of the complete graph with k vertices: given such an orientation, we assign a k-component link in $\mathbb{R}^3$ to any k-ornament in such way that to equivalent ornaments there correspond homotopy equivalent links.

Indeed, let us fix such an orientation. Make a link diagram from the (image of) our regular ornament in the following way: the ith string goes everywhere under the jth at their crossing points if the edge (ij) of the complete graph is oriented from the ith vertex to the jth. At the selfintersection points of the same component the over/undercrossings may be chosen in an arbitrary way.

THEOREM 2. *If two ornaments are equivalent, then the links assigned to them by the above rule (based on an arbitrary orientation of the complete graph) are homotopy equivalent.*

Indeed, by Theorem 1, any Reidemeister move can be lifted to an admissible move of a link which preserves its homotopy class, and the resulting link diagram again satisfies the above rule for over/undercrossings.

In particular, homotopy invariants of links provide the invariants of ornaments. These invariants can be nontrivial: for instance, any cyclic orientation of the complete graph with three vertices transforms the 3-ornament from Figure 1 into the Borromean link.

1.4. Index-type invariants. Recall that any closed oriented immersed curve c in $\mathbb{R}^2$ defines an integer-valued function ind_c on its complement: for any point t of the complement, $\mathrm{ind}_c(t)$ equals the rotation number of the vector (t, x) when x runs once around c.

To any regular k-ornament we assign $\binom{k}{2}$ functions $I_{i,j} = I_{i,j}(b_1, \dots, b_k)$, $1 \le i < j \le k$, with integer values and arguments; these functions are invariant under the moves from Theorem 1 and hence define invariants of ornaments.

To do this, to any (transversal) intersection point x of the ith and jth curves we assign k integers $b_1(x), \dots, b_k(x)$ and a sign $\sigma(x)$ in the following way.

If $l \neq i, j$, then $b_l(x)$ is just the number $\operatorname{ind}_l(x)$, the index of x with respect to the lth curve. Now, close to any regular point of the ith curve (in particular, to the intersection point x) the values of the corresponding function $\operatorname{ind}_i(\cdot)$ take two neighboring integer values on different sides of the curve. Define the number $b_i(x)$ as the smallest of these values at the neighboring points to x. The number $b_j(x)$ is defined in the same way by means of ind_j. Finally, $\sigma(x)$ equals 1 if the tangent vectors of the ith and jth curves at the point x define a positive frame (with respect to a fixed orientation of $\mathbb{R}^2$) and equals -1 if this frame is negatively oriented.

Given a regular k-ornament and k integers $b_1, \dots, b_k$, define the number $I_{i,j}(b_1, \dots, b_k)$ as the number of transversal intersection points x of the ith and jth curves of our ornament such that $b_1(x) = b_1, \dots, b_k(x) = b_k$ and $\sigma(x) = 1$, minus the number of similar points with $\sigma(x) = -1$.

Theorem 3. *All the functions $I_{i,j}$, $1 \leq i < j \leq k$, are invariant under all the Reidemeister moves of Figure 2.*

Proof. Immediate.

The functions $I_{i,j}$ are not independent. For instance, for $k = 3$ let us define the numbers $i_{1,2}, i_{2,3}, i_{3,1}$ as the sums

$$\sum_{b_1, b_2, b_3=-\infty}^{\infty} b_3 I_{1,2}(b_1, b_2, b_3), \qquad \sum_{b_1, b_2, b_3=-\infty}^{\infty} b_1 I_{2,3}(b_1, b_2, b_3),$$
$$\sum_{b_1, b_2, b_3=-\infty}^{\infty} -b_2 I_{1,3}(b_1, b_2, b_3),$$

respectively.

Proposition 1. *All three numbers $i_{1,2}, i_{2,3}, i_{3,1}$ coincide.*

Proof. Indeed, for an unlinked ornament all three numbers are equal to 0, and any forbidden move from Figure 3 simultaneously increases or decreases by 1 all three numbers.

This number $i_{1,2}$ is called the *index* of the 3-ornament φ and denoted by $i(\varphi)$.

More generally, for any $k \geq 3$ and any k-ornament φ, define the *index* $i(\varphi)$ of φ, as the number

$$\frac{1}{3} \sum_{1 \leq i < j \leq k} \sum_{b_1, \dots, b_k=-\infty}^{\infty} (b_1 + \dots + b_{i-1} - b_{i+1} - \cdots - b_{j-1} + b_{j+1} + \dots + b_k) I_{i,j}(b_1, \dots, b_k)$$

This number $i(\varphi)$ is always an integer: again, any elementary surgery of Figure 3 decreases or increases the previous double sum by 3.

In a similar way, given a regular k-ornament, for any k integer nonnegative exponents $\beta_1, \dots, \beta_k$ we can define the corresponding *momenta*

$$M_{i,j}(\beta_1, \dots, \beta_k) = \sum_{b_1, \dots, b_k = -\infty}^{\infty} b_1^{\beta_1} \cdots b_k^{\beta_k} I_{i,j}(b_1, \dots, b_k).$$

It is natural to call the function $M_{i,j}$ the *Laplace transform* of $I_{i,j}$.

Since all the functions $I_{i,j}$ are finite, they can be reconstructed from their Laplace transforms.

Here are some other relations on the indices $I_{i,j}$ and their momenta.

PROPOSITION 2. *For any* $1 \le i < j \le k$ *and any two values* b_i *and* b_j, *the sum*

$$\sum_{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{j-1}, b_{j+1}, \dots, b_k = -\infty}^{\infty} I_{i,j}(b_1, \dots, b_k)$$

equals 0.

PROOF. The curves other than the ith and jth ones actually do not participate in these sums; after they are removed the statement becomes trivial.

Here is an equivalent reformulation of this proposition in terms of momenta.

PROPOSITION 2$'$. *If* $\beta_l = 0$ *for all* l *other than* i *or* j, *then* $M_{i,j}(\beta_1, \dots, \beta_k) = 0$.

REMARK. The construction of the invariants $I_{i,j}$ and $M_{i,j}$ can be immediately extended to that of invariants which distinguish maps of collections of $(n-1)$-dimensional manifolds in $\mathbb{R}^n$, no $n+1$ of which intersect at the same point; the corresponding functions I and M in this case have n lower indices.

REMARK. I expect that there are many other elementary invariants of ornaments, and the spectral sequence of §4 can be considered as a regular method of guessing them: for instance, I guessed the invariants $I_{i,j}$ and $M_{i,j}$ after calculating the terms $E^{-i,i}$ of the sequence with $i = 2, 3, 4$.

For a generalization of these index-type invariants see §§3, 4 in [Mx].

1.5. Examples. A. The simplest picture of the nontrivial ornament (see Figure 4) has 16 nonequivalent realizations, depending on the orientation and ordering (coloring) of circles. All of them can be distinguished by the functions $I_{i,j}$. The Fenn-Taylor invariants split these 16 ornaments into two groups, with 8 ornaments in each (and are constant on any of these two groups): indeed, all ornaments in any of these groups are equivalent as ornaments on a sphere.

B. For the ornament in Figure 5, all invariants $I_{i,j}$ vanish. However, this ornament is nontrivial because so is its Fenn-Taylor invariant.

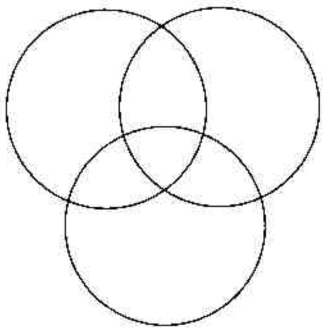

FIGURE 4.

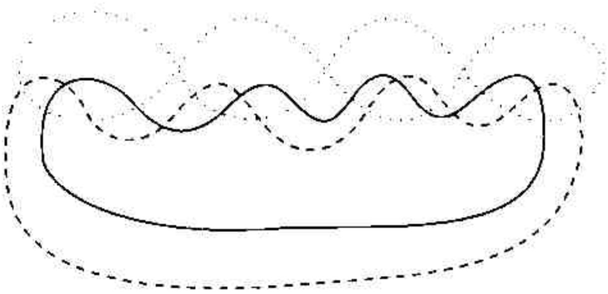

FIGURE 5. An ornament which annihilates all index-type invariants

C. For the ornament in Figure 5 of the work [Mx], both Fenn-Taylor invariant and invariants $I_{i,j}$ vanish. (This example is due to Merkov, who also proved the nontriviality of this ornament by using a refinement of the Fenn-Taylor invariant, see [Mx].) It seems also likely that the link obtained from this ornament by the construction of 1.3 is homotopy nontrivial, which also proves the nontriviality of this ornament.

§2. Elementary definition of finite-order invariants

DEFINITION AND NOTATION. A *quasiornament* is any C^∞-smooth map $C_k \to \mathbf{R}^2$. The space of all k-quasiornaments is denoted by κ_k. The *discriminant* $\Sigma \subset \kappa_k$ is the space of all quasiornaments that are not ornaments, i.e., have triple points.

The discriminant is a singular subvariety in κ_k. Its regular points are the quasiornaments having only one forbidden triple point such that the three local branches at the triple point are smooth and pairwise nontangent; singular points of Σ correspond to quasiornaments with several singularities or with more complicated singularities. A natural stratification of the discriminant is provided by the classification of these singularities.

Any numerical invariant of ornaments can be expressed in terms of the discriminant. Indeed, to each nonsingular piece of the discriminant (i.e., a connected component of the set of its nonsingular points) we can assign its *index*, i.e., the difference of values of this invariant on two neighboring topologically different ornaments, taken in an appropriate order.

The value of the invariant on any ornament can be reconstructed from this system of indices (under the assumption that any invariant vanishes on the trivial ornament): to do this, we connect our invariant with the trivial one by a generic homotopy in the space $\mathcal{K}_k$ and count all the indices of all quasiornaments at which this homotopy intersects the discriminant.

Conversely, suppose that to each nonsingular component of the discriminant we have assigned a numerical index. In order that this collection of indices define an invariant of ornaments, it must satisfy a homological condition: the sum of these components taken with the appropriate coefficients (i.e., their indices) must have no boundary in the space of all quasiornaments. Enumeration of such admissible collections is a problem in homology theory and can be solved by standard methods of this theory. A partial solution to this problem is described in §§4–8; in the present section we give an elementary characterization of the invariants thus obtained.

DEFINITION. A *degree j standard singularity* of ornaments is a pair of the form

$$\{\text{a quasiornament } \varphi\colon C_k \to \mathbb{R}^2;\ \text{a point } x \in \mathbb{R}^2\}$$

such that $\varphi^{-1}(x)$ consists of exactly $j+1$ points $z_1, \dots, z_{j+1}$, at least three of which belong to different components of C_k, the map φ close to all these points is an immersion, and the corresponding $j+1$ local branches of $\varphi(C_k)$ are pairwise nontangent at x. A quasiornament is called a *regular quasiornament of complexity i* if all its forbidden points (i.e., the points at which at least three different components meet) are standard singular points, and the sum of degrees of these singularities equals i.

Given an invariant of ornaments, to any regular quasiornament of finite complexity there corresponds a collection of *characteristic numbers* which we define below; the invariant is *of order i* if an only if all such numbers corresponding to all quasiornaments of complexity $> i$ are equal to zero.

Let us define the characteristic numbers. Let φ be a regular quasiornament with m singular points $x_1, \dots, x_m \in \mathbb{R}^2$, with inverse images in C_k of these points being

$$z_{1,1}, \dots, z_{1,j_1} \in \varphi^{-1}(x_1), \dots, z_{m,1}, \dots, z_{m,j_m} \in \varphi^{-1}(x_m).$$

A *degeneration mode* corresponding to the regular quasiornament φ is some arbitrary order of marking all these points $z_{1,1}, \dots, z_{m,j_m}$, satisfying the following conditions: at any step we mark either some three points of a group $\varphi^{-1}(x_l)$ belonging to some three different components of C_k (if no point of the same group is already marked) or one point (if some three or more other points of the same group are already marked).

EXAMPLE. Suppose that our 3-quasiornament has one singular point at which four points of C_3 meet: two from the first component of C_3, one from the second and one from the third (see Figure 6). Then there are two different degeneration modes (see Figure 7). If our quasiornament has exactly one additional singular point at which 3 points meet, then there are 6 different degeneration modes: for any case from Figure 7, we can mark the whole second group before, after, or in between the steps taken while marking the points from the first group.

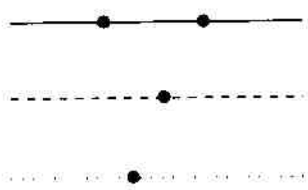

FIGURE 6. An A-configuration for $A = (2, 1, 1)$

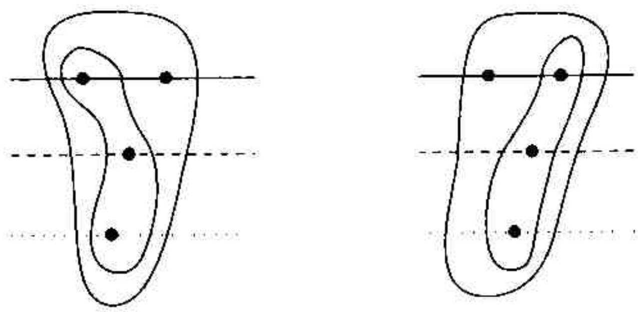

FIGURE 7. Two degeneration modes for the configuration of Figure 6

To any degeneration mode there corresponds a *characteristic number* of our invariant (and of the quasi-singularity φ). This number will be defined by induction over the process of marking.

Base of induction: for a nonsingular ornament φ (and the empty degeneration mode) the characteristic number equals the value which our invariant takes on φ.

Suppose that the last step of degeneration consists in marking certain three points z, z', z'' (and hence the corresponding group $\varphi^{-1}(\cdot)$ consists of these three points only). Let $j < l < n$ be the numbers of the components of C_k containing these points. Then we can move our map φ slightly in two ways so that all the critical points other than $\varphi(z)$ stay in place, and the triple point

$\varphi(z)$ is resolved in the two ways shown on Figure 8. These two resolutions are not equal: we can always call one of them *positive*, and the other will be *negative*. Indeed, to any of these two pictures there correspond three integers: the first of them is the index of the intersection point $*_{j,l}$ of the jth and lth curve with respect to the nth curve, multiplied by the sign $\sigma(*_{j,l})$ (see 1.4); the second number is defined in a similar way by the point $*_{l,n}$ and sign $\sigma(*_{l,n})$, and the third by $*_{j,n}$ and the sign $-\sigma(*_{j,n})$. It is easy to prove that for one of our two resolutions all these three numbers are one more than the corresponding numbers for the other; this resolution is the one called positive. Then the characteristic number that our invariant assigns to the quasiornament φ and the degeneration mode, is equal to the characteristic number defined by the same invariant for the positive resolution and for the same degeneration mode without the last step (this value is already known by the inductive assumption) minus a similar characteristic number for the negative resolution.

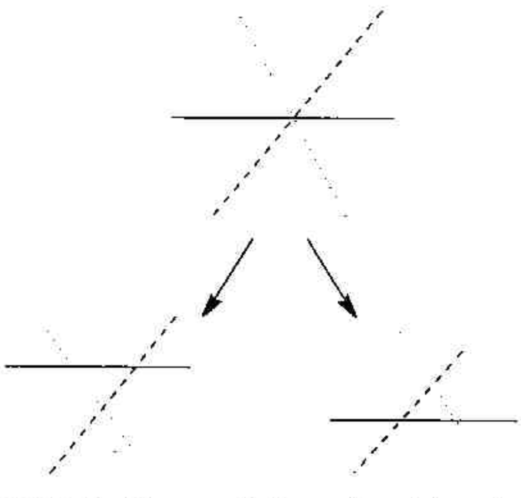

FIGURE 8. Two resolutions of a triple point

Now, let the last step of degeneration consists in adding only one point $z_{l,j} \in \varphi^{-1}(x_l)$. In this case, we can partially resolve the quasiornament φ in two topologically different ways φ', φ'' so that the remaining singularity glues together the same points $z_{i,j'}$ as φ, except only for this one. Indeed, we preserve our map φ everywhere outside a small neighborhood of $z_{l,j}$, and in this neighborhood change it in such a way that the corresponding local branch of (the image of) the ornament translates parallel to itself to one side or to the other (see Figure 9). Again, one of these resolutions can be invariantly called positive and the other negative. To do this, take an arbitrary point in $\mathbb{R}^2$ not on the ornament and very close to $\varphi(z_{l,j})$ (much

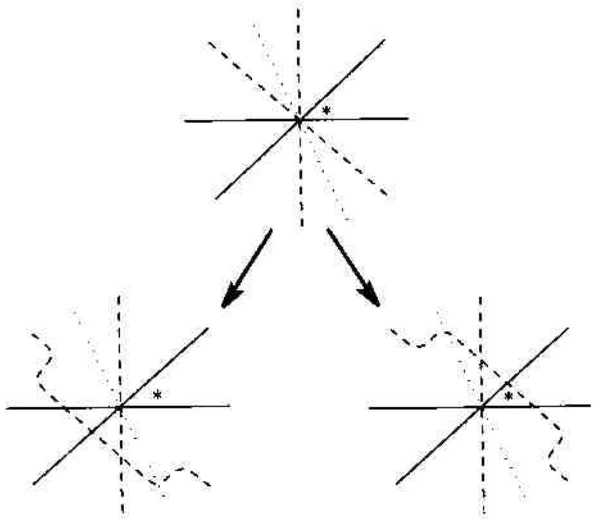

FIGURE 9. Two partial resolutions of a multiple point

closer than any of the points $\varphi'(z_{l,j})$, $\varphi''(z_{l,j})$: for instance, the point $*$ in Figure 9). Then the positive resolution of φ is the one of the two maps φ', φ'' for which the index of the point $*$ with respect to the component of C_k containing $z_{l,j}$ is greater. Again, the characteristic number corresponding to φ (and to our degeneration mode) equals the characteristic number of the one of the resolutions φ', φ'' (with the previous degeneration mode) which is positive, minus the characteristic number of the negative resolution.

DEFINITION. An invariant of ornaments is *of order* i if all corresponding characteristic numbers for any regular quasiornament of complexity $> i$ vanish.

An equivalent definition will be given in §4.

REMARK. Of course, the characteristic numbers corresponding to different degeneration modes of the same regular quasiornament satisfy some natural relations; the study of these relations is closely related to the theory of arrangements, order complexes, etc., see §§5–7.

For instance, if two degeneration modes differ only by a permutation of markings preserving the order of marking the points in the group $\varphi^{-1}(x_l)$ for any $l = 1, \dots, m$, then the corresponding characteristic numbers coincide.

THEOREM 4. *Any invariant* $M_{i,j}(\beta_1, \dots, \beta_k)$ *from* 1.4 *is an invariant of order* $\beta_1 + \dots + \beta_k + 1$.

PROOF. Immediate, by induction over the process of degeneration.

Let A be a finite series of integers, $A = (a_1 \geq a_2 \geq \dots \geq a_m \geq 3)$. Denote by $|A|$ the number $a_1 + \dots + a_m$, and by $\#A$ the number of elements a_l of the series (denoted in the previous line by m).

DEFINITION. An *A-configuration* is a collection of $|A|$ pairwise different points in C_k divided into groups of cardinalities $a_1, \ldots, a_{\#A}$ in such a way that any group contains points of at least three different components of C_k. Two A-configurations are *equivalent* if they can be transformed one into the other by a diffeomorphism $C_k \to C_k$ which preserves ordering and orientations of all components of C_k. A quasiornament $\varphi: C_k \to \mathbb{R}^2$ *respects* an A-configuration if it sends any of the corresponding $\#A$ groups of points into one point in $\mathbb{R}^2$. The quasiornament φ *strictly respects* the A-configuration if, moreover, all these $\#A$ points in $\mathbb{R}^2$ are distinct, have no preimages other than these $|A|$ points, and φ has no extra points in $\mathbb{R}^2$ at which the images of three or more different components of C_k intersect. A *degeneration mode* of an A-configuration is a degeneration mode of an arbitrary quasiornament strictly respecting it.

Obviously, the space of all quasiornaments which respect a given A-configuration J is a linear subspace of codimension $2(|A| - \#A)$ in the space of all quasiornaments. We shall denote this subspace by $\chi(J)$. The set of all quasiornaments that strictly respect this configuration is an open dense subset in this subspace.

Let M be an invariant of ornaments and J an A-configuration.

THEOREM 5. *If M is an invariant of order i, and $|A| - \#A = i$, then for any regular quasiornament φ that strictly respects the A-configuration J all characteristic numbers defined by M and φ depend on the configuration J only.*

PROOF. Any two regular quasiornaments φ, φ' that strictly respect the same A-configuration J can be transformed one into the other by some homotopy $\varphi: C_k \times [0, 1] \to \mathbb{R}^2$, $\varphi \equiv \varphi(\cdot, 0)$, $\varphi' \equiv \varphi(\cdot, 1)$, such that

1) any quasiornament $\varphi_t \equiv \varphi(\cdot, t)$, $t \in [0, 1]$, respects the configuration J, and

2) for almost all $t \in [0, 1]$ it *strictly* respects this configuration, and only at a finite number of instants $t_i \in (0, 1)$ the topological picture of the set $\varphi(C_k)$ undergoes one of the following local surgeries:

a) one of the permitted surgeries from Figure 2 away from the "bad" points of $\varphi(t)$,

b) the local move connecting the two lower pictures on Figure 8,

c) the local move connecting the two lower pictures on Figure 9,

d) the local move preserving all local branches of $\varphi(C_k)$ at all its singular points but one, and changing the last branch as shown in Figure 10.

The invariance of the characteristic numbers under one of the surgeries of type a) is obvious, and the invariance under the surgeries b) and c) follows from the definition of invariants of order i (indeed, the difference of the characteristic numbers of the quasiornaments at the left and right sides of Figure 8 or 9 is just the characteristic number of the upper quasiornament on

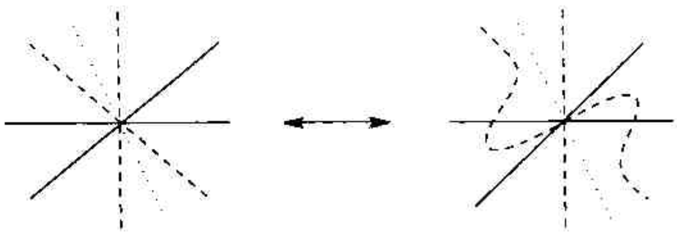

FIGURE 10.

the same picture, and it is zero because the complexity of this quasiornament is greater than i). Finally, for surgery d) the invariance follows easily from the definitions, and the theorem is proved.

Any equivalence of A-configurations establishes an one-to-one correspondence between their degeneration modes.

THEOREM 5′. *Any invariant of order i assigns equal characteristic numbers to any two equivalent A-configurations and their degeneration modes corresponding to each other via this equivalence.*

PROOF. The proof is obvious.

COROLLARY. *For any natural k and i, the space of order i invariants of k-ornaments is finite-dimensional.*

PROOF. Indeed, for any k there is only a finite number of equivalence classes of configurations of a given complexity, as well as of their degeneration modes.

In §§4–8 we show how to calculate all such invariants.

§3. Coding finite-order invariants and calculating their values on knots

Any invariant of order i can be encoded by its *actuality table* which we now describe.

This table has $i+1$ levels numbered by $0, 1, \ldots, i$. The lth level consists of several cells which are in one-to-one correspondence with all possible pairs consisting of

a) an equivalence class of A-configurations of complexity l,

b) a degeneration mode of this A-configuration.

In each cell we put

a) a picture (or a code) representing a "model" regular quasiornament which respects some A-configuration from the corresponding equivalence class (this picture is the same for all invariants), and

b) a number (called the *actuality index* corresponding to this picture) which is just the characteristic number that our invariant and the degeneration mode corresponding to the cell assign to this quasiornament.

By Theorem 5′, we need not draw the pictures in the cells of the highest

(ith) level of the table: indeed, the corresponding characteristic numbers depend only on the data indexing the cell.

For instance, the 0th level consists of the trivial (disjoint) ornament, and the corresponding characteristic number equals 0 (recall that we assume that all invariants vanish on the trivial ornament). The first level is empty, because there are no configurations of complexity 1.

In order to calculate the value of our invariant on some ornament, we join this ornament with the trivial one by a generic path in the space κ_k. This path has only a finite number of transversal intersections with the discriminant in its nonsingular points, i.e., in some regular quasiornaments with only one simplest triple point. The value of the invariant on our ornament equals the sum of characteristic numbers of these quasiornaments taken with the signs depending on the direction in which we traverse the discriminant at the corresponding points (i.e., from the negative side to the positive on or in the opposite direction).

To calculate these characteristic numbers, we use an inductive process, the general (lth) step of which consists in the following.

Before this step, we have reduced our problem to the calculation of the characteristic numbers which our invariant assigns to several regular quasiornaments of complexity $\geq l$, taken together with certain degeneration modes of the configurations strictly respected by them.

To calculate such a number for some quasiornament φ of this list, we choose the cell in the actuality table corresponding to the (equivalence class of the) A-configuration strictly respected by this quasiornament, and to the degeneration mode. (Without loss of generality, we shall assume that the quasiornament encoded in this cell respects exactly the same A-configuration as φ: the corresponding reduction can be always done by a reparametrization of C_k.) Then we join these two quasiornaments by a generic path in the space of all quasiornaments that respect this A-configuration. For almost all points of the path, the corresponding quasiornaments respect this configuration strictly, and only at a finite number of instants they undergo one of the local surgeries shown in Figures 8, 9, 10. The surgery in Figure 10 may be disregarded. At the instant of any other surgery, we get a regular quasiornament that strictly respects some A'-configuration whose complexity $|A'| - \#A'$ is strictly greater (by two in the case in Figure 8, and by one in the case in Figure 9) than that for the configuration A. Also a degeneration mode for this A'-configuration is well defined: it is obtained from the previous mode for the A-configuration by adding one more step: marking all additional points belonging to the A'-configuration but not belonging to the A-configuration. The characteristic number of the original quasiornament φ and the original degeneration mode equals the similar characteristic number for the table quasiornament respecting the same A-configuration and for the same degeneration mode, plus the sum of characteristic numbers for the regular quasiornaments (and the degeneration modes just defined) respecting more complicated configurations that we meet along the path: these characteristic numbers in the sum must be taken with the coefficient 1 (resp., -1) if the corresponding local surgery goes from the negative picture to the positive one (see §2) (resp., from the positive picture to the negative).

Thus we have reduced the calculation of characteristic numbers for some regular quasiornament respecting an A-configuration to that for several quasiornaments respecting certain A'-configurations, where the complexities of all A'-configurations are strictly greater than those for A. Since our invariant is of order i, this process stops when the complexities of all such configurations attain i.

§4. Discriminants and their resolutions

In this section, we begin the systematic topological study of the discriminant $\Sigma \subset \kappa_k$; in particular, we present a method of calculation of all finite-order invariants (and show why this class of invariants is natural).

We construct a spectral sequence $E_r^{p,q}$ that calculates the cohomology of the space of ornaments, $\kappa_k - \Sigma$. This construction is based on the natural stratification of the discriminant by the types of degeneration of quasiornaments. For $r \geq 1$ this spectral sequence lies in the domain $\{p, q \mid p < 0, p+q \geq 0\}$, see Figure 11. The invariants of ornaments correspond to the elements of the groups $E_\infty^{-i,i}$, $i \geq 1$. The invariants that appear from our spectral sequence are exactly the invariants of finite order described in §2: they are, in a sense, exactly the invariants that can be expressed in terms of strata of finite codimension in the discriminant.

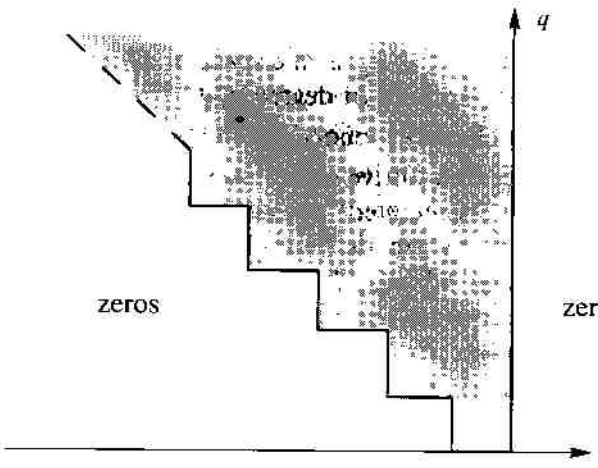

FIGURE 11. The principal spectral sequence

The description of invariants in terms of the discriminant, given in §2, can be intuitively considered as a version of the Alexander duality formula:

$$H^0(\kappa_k - \Sigma) \cong \overline{H}_{\infty-1}(\Sigma). \tag{1}$$

In this (strictly speaking, meaningless) formula and everywhere below $\overline{H}_*$ denotes the closed homology, i.e., the homology of the one-point compactification reduced modulo the compactifying point; ∞ is the notation for the dimension of κ_k, and the right part $\overline{H}_{\infty-1}(\Sigma)$ is just the group of all described above linear combinations of smooth pieces of Σ satisfying certain homological concordance constraints, see the beginning of §2.

The spectral sequence we construct in this section also provides elements of the higher cohomology groups $H^i(\kappa_k - \Sigma)$, $i > 0$. Indeed, the formula (1) can be extended to the general Alexander duality formula

$$H^i(\kappa_k - \Sigma) \cong \overline{H}_{\infty-i-1}(\Sigma), \tag{2}$$

which can also be given an exact meaning in terms of the discriminant and its strata. To justify these formulas, we proceed as follows:

a) choose an increasing sequence of affine subspaces Γ_k^d, $d \to \infty$, in κ_k such that any compact subset in κ_k can be approximated arbitrarily well by points of an appropriate subspace of this sequence;

b) for any d, construct a spectral sequence $E_r^{p,q}(d) \to H^{p+q}(\Gamma_k^d \setminus \Sigma)$,

c) prove the stabilization of these spectral sequences.

The groups $E_\infty^{p,q}(\infty)$ of the stabilized spectral sequence provide the elements of the group $H^{p+q}(\kappa_k - \Sigma)$; as usual, these elements are well defined modulo similar elements coming from the groups

$$E_\infty^{p+l,q-l}(\infty), \quad l \geq 1.$$

4.1. Finite-dimensional approximations of the space of ornaments. For any $d \geq 1$, define the subspace $\tilde{\Gamma}_k^d \subset \kappa_k$ as the space of all maps $C_k \to \mathbb{R}^2$ given by $2k$ trigonometric polynomials of degree $\leq d$.

The space $\tilde{\Gamma}_k^d$ is tautologically embedded in all spaces $\tilde{\Gamma}_k^{d'}$, $d' \geq d$.

The spaces $\tilde{\Gamma}_k^d$ are in an intuitive sense "nongeneric" in κ_k: for instance, they contain "infinitely degenerated" constant maps of C_k: of course, this situation cannot arise in any generic finite-dimensional family of maps. For our approximating spaces Γ_k^d we use small perturbations of $\tilde{\Gamma}_k^d$ which are generic in the following precise sense.

For any A-configuration $J \subset C_k$ and any affine subspace $\Gamma \subset \kappa_k$ denote by $\chi(\Gamma, J)$ the space $\Gamma \cap \chi(J)$, i.e., the space of all maps $\varphi \in \Gamma$ that respect J.

PROPOSITION 3. *In the space of all affine subspaces of fixed finite dimension in κ_k, a residual (in particular, dense) subset consists of planes Γ such that for any index $A = (a_1 \geq \ldots \geq a_{\#A} \geq 3)$ and any A-configuration $J \subset C_k$, the following assertions hold:*

(A) *For almost any A-configuration* J' *equivalent to* J, *the set* $\chi(\Gamma, J)$ *is an affine subspace of codimension* $2(|A| - \#A)$ *in* Γ (*in particular, it is empty if* $2(|A| - \#A) > \dim \Gamma$).

(B) *Denote the number* $|A| - \#A$ *by* i *and suppose that* $2i \leq \dim \Gamma$. *Then, in the set of all configurations equivalent to* J, *the subset of those configurations* J' *for which* $\chi(\Gamma, J')$ *is empty, is of codimension* $\geq \dim \Gamma - 2i + 1$, *while the set of* J' *such that the codimension of* $\chi(\Gamma, J')$ *in* Γ *equals* $2i - l$, $l \geq 1$, *is a subset of codimension* $\geq l(\dim \Gamma - 2i + l + 1)$. *In particular, if* $i < 2(\dim \Gamma + 1)/7$, *then the codimension of any set* $\chi(\Gamma, J)$ *defined by any A-configuration* J *with* $|A| - \#A = i$ *is exactly equal to* $2i$.

(C) *Suppose that* $|A| - \#A = i \geq \dim \Gamma/2$. *Then in the set of configurations equivalent to* J *the set of all configurations* J' *such that* $\dim \chi(\Gamma, J') = l \geq 0$ *is either empty or has codimension at least* $(l+1)(2i - \dim \Gamma + l)$. *In particular, the set of all* J' *such that* $\chi(\Gamma, J')$ *is nonempty, is of codimension* $\geq 2i - \dim \Gamma$ *and is empty when* $\dim \Gamma < i$.

PROOF. This proposition is a corollary of the Thom transversality theorem, see [GG].

DEFINITION. The subspaces Γ in κ_k satisfying the conditions of this proposition are called *Σ-nondegenerate.*

In what follows Γ_k^d denotes a Σ-nondegenerate affine subspace in κ_k that is sufficiently close to the subspace $\tilde{\Gamma}_k^d$ in the space of all subspaces of the same dimension.

By the Weierstrass approximation theorem, any quasiornament and any compact subset in the space of quasiornaments can be approximated arbitrarily close (in any prescribed C^m-topology) by quasiornaments lying in an appropriate space Γ_k^d.

4.2. Geometrical resolution of the discriminant. Here we construct a resolution of the space $\Sigma \cap \Gamma_k^d$, i.e., a semialgebraic set σ together with a proper projection $\sigma \to \Sigma \cap \Gamma_k^d$ such that the induced map $\overline{H}_*(\sigma) \to \overline{H}_*(\Sigma \cap \Gamma_k^d)$ is an isomorphism.

Denote by Ψ the disjoint union of all $\binom{k}{3}$ possible three-dimensional tori $T^3_{\alpha\beta\gamma}$, $1 \leq \alpha < \beta < \gamma \leq k$, that are the direct products of three different components of the manifold C_k.

Let N be a very large natural number, and $\lambda\colon \Psi \to \mathbb{R}^N$ a smooth embedding. Denote by σ_2 the subset in $\Gamma_k^d \times \mathbb{R}^N$ consisting of all possible pairs of the form

$$(\text{a map } \varphi \in \Gamma_k^d;\ \text{a point } \lambda(x, y, z) \in \mathbb{R}^N) \tag{3}$$

such that x, y, z are points of three different components of C_k and $\varphi(x) = \varphi(y) = \varphi(z)$.

PROPOSITION 4. *If* N *and* d *are sufficiently large, the map* λ *is generic,* *and* Γ_k^d *satisfies the conditions of Proposition* 3, *then* σ_2 *is a smooth manifold with a natural structure of the space of an orientable (and even stably trivial)* $(\dim \Gamma_k^d - 4)$*-dimensional affine bundle over* Ψ, *the projection in this bundle being defined by forgetting the first elements* φ *in the pairs* (3).

PROOF. This follows immediately from the construction.

The obvious map $\sigma_2 \to \Sigma \cap \Gamma_k^d$ is a smooth normalization of $\Sigma \cap \Gamma_k^d$; its inverse image over a nonsingular point consists of only one point, while the inverse images of singular points can consist of several points.

For any point $\varphi \in \Sigma \cap \Gamma_k^d$, let us take all possible points $(x, y, z) \in \Psi$ such that $\varphi(x) = \varphi(y) = \varphi(z)$; let $\tilde{\Delta}(\varphi)$ be the convex hull in $\mathbb{R}^N$ of all points $\lambda(x, y, z)$ where the point (x, y, z) satisfies the following condition.

PROPOSITION 5. *Suppose that* Γ_k^d *satisfies the conditions of Proposition* 3, N *is sufficiently large and the embedding* $\lambda\colon \Psi \to \mathbb{R}^N$ *is generic. Then, for any* $\varphi \in \Sigma \cap \Gamma_k^d$, *the polyhedron* $\tilde{\Delta}(\varphi)$ *is a simplex whose vertices are all the points* $\lambda(x, y, z)$ *such that* $\varphi(x) = \varphi(y) = \varphi(z)$.

PROOF. It follows from the Thom multijet transversality theorem, see [GG].

Below we shall assume that Γ_k^d, N and λ satisfy the conditions of this proposition.

For any $\varphi \subset \Sigma \cap \Gamma_k^d$, denote by $\Delta(\varphi)$ the simplex $\varphi \times \tilde{\Delta}(\varphi) \subset \Gamma_k^d \times \mathbb{R}^N$, and by σ the union of all simplices $\Delta(\varphi)$ over all φ. The topology of the space σ is induced from the ambient space $\Gamma_k^d \times \mathbb{R}^N$. The obvious projection $\Gamma_k^d \times \mathbb{R}^N \to \Gamma_k^d$ maps σ onto $\Sigma \cap \Gamma_k^d$.

PROPOSITION 6. *Suppose that the conditions of Proposition* 5 *are satisfied. Then the projection* $\pi\colon \sigma \to \Sigma \cap \Gamma_k^d$ *is proper and induces a homotopy equivalence of the one point compactifications of these two spaces, and in particular, an isomorphism* $\overline{H}_*(\sigma) \to \overline{H}_*(\Sigma \cap \Gamma_k^d)$.

PROOF. Indeed, the fact that π is proper follows from the construction; the induced map of compactifications is a piecewise-algebraic map of semi-algebraic compact sets with contractible fibers. This implies the assertion about the homotopy equivalence, see [D].

The space σ together with the projection π is called the *geometrical resolution* of $\Sigma \cap \Gamma_k^d$.

4.3. Filtration on the space of the geometrical resolution. Restrict the obvious projection $\Gamma_k^d \times \mathbb{R}^N \to \mathbb{R}^N$ to the space σ. If N is sufficiently large and λ is generic, then the inverse image of any point θ in $\mathbb{R}^N$ under this projection is an affine subspace of the form $\chi(\Gamma_k^d, J) \times \theta$, where J is some A-configuration.

Define an increasing filtration $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_{\mathrm{last}} = \sigma$ on the set σ by assigning to σ_i the union of all subspaces of the form $\chi(\Gamma_k^d, J) \times \theta$ where

J is an A-configuration with $|A| - \#A \leq i$. By Proposition 3, the number of *last* elements of the filtration does not exceed $\dim \Gamma_k^d$.

Consider the homology spectral sequence $E^r_{p,q}(d)$ converging to the group $\overline{H}_*(\sigma)$ and generated by this filtration. By definition, the term $E^1_{p,q}$ of this spectral sequence equals $\overline{H}_{p+q}(\sigma_{-p} - \sigma_{-p-1})$.

PROPOSITION 7. *If i is in the stable range, $i < 2(\dim \Gamma_k^d + 1)/7$, then the term $\sigma_{-p} - \sigma_{-p-1}$ of our filtration is the space of an affine fiber bundle (whose fibers are the fibers of the projection $\Gamma_k^d \times \mathbb{R}^N \to \mathbb{R}^N$ restricted to σ); this bundle is orientable.*

PROOF. Indeed, the fact that this projection is an affine fiber bundle follows from Proposition 3. This bundle can be regarded as a subbundle of the trivial bundle with fiber Γ_k^d. The fiber of this subbundle at each point is distinguished in Γ_k^d by several conditions of type $\varphi(x) = \varphi(y)$, where the pairs of points x, y are defined by the point of the base. For stable i, all these conditions are linearly independent. Hence, the quotient bundle of our subbundle at each point splits into the direct sum of several copies of $\mathbb{R}^2$, with the canonical two-dimensional coordinate $\varphi(x) - \varphi(y)$ on each. This splitting is defined invariantly, up to a possible permutation of summands. Such permutations do not change the orientation of the sum, and the proposition is proved.

Thus, by the Thom isomorphism, the closed homology group $\overline{H}_*$ of the space $\sigma_{-p} - \sigma_{-p-1}$ reduces to that of the base of this bundle, which can be, in principle, described in combinatorial terms. For the first calculations see §8 below.

Make our spectral sequence $E^r_{p,q}(d)$ a cohomological spectral sequence by renaming the term $E^r_{p,q}(d)$ as $E_r^{-p, \dim \Gamma_k^d - 1 - q}(d)$. This spectral sequence is called the *main spectral sequence*, and the previous one the *main homological spectral sequence*. By the Alexander duality theorem, the spectral sequence $E_r^{p,q}$ converges to $H^*(\Gamma_k^d \setminus \Sigma)$.

By construction, this spectral sequence lies in the region $\{p, q \mid p < 0\}$, and its term $E_\infty^{p,q}$ can be nontrivial only if $p + q \geq 0$.

4.4. Main properties of the main spectral sequence.

THEOREM 6. *For any choice of the space Γ_k^d satisfying the conditions of Proposition 3,*

(A) *If the term $E_1^{p,q}(d)$ of our spectral sequence is nontrivial, then*
   (1) (i) $p + q \geq 0$;
   (2) (ii) $p \geq -\dim \Gamma_k^d$;

(B) *For any $d' \geq d$ and any space $\Gamma_k^{d'}$ satisfying the assertions of Proposition 3, the corresponding spectral sequence $E_r^{p,q}(d')$ coincides with*

$E_r^{p,q}(d)$ *for following values of* p, q *and* r:

(3) (i) *for* $r = 1$ *and* p, q *in the "stable" region* $\{p+q \geq 0, -p < 2(\dim \Gamma_k^d + 1)/7\}$;

(4) (ii) *for any* $r > 1$ *and* p, q *such that the differentials* d^t, $t < r$, *do not act into the cell* $E_t^{p,q}$ *from the unstable region.*

COROLLARY 1. *If* Γ_k^d *and* $\Gamma_k^{d'}$ *are in general position, then* $E_r^{-i,i}(d) \cong E_r^{-i,i}(d')$ *for any* $r = 1, 2, \dots, \infty$ *and any* $d' \geq d > ((7i/2-1)/2k-1)/2$.

COROLLARY 2. *For any* i *the inclusion homomorphism* $\overline{H}_{\dim \Gamma_k^d - 1}(\sigma_i) \to \overline{H}_{\dim \Gamma_k^d - 1}(\sigma)$ *is a monomorphism.*

The restriction (ii) in part (A) of the theorem follows from Proposition 3. Part (i) will be proved in §6 (and, in a different way, in §7).

Let $s = 2(\dim \Gamma_k^d + 1)/7$; then the identical embedding $\Gamma_k^d \to \Gamma_k^{d'}$ induces an embedding $\sigma_s(d) \to \sigma_s(d')$ of sth terms of the corresponding resolutions; this embedding respects the above-defined filtrations on both spaces.

LEMMA. *The last embedding can be extended to a homotopy equivalence of the* $4k(d'-d)$*-fold suspension of the one-point compactification of* $\sigma_s(d)$ *onto the one-point compactification of* $\sigma_s(d')$ *such that for any* $u < s$ *the restriction of this homotopy equivalence to the* $4k(d'-d)$*-fold suspension of the compactification of* $\sigma_u(d)$ *is a homotopy equivalence of this space onto the compactification of* $\sigma_u(d')$.

PROOF. The proof repeats that of Theorem 4.2.4 in Chapter III of the book [V2]; this lemma implies part (B) of Theorem 6.

4.5. The stable spectral sequence. Part (B) of Theorem 6 allows us to define the *stable spectral sequence* $E_r^{p,q} = E_r^{p,q}(\infty)$: its term $E_r^{p,q}$ is equal to the common term $E_r^{p,q}(d)$ of all spectral sequences corresponding to sufficiently large d.

PROPOSITION 8. *For any* i, *there exists* $d = d(i)$ *such that for all* $d' \geq d$ *and all* r *we have* $E_r^{-i,i}(d') \cong E_r^{-i,i}(d)$, *in particular* $E_r^{-i,i} \cong E_r^{-i,i}(d)$. *Namely, it is sufficient to take* $d > [(7i/2-1)/2k-1]/2$.

PROOF. This follows immediately from the structure of the spectral sequence (see Theorem 6) and the stabilization properties of the strata of $\Sigma \cap \Gamma_k^d$ (see Proposition 3).

The isomorphism $E_\infty^{-i,i}(d') \cong E_\infty^{-i,i}(d)$ from this proposition agrees with the map in homology: for $d' > d$ sufficiently large with respect to i, there is a canonical isomorphism

$$\overline{H}_{2k(2d+1)-1}(\sigma_i(d)) \cong \overline{H}_{2k(2d'+1)-1}(\sigma_i(d')). \tag{4}$$

This isomorphism is compatible with the Alexander duality isomorphism: let θ and θ' be equivalent ornaments in κ_k, and $\theta \in \Gamma_k^d$, $\theta' \in \Gamma_k^{d'}$, then the elements of groups (4) corresponding to each other by the isomorphism (4) have the same values on these knots.

DEFINITION. A *stable invariant of order* i is an element of the homology group $\overline{H}_{\dim \Gamma_k^d - 1}(\sigma_i(d))$, where d is sufficiently large. Two elements of such groups corresponding to different d define the same invariant of order i if they correspond to each other via the isomorphism (4).

Any such stable invariant of finite order i can be regarded as a well-defined invariant of ornaments: this follows from the stability property and from the fact that any homotopy that realizes the equivalence of two ornaments can be approximated arbitrarily well by some homotopy that also avoids the discriminant in an appropriate space Γ_k^d.

DEFINITION. An invariant of ornaments is *of order* i if it is obtained by the above construction from some stable invariant of order i.

THEOREM 7. *This definition is equivalent to the definition from* §2.

A proof will be given in §7.

In the next §§5, 6 we begin the study of the term E_1 of the stable spectral sequence, and in §8 we present the results of the first calculations.

§5. Complexes of connected hypergraphs

Recall that for i in stable range, $i < 2(\dim \Gamma_k^d + 1)/7$, the space $\sigma_i - \sigma_{i-1}$ is the space of an oriented affine bundle over a stratified variety, in particular the closed homology group $\overline{H}_*(\sigma_i - \sigma_{i-1})$ reduces to a similar group of that base variety.

To study the last group (and to estimate similar groups for i in the non-stable domain) we need some homological preliminaries.

Let θ be a finite set given together with some subdivision into k disjoint subsets, $\theta = (\theta_1, \dots, \theta_k)$. Denote by $\Delta[\theta]$ (or $\Delta[\theta_1, \dots, \theta_k]$) the simplex whose vertices correspond to all triples of points in θ belonging to different subsets $\theta_j, \theta_l, \theta_m$, $1 \le j < l < m \le k$.

DEFINITION. A collection of vertices of the simplex $\Delta[\theta]$ is called *connecting* if any two points of θ can be joined by a chain of points such any two neighboring points belong to one triple corresponding to some vertex of the collection. A face of $\Delta[\theta]$ is called *connecting*, if the collection of its vertices is connecting.

The *connecting part* of the simplex $\Delta[\theta]$ is the union of interior points of all its connecting faces (including the simplex itself, if there is at least 3 nonempty sets θ_j).

Now, let $\Theta = (\theta^1, \dots, \theta^{\#A})$ be a collection of $\#A$ sets θ^j, $j = 1, \dots, \#A$, of cardinalities $a_1, \dots, a_{\#A}$, any of which is divided into some subsets $\theta_1^j, \dots, \theta_k^j$ (k is the same for all j).

Consider the join of the simplices $\Delta[\theta^j]$ over all $j = 1, \dots, \#A$, i.e., the simplex whose vertices are all the vertices of these $\#A$ simplices. Denote this simplex by $\Delta[[\Theta]]$.

DEFINITION. A face of the simplex $\Delta[[\Theta]]$ is *essential* if for any $j = 1, \ldots, \#A$, the vertices of this face belonging to the simplex $\Delta[\theta^j]$ span a connecting face of this simplex. The *essential part* of the simplex $\Delta[[\Theta]]$ is the union of interior points of all its essential faces.

The union of all nonessential faces in the simplex $\Delta[[\Theta]]$ is obviously a subcomplex of its natural triangulation. Denote by $\Xi(\Theta)$ the corresponding quotient complex. The (reduced modulo a point) homology group of this quotient complex is another realization of the closed homology group of the essential part of the simplex.

CONVENTION. From now on, we consider homology with coefficients in the field $\mathbb{R}$ only.

EXAMPLES. Let $\#A = 1$, so that $\Theta = (\theta^1)$, and $k = 3$, so that θ^1 consists of 3 subsets of cardinalities α_1, α_2, α_3.

1. Let $\alpha_2 = \alpha_3 = 1$. Then the simplex $\Delta[[\Theta]]$ is of dimension $(\alpha_1 - 1)$ and has only one essential face: the simplex itself. In particular, the group $H_i(\Xi(\Theta))$ equals $\mathbb{R}$ if $i = \alpha_1 - 1$ and is trivial for all other i.

2. Let $\alpha_1 = \alpha_2 = 2$, $\alpha_3 = 1$. Then θ_1 consists of the points x_1 and x_2, while θ_2 consists of the points y_1 and y_2, and θ_3 has only one point z. The dimension of the simplex $\Delta[[\Theta]]$ equals 3, and its vertices are called

$$(x_1, y_1, z), (x_1, y_2, z), (x_2, y_1, z), (x_2, y_2, z),$$

see Figure 12. The essential faces of this simplex are: the simplex itself, all its two-dimensional faces, and two edges $((x_1, y_1, z); (x_2, y_2, z))$, $((x_1, y_2, z); (x_2, y_1, z))$. The group $\overline{H}_i(\Xi(\Theta))$ equals $\mathbb{R}$ for $i = 2$ and is trivial for other i. The group $\overline{H}_2(\Xi(\Theta))$ is generated by either of two chains

$$((x_2, y_1, z); (x_2, y_2, z); (x_1, y_1, z)) + ((x_1, y_1, z); (x_2, y_2, z); (x_1, y_2, z))$$

or

$$((x_1, y_2, z); (x_2, y_1, z); (x_1, y_1, z)) + ((x_2, y_2, z); (x_2, y_1, z); (x_1, y_2, z)),$$

the sum of these chains being homologous to zero.

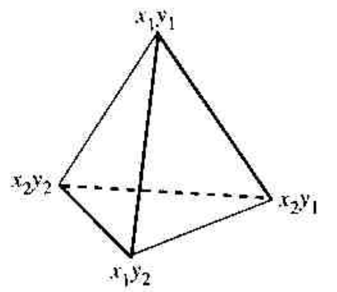

FIGURE 12. Simplex $\Delta(\theta)$ for $\alpha_1 = \alpha_2 = 2$, $\alpha_3 = 1$

The case when Θ consists of more than one collection θ^j can be reduced to the case $\#A = 1$ by the following assertion.

PROPOSITION 9. *For any $\Theta = (\theta^1, \dots, \theta^{\#A})$, there is natural isomorphism*

$$H_{*-\#A+1}(\Xi(\Theta)) = \otimes_{j=1}^{\#A} H_*(\Xi(\theta^j)).$$

PROOF. Indeed, this follows from the formula for the homology of a join.

THEOREM 8. *The complex $\Xi(\Theta)$ is acyclic in dimensions greater than $|A| - 2\#A - 1$, where $|A|$ is the total number of elements in all sets θ^j.*

PROOF. The proof is based on the Goresky–MacPherson formula (see [GM]) for the homology of subspace arrangements, cf. [BW, V3].

Indeed, by Proposition 9 it is sufficient to prove this theorem in the case when $\#A = 1$, i.e., we have only one group of points $\theta = \theta^1$ divided into k subsets of cardinalities $\alpha_1, \dots, \alpha_k$. Consider the space $\mathbb{R}^{\alpha_1} \oplus \dots \oplus \mathbb{R}^{\alpha_k}$ with fixed linear coordinates $x_{1,1}, \dots, x_{1,\alpha_1}$ in $\mathbb{R}^{\alpha_1}$ and so on. Consider the collection of all subspaces in this space distinguished by all possible systems of equations of the form $x_{i,\beta} = x_{j,\gamma} = x_{l,\delta}$ where $i \neq j \neq l \neq i$. By the Goresky–MacPherson formula (in the form proposed in Theorem 4 of [V3]) and the Alexander duality theorem, the homology group of the one-point compactification of the union of all such planes splits into the direct sum of homology groups of certain cell complexes corresponding to all possible intersections of these planes. In particular, the complex corresponding to the intersection of all these planes (i.e., to the line $x_{1,1} = \dots = x_{k,\alpha_k}$) is nothing but the suspension of our quotient complex $\Xi(\Theta)$. Therefore, the i-dimensional homology group of this quotient complex enters as a direct summand in the $(i+1)$-dimensional homology group of some $(|A| - 2)$-dimensional topological space. This implies the theorem.

§6. Structure of the space $\sigma_i - \sigma_{i-1}$

6.1. J-blocks and complexes of connected hypergraphs. The space $\sigma_i - \sigma_{i-1} \subset \Gamma_k^d \times \mathbb{R}^N$ splits in a natural way in a union of subspaces corresponding to different equivalence classes of A-configurations of complexity i. Indeed, let us fix some index $A = (a_1 \geq \dots \geq a_{\#A})$ and some A-configuration J in C_k. To this configuration we assign the subset in $\Gamma_k^d \times \mathbb{R}^N$ which is the direct product of the affine subspace $\chi(\Gamma_k^d, J) \subset \Gamma_k^d$ (see 4.1) and the simplex $\tilde{\Delta}(\varphi(J)) \subset \mathbb{R}^N$ (see 4.2), where $\varphi(J)$ is an arbitrary generic point of this subspace (i.e., a map that strictly respects J); this simplex depends on J only.

DEFINITION. For any A-configuration J of complexity i in C_k, the *J-block* in σ_i is the union of all products

$$\chi(\Gamma_k^d, J') \times \tilde{\Delta}(\varphi(J')) \subset \Gamma_k^d \times \mathbb{R}^N \tag{5}$$

where J' is some A-configuration equivalent to J, and $\varphi(J')$ is arbitrary map $C_k \to \mathbb{R}^2$ strictly respecting J'.

Obviously, the closure of the space $\sigma_i - \sigma_{i-1}$ in σ_i coincides with the union of all J-blocks over all (equivalence classes of) configurations J of complexity i.

For any A-configuration $J = (\theta^1, \dots, \theta^{\#A})$ with $\operatorname{card}\theta^j = a_j$, the simplex $\tilde{\Delta}(\varphi(J))$ can be identified in an obvious way with the simplex $\Delta[J]$ defined in §5; indeed, to the vertex $(x, y, z) \in \Delta[J]$ there corresponds the vertex $\lambda(x, y, z) \in \tilde{\Delta}(\varphi(J))$, and this correspondence extends inside the simplices by linearity.

PROPOSITION 10. *A point of the left product in* (5) *belongs to the term* σ_{i-1} *of our filtration if an only if its projection onto the factor* $\tilde{\Delta}(\varphi(J'))$ *lies in a nonessential face of this simplex.*

PROOF. This follows directly from the definitions.

6.2. Auxiliary spectral sequence. Define the *auxiliary filtration* on the space $\sigma_i - \sigma_{i-1}$ by assigning to its lth term the union of points of J-blocks over all A-configurations J with $|A| \le l$.

This filtration defines a new spectral sequence, $G^r_{p,q}$, converging to the group $\overline{H}_*(\sigma_i - \sigma_{i-1})$ which is (up to a shift of indices) just the column $E^1_{i,*}$ of the main homological spectral sequence from §4.

We call this spectral sequence G the *auxiliary* spectral sequence.

The term $G^1_{p,q}$ of this spectral sequence is the direct sum of the homology groups $\overline{H}_{p+q}((J\text{-block}) \setminus \sigma_{i-1})$ taken over all equivalence classes J of A-configurations with $|A| - \#A = i$, $|A| = p$.

6.3. Proof of part A(i) of Theorem 6. Consider an arbitrary J-block in σ_i, where J is an A-configuration of complexity i. The intersection of this J-block with the set $\sigma_i - \sigma_{i-1}$ can be considered as a fiber bundle whose base is the space of pairs of the form {a configuration J' equivalent to J; a map $\varphi\colon C_k \to \mathbb{R}^2$ respecting J'}, and the fiber is the essential part of the simplex $\Delta[J']$. By Proposition 3, the dimension of the base is no greater than $|A| + \dim \Gamma^d_k - 2(|A| - \#A)$, and the closed homology group of the fiber, $\overline{H}_l$, is trivial for $l > |A| - 2\#A - 1$. Hence, the term $G^1_{p,q}$ of the auxiliary spectral sequence is trivial for $p + q > \dim \Gamma^d_k - 1$. This completes the proof of Theorem 6.

6.4. On the calculation of the stable spectral sequence. If i is in the stable domain, $i < 2(\dim \Gamma^d_k + 1)/7$, and J is an A-configuration with $|A| - \#A = i$, then the corresponding J-block in $\sigma_i - \sigma_{i-1}$ is a fiber bundle whose base is the space of all A-configurations equivalent to J and the fiber is the product of a (canonically oriented) affine space and the essential part of the simplex

$\Delta(\varphi(J)) = \Delta[J]$. By the Thom isomorphism, we can forget about the first factor in the fiber (and, moreover, about all such factors for all such J-blocks simultaneously) and calculate the terms $E^1_{i,*}$ of the principal spectral sequence by investigating the remaining bundles and their interrelations for different J. For the first calculations see §8.

§7. Proof of Theorem 7

To prove this theorem, we construct one more resolution of the discriminant variety $\Sigma \subset \Gamma^d_k$, based on the notion of order complex. This resolution will be called the *visible resolution* of the discriminant and denoted by σv.

The space of this resolution again has a natural filtration equivalent to that of the resolution σ considered above (i.e., there is a natural proper embedding $\sigma v \mapsto \sigma$ preserving the filtrations and such that the induced morphisms

$$\overline{H}_*(\sigma v_i - \sigma v_{i-1}) \to \overline{H}_*(\sigma_i - \sigma_{i-1})$$

are isomorphisms, in particular this embedding establishes an isomorphism of the corresponding spectral sequences, starting from their terms E_1). All the notions in the definition of invariants of order i given in §2 (i.e., invariants, regular quasiornaments, degeneration modes, and characteristic numbers) can be naturally interpreted in terms of this resolution.

For example, the degeneration modes can be interpreted as follows. Again, the spaces $\sigma v_i - \sigma v_{i-1}$ are divided into J-blocks (which are compatible with the J-blocks in $\sigma_i - \sigma_{i-1}$ by means of the previous embedding). These J-blocks are fibered into certain simplicial complexes $\Delta v(J)$ (instead of simplices $\Delta(\varphi(J))$ for the resolution σ). For any A-configuration J, the topology of the corresponding complex $\Delta v(J)$ depends only on A, the dimension of this complex equals $|A| - 2\#A - 1$, and its simplices of highest dimension $|A| - 2\#A - 1$ are in one-to-one correspondence with the degeneration modes of J. Thus, the dimension of the J-block equals exactly $\dim \Gamma^d_k - 1$; this implies a new proof of assertion (A(i)) of Theorem 6.

After this interpretation of these notions, the equivalence of two definitions of the orders of invariants becomes almost tautological, see Proposition 17.

7.1. Two concepts of order complex of a collection of intersecting sets. Suppose that we have a collection of sets $V_1, \dots, V_l$. To such collection there correspond two (homotopy equivalent) simplicial complexes: the formal order complex and the visible order complex; let us define them.

Denote by L the set $(1, \dots, l)$ of indices of the sets V_i, by 2^L the set of all subsets in L, by V the union $V_1 \cup \dots \cup V_l$ and, for any $\alpha \in 2^L$, by V_α the intersection $\bigcap_{j\in\alpha} V_j$.

DEFINITION. The *formal order complex* related to the collection $V_1, \dots, V_l$ is the abstract simplicial complex whose i-dimensional simplices are sequences $(\alpha_0, \alpha_1, \dots, \alpha_i)$ of subsets in L such that

a) α_j strictly contains α_{j+1} for any $j = 0, \dots, i-1$;

b) the set V_{α_0} is nonempty.

The *visible order complex* related to the same collection is the abstract simplicial complex whose i-dimensional simplices are sequences

$$(V_{\alpha_0}, \dots, V_{\alpha_i}) \tag{6}$$

such that

a) V_{α_k} is strictly contained in $V_{\alpha_{k+1}}$ for any $k = 0, \dots, i-1$;

b) the set V_{α_0} is nonempty.

The notations for the formal and visible order complexes are

$$\Delta f(V_1, \dots, V_l) \quad \text{and} \quad \Delta v(V_1, \dots, V_l),$$

respectively.

Note that the same simplex (or even vertex) of the visible order complex can have different expressions in the form (6) if $V_\alpha = V_\beta$ for some $\alpha \neq \beta$.

EXAMPLE. Let $l = 3$ and the intersection $V_1 \cap V_2 \cap V_3$ be nonempty. Then, the formal and visible order complexes are shown on Figures 13 and 14 respectively. The picture from Figure 13 does not depend on which sets V_α, $V_{\alpha\cup\beta}$ are *strictly* incident, whereas the complex Δv depends strongly on this circumstance. In general, for arbitrary l, if the intersection of the sets $V_1, \dots, V_l$ is nonempty, then the corresponding formal order complex can be naturally identified with the first barycentric subdivision of the $(l-1)$-dimensional simplex whose vertices correspond to the sets $V_1, \dots, V_l$.

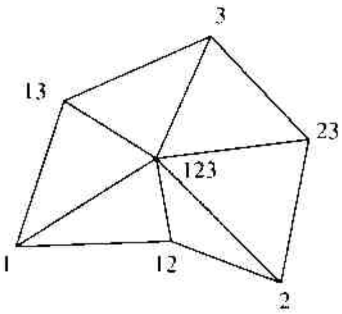

FIGURE 13. Formal order complex of three sets with nonempty intersection

NOTATION. For any $\alpha \subset L$, let $\bar{\alpha}$ be the maximal possible set in the family of all subsets $\alpha' \subset L$ such that $V_{\alpha'} = V_\alpha$.

PROPOSITION 11. *The visible order complex of any collection of sets $(V_1, \dots, V_l)$ can be naturally regarded as a subcomplex of the formal order complex of the same collection. Moreover, this subcomplex is a deformation retract of the formal order complex.*

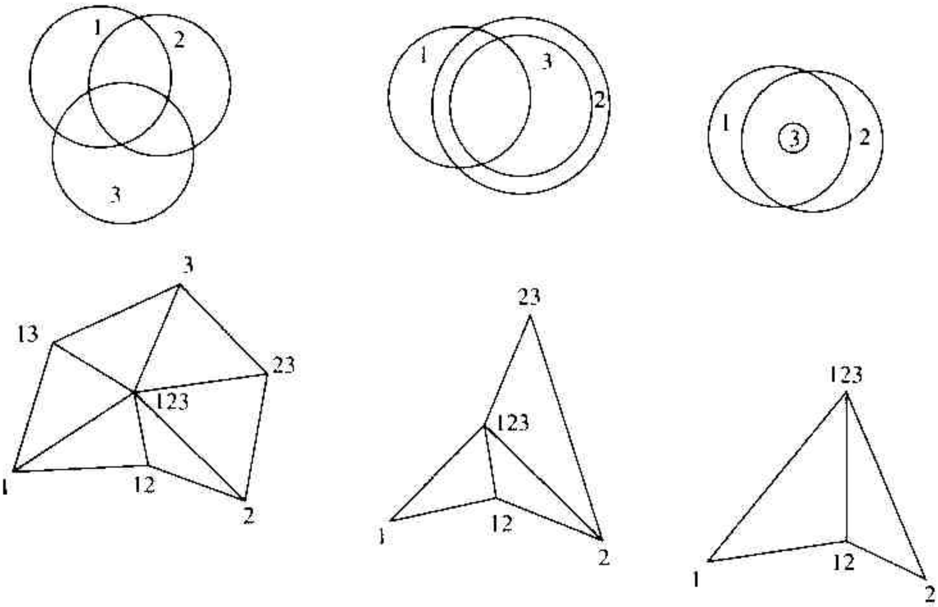

FIGURE 14. Visible order complexes for different three-element sets with nonempty intersections

PROOF. The 0-dimensional skeleton of the complex Δv is naturally embedded into that of Δf: to any point in Δv (i.e., to a set of the form V_α, $\alpha \in 2^L$) we assign the set $\overline{\alpha}$. This embedding I of 0-skeletons extends to a map of the whole complex Δv by linearity on any simplex, and the resulting map is, obviously, again an embedding: its image coincides with the union of all simplices $(\alpha_0, \dots, \alpha_i)$ such that $\alpha_j = \overline{\alpha}_j$ for any $j = 0, \dots, i$.

Now, let us construct the retraction $\Delta f \to \Delta v$.

We map any point $\alpha \in sk^0(\Delta f)$ to the point $\overline{\alpha} \in sk^0(I(\Delta v))$ and extend this map of 0-skeletons to a map $\Delta f \to I(\Delta v)$ by linearity on the simplices. This map is obviously continuous and is a deformation retraction: indeed, the inverse image of any point x in $I(\Delta v)$ consists of a family of segments in Δf which have no intersection points except for their endpoints (which all coincide with x).

7.2. Visible resolution of the discriminant.

LEMMA. *If the space Γ_k^d is Σ-nondegenerate, then there exists an affine subspace $\Gamma \subset \kappa_k$ containing Γ_k^d and such that for any A and any A-configuration J which is respected by at least one element of Γ_k^d, the codimension of the corresponding subspace $\chi(\Gamma, J)$ in Γ equals exactly $2(|A| - \#A)$ and has no other representations in the form $\chi(\Gamma, J')$, $J' \neq J$.*

PROOF. The proof is trivial.

Obviously, if some space Γ satisfies the conditions of this lemma, then any affine space containing Γ also satisfies these conditions.

Let μ be the largest possible complexity $|A| - \#A$ of configurations respected by the maps $\varphi \in \Gamma_k^d$.

Let Γ be a space satisfying the previous lemma, let $G^4(\Gamma), G^6(\Gamma), \dots, G^{2\mu}(\Gamma)$ be all the "affine" Grassmann manifolds whose points are affine subspaces of codimensions $4, 6, \dots, 2\mu$ in Γ. Let $h^4(\Gamma), h^6(\Gamma), \dots, h^{2\mu}(\Gamma)$ be the topological subspaces in these manifolds whose points are the subspaces of the form $\chi(\Gamma, J)$ for some A-configurations J. Consider the join $h^4(\Gamma) * \dots * h^{2\mu}(\Gamma)$ of these subspaces, i.e., roughly speaking, the (suitably topologized) union of all simplices whose vertices are the points of these spaces $h^{2l}(\Gamma)$. Denote this join by $X(\Gamma)$. The visible resolution will be regarded as a subset in the direct product $\Gamma_k^d \times X(\Gamma)$.

Definition. For any A-configuration J, the *feud* of J is a subset in $X(\Gamma)$ defined as the union of simplices whose vertices are points $g_{l_1} \in h^{2l_1}(\Gamma)$, $g_{l_2} \in h^{2l_2}(\Gamma), \dots$ with $l_1 < l_2 < \dots$, such that the corresponding planes of codimensions $2l_1, 2l_2, \dots$ in Γ form a flag (i.e., are all incident to each other) and all belong to $\chi(\Gamma, J)$.

Obviously, the feud of any A-configuration J is a contractible space: it is the union of simplices with one common vertex $g_{|A|-2\#A} = \{\chi(\Gamma, J)\}$.

For any point $\varphi \in \Sigma \subset \Gamma_k^d$, consider the A-configuration $J(\varphi)$ strictly respected by φ, and take the subset

$$\varphi \times \operatorname{feud}(J(\varphi)) \subset \Gamma_k^d \times X(\Gamma) \tag{7}$$

Define the *visible resolution* $\sigma v \subset \Gamma_k^d \times X(\Gamma)$ of the set $\Sigma \subset \Gamma_k^d$ as the union of sets like the left-hand side of (7) over all $\varphi \in \Sigma$.

Proposition 12. *The obvious projection $\sigma v \to \Gamma_k^d$ is proper and establishes a homotopy equivalence between the one-point compactifications of σv and Σ.*

Proof. The proof is the same as for Proposition 6.

7.3. Embedding the visible resolution into the (formal) resolution constructed in §4. Here is another description of the feud of $J(\varphi)$. Consider all points $(x_j, y_j, z_j) \in \Psi$ (where x, y and z are points of three different components of C_k) such that $\varphi(x_j) = \varphi(y_j) = \varphi(z_j)$. For any such point (x_j, y_j, z_j) denote by V_j the space $\chi(\Gamma, J_j)$, where J_j is the (3)-configuration (x_j, y_j, z_j).

Proposition 13. *The feud of $J(\varphi)$ is a finite simplicial complex and can be naturally identified with the visible order complex of all spaces $V_j = \chi(\Gamma, J_j)$.*

Proof. This is a tautology.

On the other hand, the simplex $\Delta(\varphi)$ which participated in the construction of the resolution σ in §4, can be naturally regarded as the support of the formal order complex of the same collection of spaces V_j (and becomes this formal order complex after barycentric subdivision). Therefore, we get a natural embedding $I: \sigma v \to \sigma$. Indeed, for any $\varphi \in \Sigma$ the restriction of this embedding on the set $(\varphi \times \text{feud}(J(\varphi)) \subset \sigma v$ is the composition of the identification of $\text{feud}(J(\varphi))$ with the visible order complex from Proposition 13, the embedding of this visible complex into the formal one (see Proposition 11) and identification of the support of the later complex with the set $\varphi \times \dot{\Delta}(\varphi) \subset \sigma$.

Since the set σv is a union of parts at the left-hand side of (7) over all $\varphi \in \Sigma$, these embeddings define a general embedding $I: \sigma v \to \sigma$.

Proposition 14. *This embedding I is continuous and proper, commutes with the natural projections onto Σ, and defines a homotopy equivalence of the one point compactifications of the spaces σv and σ.*

Proof. All assertions of this proposition but the last one (about the homotopy equivalence) follow immediately from the construction. Further, a deformation retraction $\sigma \to I(\sigma v)$ is well defined: it is the union of retractions from Proposition 11 applied to all fibers of the projection $\sigma \to \Sigma$. This retraction extends to a retraction of one-point compactifications of these spaces and establishes the desired homotopy equivalence.

On the space σ the filtration $\sigma_2 \subset \sigma_3 \subset \cdots$ is defined (see 4.3) as well as the decomposition of the sets $\sigma_i - \sigma_{i-1}$ into J-blocks. The embedding I just constructed induces similar structures on the space σv: they are defined as the inverse images of corresponding sets in σ. In particular, the induced filtration $\sigma v_2 \subset \sigma v_3 \subset \cdots$ defines in a standard way a spectral sequence $vE^r_{p,q}(d) \to \overline{H}_*(\sigma v)$.

Proposition 15. *The embedding $I: \sigma v \to \sigma$ induces an isomorphism of spectral sequence $vE^r_{p,q}(d)$ with the sequence $E^r_{p,q}(d)$ (constructed in 4.3) beginning with the terms vE^1, E^1.*

Proof. Indeed, the retraction $\sigma \to I(\sigma v)$ from Proposition 14 respects the filtration by the sets σ_i, σv_i, and the proposition follows.

Corollary. *The order i invariants could be defined via the filtration of the space σv, not of σ, as it was done in 4.5.*

7.4. Homology in σv_i and characteristic numbers.

Proposition 16. *Let φ be a point in Σ, and $J(\varphi)$ an A-configuration strictly respected by φ, $i = |A| - \#A$. Then the simplices of the simplicial complex* $\text{feud}(J(\varphi))$ *that do belong not only to σv_i, but also to σv_{i-1}, are exactly those which do not contain the point $\{\chi(\Gamma, J(\varphi))\} \in h^{2i}(\Gamma)$. The dimension of this simplicial complex equals $|A| - 2\#A - 1$, and its simplices* *of the highest dimension are in a natural one-to-one correspondence with the degeneration modes of the configuration* J.

PROOF. The one-to-one correspondence is defined as follows. Given a degeneration mode of $J(\varphi)$, for any $l = 1, 2, \ldots, |A| - 2\#A$ denote by $J_{(l)}$ the subconfiguration in $J(\varphi)$ consisting of all points marked on (and before) the lth step of degeneration. Then, to any such degeneration mode, there corresponds a simplex in $\mathrm{feud}(J(\varphi))$, whose lth vertex is the affine plane in Γ consisting of all maps $\varphi: C_k \to \mathbb{R}^2$, $\varphi \in \Gamma$, that respect the subconfiguration $J_{(l)}(\varphi)$. All the assertions of this proposition follow immediately from the definitions.

As a corollary, we get a new proof of Theorem 8.

Let M be an arbitrary invariant of ornaments. The class in $\overline{H}_{\dim \Gamma_k^d - 1}(\sigma v) = \overline{H}_{\dim \Gamma_k^d - 1}(\Sigma)$ that is dual to the restriction of the invariant M to $\Gamma_k^d - \Sigma$ can be regarded as the (uniquely determined) linear combination of fundamental cycles of maximal strata of σv.

Let us describe these strata.

Each of them belongs entirely to some space $\sigma v_i - \sigma v_{i-1}$, and, moreover, to some J-block in this space. Suppose that i is in the stable range, $i < 2(\dim \Gamma_k^d + 1)/7$, and J is any A-configuration with $|A| - \#A = i$. Then our J-block in $\sigma v_i - \sigma v_{i-1}$ can be naturally identified with the space of all triples

$$(J', \varphi, x), \tag{8}$$

where J' is a configuration equivalent to J, φ is a quasiornament which respects J', and x is a point of the set $\varphi \times \mathrm{feud}(J') \subset \Gamma_k^d \times X(\Gamma)$. The strata of maximal dimension of σv that belong to $\sigma v_i - \sigma v_{i-1}$ consist exactly of points of the form (8) such that φ strictly respects J' and x is a point of a maximal (i.e., of dimension $|A| - 2\#A - 1$) simplex in the complex $\mathrm{feud}(J')$.

Recall that the characteristic numbers of order i were defined in §2 as the functions of the following data: an invariant; an A-configuration J' with $|A| - \#A = i$; a regular quasiornament strictly respecting J'; a degeneration mode of this quasiornament (or, what is the same, of the configuration J'). Now we have interpreted all these data in terms of the resolution σv. Namely, the invariant is a linear combination of fundamental cycles of maximal strata, and all other arguments can be encoded by an appropriate point of any such stratum: indeed, regular quasiornament φ and configuration J' respected by φ are two first elements of the expression of this point in the form (8), and degeneration mode of this quasiornament is the maximal simplex in $\varphi \times \mathrm{feud}(J')$ containing the element x of this expression. Now we give an interpretation of the characteristic number in terms of these data.

First, note that the maximal strata in $\sigma v_i - \sigma v_{i-1}$ are naturally oriented. Indeed, given a point (8) of such a stratum, consider three sets of neighboring points: the set of all $\tilde{J}' \approx J'$; for a given $\tilde{J}' \approx J'$, the set of all $\tilde{\varphi} \approx \varphi$

respecting $\tilde{J}'$; and, for given such $\tilde{J}'$ and φ, the set of all $\tilde{x} \approx x$ such that $(\tilde{J}', \tilde{\varphi}, \tilde{x})$ is again a point of our stratum. All these three sets are invariantly oriented; let us define these orientations.

The set of allowed $\tilde{x} \approx x$ is a neighborhood of x in the maximal simplex of the complex $\mathrm{feud}(J')$, or, what is the same by Proposition 13, of a visible order complex; the vertices of this simplex (and hence also the simplex itself) are naturally ordered by the definition of this complex, the simplex is thus naturally oriented.

The set of neighboring $\tilde{\varphi} \approx \varphi$ is oriented by Proposition 7.

Finally, an orientation of the space of all $\tilde{J}' \approx J'$ is nothing but some ordering of the points in J'. A partial ordering of these points is defined by the degeneration mode of J' corresponding to the simplex containing x; this partial ordering does not distinguish only the points in the triples that correspond to the same step of this degeneration. But these points in the triples are ordered by the numbers of components of C_k that contain them. So, we have defined a canonical orientation of our maximal stratum, and hence our invariant (i.e., a linear combination of fundamental cycles of such strata) assigns a number to any such stratum (and, in particular, to any of its point (J', φ, x)), namely, the coefficient with which this stratum (taken with the orientation just defined) participates in this linear combination.

PROPOSITION 17. *Suppose that the point* (8) *belongs to a maximal stratum of* $\sigma v_i - \sigma_{i-1}$ *and the quasiornament* φ *in* (8) *is regular. Then the number just assigned to any invariant and the point* (8), *coincides* (*up to a sign*) *with the characteristic number defined as in* §2 *by our invariant, by the quasiornament* φ, *and by the degeneration mode corresponding to the simplex in* $\mathrm{feud}(J')$ *that contains the point* x.

PROOF. We prove this proposition by induction over the degeneration mode corresponding to this simplex. Consider the $(|A|-2\#A-2)$-dimensional face (of this simplex) that does not contain the maximal vertex $\{\chi(\Gamma, J')\}$ of the complex $\mathrm{feud}(J')$; let x' be arbitrary interior point of this face. The point $(\varphi, x') \subset \Gamma_k^d \times X(\Gamma)$ belongs to σv_{i-1} (or even to σv_{i-2} if the last step of the degeneration mode is the marking of a triple of points). It does not belong to a maximal stratum in σv_{i-1} (resp., in σv_{i-2}), because φ respects a too complicated configuration.

Close to this point (φ, x') there are exactly three maximal strata of σv: one of them is the previously considered stratum in $\sigma v_i - \sigma v_{i-1}$ containing the point (φ, x), and the two others lie in σv_{i-1} (in σv_{i-2}) and correspond to two different resolutions of φ and to their degeneration mode which is just the degeneration mode corresponding to x without the last step.

Now the assertion of the proposition follows from the inductive assumption applied to these two resolutions and from the fact that our linear combination of maximal strata is a cycle in σv.

This proposition implies Theorem 7.

§8. The first calculations in the stable spectral sequence

In this section we consider only the case when $k = 3$ and d is sufficiently large, so that for all A-configurations undert consideration the corresponding spaces $\chi(\Gamma_k^d, J)$ have codimension $2(|A| - \#A)$ in Γ_k^d.

The main result of this section is the following theorem.

THEOREM 9. *There is no order* 1 *invariants of* 3*-ornaments, exactly one (up to multiplicative constant) invariant of order* 2, *exactly three more linearly independent invariants of order* 3, *and exactly seven more linearly independent invariants of order* 4.

All these invariants can be reduced to those from 1.4, for this reduction see 8.5.

We only outline the calculations which prove this theorem.

Recall that all J-blocks are the spaces of certain affine bundles. Since the exact value of d is not significant, we indicate the *codimensions* of these bundles, i.e., the differences between the dimensions of Γ_k^d and of the fibers of the bundles. For any J-block in $\sigma_i - \sigma_{i-1}$ this codimension equals $2i$.

8.1. The term σ_1 of the filtration of the space σ is empty: the simplest singularity is the triple point, for which $|A| - \#A = 3 - 1 = 2$.

8.2. The term σ_2 is the space of an oriented affine fiber bundle of codimension 4 over a 3-dimensional torus, therefore the groups $E_1^{-2,q}$ equal $\mathbb{R}$ for $q = 2, 5$, equal $\mathbb{R}^3$ for $q = 3, 4$, and are trivial for all other q.

The group $E_1^{-2,2}$ obviously stabilizes at that term, the corresponding invariant of order 2 is just the *index* $i(\varphi) = M_{1,2}(0, 0, 1)$, see 1.4.

8.3. The term $\sigma_3 - \sigma_2$ consists of three J-blocks, where J is an A-configuration with $A = (4)$, and J has two points on certain component of C_3 and one point on any of the other two components. Any of these blocks is the space of a fiber bundle whose base is the space of all such configurations, and the fiber is the product of a canonically oriented affine space of codimension 6 and an open interval (the missing endpoints of these intervals lie in the boundary of $\sigma_3 - \sigma_2$ in σ_2).

The base of this bundle is obviously diffeomorphic to the direct product of the two-dimensional torus and an open Möbius band; the bundle of intervals changes its orientation after traversing exactly the same loops in the base that destroy the orientation of the base. Hence, the space of this bundle of intervals is diffeomorphic to the direct product of a 3-torus and an open two-dimensional disk.

In particular, the contribution of any of these three J-blocks into the group $E_1^{-3,q}$ is isomorphic to $\mathbb{R}$ for $q = 3, 6$, to $\mathbb{R}^3$ for $q = 4, 5$, and is trivial for all other q.

8.4. The term $\sigma_4 - \sigma_3$ consists of seven J-blocks.

8.4.1. The three simplest blocks correspond to the A-configurations J, where $A = (5)$ and J has three points on one component, and one point on any of the other two. This J-block is the space of a bundle whose base is diffeomorphic to the direct product of a 3-torus and an open two-dimensional disk, and the fiber is the product of an oriented affine space of codimension 8 and an orientable bundle whose fiber is the interior part of an triangle.

Since these J-blocks do not adjoin other blocks of the same set $\sigma_4 - \sigma_3$, the contribution of any of these blocks to the groups $E_1^{-4,q}$ equals $\mathbb{R}$ for $q = 4, 7$, equals $\mathbb{R}^3$ for $q = 5, 6$, and is trivial for all other q.

8.4.2. The second collection of three J-blocks corresponds to (5)-configurations J with one point on one of components and two points on any of two others. The base of the corresponding fiber bundle is the product of a circle and two Möbius bands, while the fiber is a product of an oriented affine space of codimension 8 and the *essential part* of the tetrahedron from Example 2 in §5, see Figure 12.

In particular, the fiber of this bundle of essential parts has nontrivial closed homology only in dimension 2 (and the corresponding homology group is one-dimensional). It is easy to calculate that the generator of this group becomes its opposite after monodromy over the orientation-reserving paths in the base.

Since these blocks have the smallest auxiliary filtration among the J-blocks with J of complexity 4, the contribution of any of these three J-blocks to $E_1^{-4,4}$ is again equal to $\mathbb{R}$.

8.4.3. Finally, one more J-block corresponds to the A-configuration J with $A = (3, 3)$. This A-block is again a fiber bundle; let us describe its base.

A two-fold covering of this base (whose leaves correspond to the orderings of two triples in the $(3, 3)$-configuration) is diffeomorphic to the direct product of three copies of the direct product of a circle and an open interval, hence is orientable. This orientation fails after projection onto the base of the covering: this two-fold covering coincides with the orientation covering of the base.

The fiber of our bundle is the direct product of a canonically oriented affine plane of codimension 8 and an interval, whose orientation fails over the orientation-reversing paths in the base. In particular, the space of the bundle is orientable, hence the contribution of our J-block to $E_1^{-4,4}$ is either $\mathbb{R}$ or zero, depending on the homology class of its boundary in the J-blocks considered in 8.4.2.

The corresponding *geometrical boundary* (i.e., the intersection of the closure of our block with the blocks from 8.4.2) coincides with the subbundle in the bundle from 8.4.2, whose fiber corresponds to the union of the edges $((x_1, y_1, z), (x_2, y_2, z))$ and $((x_1, y_2, z), (x_2, y_1, z))$ on Figure 12; it is easy to calculate that this geometrical boundary participates in the algebraic boundary twice with opposite orientations, and the auxiliary spectral sequence degenerates in the corresponding term. So, our J-block enters nontrivially into the group $E_1^{-4,4}$, which is hence equal to $\mathbb{R}^7$.

8.5. All the described elements of the groups $E_1^{-i,i}$, $i = 2, 3, 4$, do not vanish at the next steps of the spectral sequence. In particular, there exist certain invariants of order i mapped into these elements by the obvious reductions mod σ_{i-1}.

All these invariants are the invariants $M_{i,j}$ from 1.4 or functions of them.

THEOREM 10. *The only generator of the group* $E_1^{-2,2}$ *coincides with the index* $i(\varphi) = M_{1,2}(0, 0, 1)$. *Three generators of group* $E_1^{-3,3}$ *are reductions mod* σ_2 *of the invariants* $M_{1,2}(0, 0, 2)$, $M_{1,3}(0, 2, 0)$ *and* $M_{2,3}(2, 0, 0)$ (*in the case of homology over* $\mathbb{R}$; *in the case of integer homology one must take similar sums of the form*

$$\sum_{b_1, b_2, b_3=-\infty}^{\infty} ((b_3^2 - b_3)/2) I_{1,2}(b_1, b_2, b_3)$$

and so on).

The seven generators of $E_1^{-4,4}$ *are the reductions mod* σ_3 *of the following invariants:* $M_{1,2}(0, 0, 3)$, $M_{1,3}(0, 3, 0)$, *and* $M_{2,3}(3, 0, 0)$ (*or, again, three sums of the form*

$$\sum_{b_1, b_2, b_3=-\infty}^{\infty} ((b_3^3 - b_3)/6) I_{1,2}(b_1, b_2, b_3)$$

in the case of integer coefficients) *for the generators from* 8.4.1; *all three possible momenta of the form* $M_{i,j}(1, 1, 1)$ *for the generators from* 8.4.2, *and* $i^2 = (M_{1,2}(0, 0, 1))^2$ *for the generator from* 8.4.3.

PROBLEM. Due to Theorem 4, all other invariants $M_{i,j}(\beta_1, \beta_2, \beta_3)$ with $\beta_1 + \beta_2 + \beta_3 \leq 3$ are linear combinations of those indicated in Theorem 10. What are the exact expressions for them?

§9. Open problems and possible generalizations

PROBLEM 1. Is the system of finite-order invariants complete, i.e., does it distinguish any two nonequivalent ornaments?

In particular, do there exist invariants of this kind proving that the ornaments from Figure 5 of the present paper and of [Mx] are nontrivial? Of which order are the simplest such invariants?

For the parallel theory of the homotopy classification of links, the answer is affirmative, see [L, BN2].

What is the smallest order of finite-order invariants that cannot be expressed as functions of the invariants $M_{i,j}$ introduced in 1.4? My guess is that this will be related to one of the following situations: a) the homology group of some complex $\Xi(\Theta)$ is more than one-dimensional in the highest possible dimension (see Theorem 8); b) there exist several nonequivalent A-configurations with the same A and all numbers of points of groups of cardinalities $a_1, \dots, a_{\#A}$ on any component. The situation b) can be realized by the $(4, 4)$-configurations such that any of two groups of 4 points constituting this configuration, have two points on the first component of C_k, one point on the second and one point on the third component. Indeed, these two pairs of points from different groups on the first component can either separate each other or not.

Problem 2. A problem similar to the classification of ornaments can be stated as follows: we consider the space of all plane curves (or collections of curves) having no triple points and no singularities obtained as degenerations of triple points (i.e., neither points where two local branches intersect and for one of them this point is a singular point with $\varphi' = 0$; nor points at which $\varphi' = \varphi'' = 0$.) The problem of classifying such objects has the same relationship with the above classification of ornaments, as isotopy classification of links has with homotopy classification. Concerning this problem see [A4].

A spectral sequence, which is a hybrid of the spectral sequences considered in §§4–6 above and in [V2], calculates the invariants of such objects. A partial problem is to study this spectral sequence explicitly.

Problem 3. In both cases considered in Problem 2 and in the main text, instead of triple intersections we can forbid (self)intersections of arbitrary multiplicity l, $l > 3$. In this case, the space of permitted ornaments is connected (and even $(l-3)$-connected), in particular the problem of classifying such objects up to homotopy is void. But the problem of calculating the higher-dimensional cohomology of the spaces of permitted objects is nontrivial and can be, in principle, solved by using a spectral sequence similar to the one from §4. In this case this spectral sequence (beginning with the term E_1) lies inside the angle

$$\{p, q | p < 0, q + (l-2)p \geq 0\};$$

in particular on any line of the form $p + q = \text{const}$ there is only a finite number of finitely generated groups, and the problem of the convergence of the spectral sequence does not arise.

Problem 4. The classification of ornaments is a model case of a big class of problems introduced in [FNRS]. Namely, let $(W_1, \dots, W_k)$ be arbitrary collection of compact manifolds of arbitrary dimensions. The problem is to study the space of all maps of (the union of) these manifolds into $\mathbb{R}^n$ having no common points of their images (or, more generally, points where the images of some t different components W_i meet, $t \leq m$).

Again, if the dimensions of the manifolds W_i are such that the space of permitted maps is dense in the space of all maps, then a spectral sequence similar to the one above presenting the cohomology classes of such spaces can be constructed. Problem: to study these spaces (and these spectral sequences).

Another problem appears if we can vary not only the maps of manifolds W_i into $\mathbb{R}^n$, but also make standard cobordisms of these manifolds, cf. [FR]. This problem has obvious application to the following well-known problem: to classify connected components of the space of homogeneous polynomial vector fields of fixed degree in $\mathbb{R}^n$ that have no singular points outside the origin.

PROBLEM 5. Do there exist invariants of ornaments arising from statistical physics in the same way as certain invariants of knots and links do?

PROBLEM 6. Does the spectral sequence from §4 degenerate at the term E_1 (at least in the case of rational coefficients, or on the main diagonal $\{p+q=0\}$? In [K1], a similar fact was established for the invariants of knots constructed in [V1].

PROBLEM 7. Does there exist a representation of our invariants by means of integrals, as it was done in [K1] for the (rational) invariants constructed in [V1]? A related question: to represent by differential forms the cohomology of complements of "k-equal" arrangements considered in [BW] and of their generalizations considered in the proof of Theorem 8, see §5.

PROBLEM 8. Brunnean ornaments. For any $k>3$, construct a k-ornament which is nontrivial, but all its $(k-1)$-subornaments are trivial. A natural candidate is the ornament constructed by Merkov, see [Mx]. The fact that any of its subornaments splits is elementary, but the nontriviality of this ornament itself is not proved yet.

PROBLEM 9. To study ornaments on any smooth surface. Note that the discriminant is invariantly cooriented even in the case of ornaments on a nonorientable surface. Indeed, any local surgery as on Figure 3 can be carried out as follows: we fix the first two local branches and move the third one parallel to itself. Then the sign of this surgery can be deduced from the comparison of two *local* orientations: the first is given by the tangent frame of first two components, and the second by the frame {tangent to the third component, direction of the translation of this component}.

PROBLEM 10. To write out all linear relations among the invariants $M_{i,j}(\beta_1, \ldots, \beta_k)$ from 1.4.

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