

## Invariants of Ornaments

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**ABSTRACT.** An *ornament* is a collection of oriented closed curves in a plane, no three of which intersect at the same point. The classification of ornaments up to natural equivalence relation is parallel to the homotopy classification of links. We construct a series of invariants of ornaments, many of which have a very classical interpretation.

The general construction leads naturally to problems in modern homological combinatorics (see [BW, Bj]) and seems to be closely related to the higher-dimensional analogues of the Chern-Simons theory and Yang-Baxter equation (see [MS, FNRS]).

### §0. Introduction

Denote by  $C_k$  the disjoint union of  $k$  circles.

**DEFINITION.** A  $k$ -*ornament* (or simply an *ornament*) is a  $C^\infty$ -smooth map  $C_k \rightarrow \mathbb{R}^2$  such that the images of no three different circles intersect at the same point in  $\mathbb{R}^2$ . Two ornaments are *equivalent* if the corresponding maps  $C_k \rightarrow \mathbb{R}^2$  can be connected by a homotopy  $C_k \times [0, 1] \rightarrow \mathbb{R}^2$  such that for any  $t \in [0, 1]$  the corresponding map  $C_k \times t \rightarrow \mathbb{R}^2$  is an ornament.

(Similar objects were considered in [FT] under the name *doodles*: the only difference is that doodles are collections of Jordan curves (without selfintersections). Of course, invariants of ornaments are also invariants of doodles; conversely, the invariants introduced in [FT] can be easily generalized to invariants of ornaments, see 1.2 in [Mx].)

In this paper we construct a series of numerical invariants of equivalence classes of ornaments. Like the knot invariants in [V1, V2], these invariants appear from the study of the *discriminant*, i.e., the space of all maps  $C_k \rightarrow \mathbb{R}^2$  which necessarily have triple intersections.

(We follow again the general strategy from [A2]: to replace the study of the soft, homogeneous space of nonsingular objects by the study of the complementary space of singular objects, which usually has a rich geometrical structure.)

Using the geometry of the discriminant, we construct a spectral sequence  $E_r^{p,q}$  which calculates the cohomology groups of the space of ornaments; in

particular the groups  $E_{\infty}^{-i,i}$  of this spectral sequence provide invariants of ornaments.

A lot of our invariants can be interpreted in absolutely classical terms; these invariants are described in 1.4 below.

As in [V1, V2], the invariants coming from the cell  $E^{-i,i}$  of this spectral sequence are called *invariants of order  $i$* , and all such invariants corresponding to different  $i$  are called *finite-order invariants*.

This theory is a model version of a wide class of problems (stated in [FNRS] in connection with the higher-dimensional generalizations of Chern-Simons theory) where a similar technique works: for example, the next problem of this class is the classification of all maps of  $k$  two-spheres (or arbitrary fixed Riemannian surfaces) in  $\mathbb{R}^3$  in such way that no four of them intersect at the same point. The construction of our invariants can be immediately generalized to these problems.

There are also many invariants of ornaments (due to Fenn, Taylor, and Merkov) which seem to be specifically "one-dimensional", see 1.2, 1.3 of the present paper and §§1, 2 in the article of A. Merkov in this volume.

In §§2, 3 we describe the elementary characterization of finite-order invariants and show how to calculate the values of these invariants on an ornament. In §§4–7 we construct and investigate the principal spectral sequence which provides such invariants. The first calculations are presented in §8. A large list of unsolved problems is given in §9.

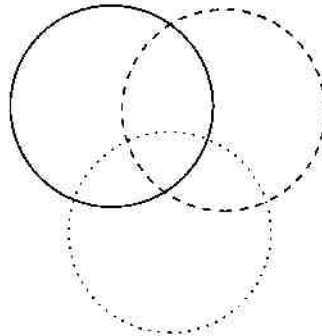


FIGURE 1. The simplest nontrivial ornament

**About the pictures.** In almost all the pictures and examples no more than three components of  $C_k$  participate. We depict these components by ordinary, dotted, and dashed lines, see Figure 1.

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§1. Elementary theory

1.1. Reidemeister moves.

DEFINITION. An ornament  $\varphi: C_k \rightarrow \mathbb{R}^2$  is *regular* if it is an immersion of  $C_k$ , and all the multiple points of the image of  $C_k$  in  $\mathbb{R}^2$  are double transversal intersection points.

THEOREM 1. Any ornament is equivalent to a regular ornament. Two regular ornaments are equivalent if and only if they can be transformed one into the other by a finite sequence of isotopies of  $\mathbb{R}^2$  (which do not change the topological picture of the image of the ornament), and of local moves shown on Figure 2 (or obtained from these moves by recoloring the strands).

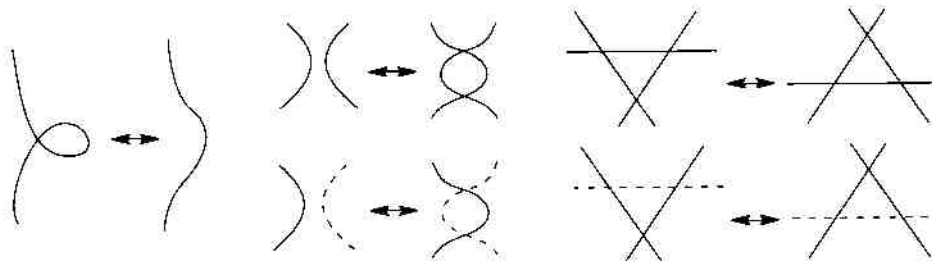


FIGURE 2. Reidemeister moves

(In other words, only the local move shown on Figure 3 is forbidden among the moves which can appear in a generic homotopy of a generic smooth map  $C_k \rightarrow \mathbb{R}^2$ ).

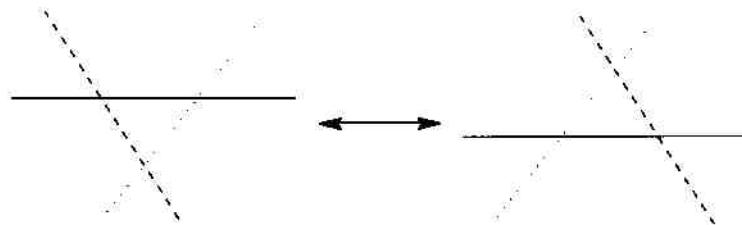


FIGURE 3. The forbidden move

PROOF. This theorem follows immediately from the Thom multijet transversality theorem, see [GG].

**1.2. On Fenn-Taylor invariants.** In [FT], Fenn and Taylor introduced an invariant of *doodles*, i.e., of collections of Jordan curves without triple intersections on a two-sphere. This invariant can be easily generalized to an invariant of ornaments on two-spheres. The values of this invariant are collections of  $k$  elements of the free group with  $k - 1$  generator considered up to cyclic permutations of symbols.

For a description and a generalization of these invariants, see [Mx].

**1.3. Reduction to the homotopy classification of links.** The classification of  $k$ -ornaments can be partially reduced in several ways to the homotopy classification of links. These reductions (which essentially also were introduced in [FT]) are numbered by the orientations of the complete graph with  $k$  vertices: given such an orientation, we assign a  $k$ -component link in  $\mathbb{R}^3$  to any  $k$ -ornament in such way that to equivalent ornaments there correspond homotopy equivalent links.

Indeed, let us fix such an orientation. Make a link diagram from the (image of) our regular ornament in the following way: the  $i$ th string goes everywhere under the  $j$ th at their crossing points if the edge  $(ij)$  of the complete graph is oriented from the  $i$ th vertex to the  $j$ th. At the selfintersection points of the same component the over/undercrossings may be chosen in an arbitrary way.

**THEOREM 2.** *If two ornaments are equivalent, then the links assigned to them by the above rule (based on an arbitrary orientation of the complete graph) are homotopy equivalent.*

Indeed, by Theorem 1, any Reidemeister move can be lifted to an admissible move of a link which preserves its homotopy class, and the resulting link diagram again satisfies the above rule for over/undercrossings.

In particular, homotopy invariants of links provide the invariants of ornaments. These invariants can be nontrivial: for instance, any cyclic orientation of the complete graph with three vertices transforms the 3-ornament from Figure 1 into the Borromean link.

**1.4. Index-type invariants.** Recall that any closed oriented immersed curve  $c$  in  $\mathbb{R}^2$  defines an integer-valued function  $\text{ind}_c$  on its complement: for any point  $t$  of the complement,  $\text{ind}_c(t)$  equals the rotation number of the vector  $(t, x)$  when  $x$  runs once around  $c$ .

To any regular  $k$ -ornament we assign  $\binom{k}{2}$  functions  $I_{i,j} = I_{i,j}(b_1, \dots, b_k)$ ,  $1 \leq i < j \leq k$ , with integer values and arguments; these functions are invariant under the moves from Theorem 1 and hence define invariants of ornaments.

To do this, to any (transversal) intersection point  $x$  of the  $i$ th and  $j$ th curves we assign  $k$  integers  $b_1(x), \dots, b_k(x)$  and a sign  $\sigma(x)$  in the following way.

If  $l \neq i, j$ , then  $b_l(x)$  is just the number  $\text{ind}_l(x)$ , the index of  $x$  with respect to the  $l$ th curve. Now, close to any regular point of the  $i$ th curve (in particular, to the intersection point  $x$ ) the values of the corresponding function  $\text{ind}_i(\cdot)$  take two neighboring integer values on different sides of the curve. Define the number  $b_i(x)$  as the smallest of these values at the neighboring points to  $x$ . The number  $b_j(x)$  is defined in the same way by means of  $\text{ind}_j$ . Finally,  $\sigma(x)$  equals 1 if the tangent vectors of the  $i$ th and  $j$ th curves at the point  $x$  define a positive frame (with respect to a fixed orientation of  $\mathbb{R}^2$ ) and equals  $-1$  if this frame is negatively oriented.

Given a regular  $k$ -ornament and  $k$  integers  $b_1, \dots, b_k$ , define the number  $I_{i,j}(b_1, \dots, b_k)$  as the number of transversal intersection points  $x$  of the  $i$ th and  $j$ th curves of our ornament such that  $b_1(x) = b_1, \dots, b_k(x) = b_k$  and  $\sigma(x) = 1$ , minus the number of similar points with  $\sigma(x) = -1$ .

**THEOREM 3.** *All the functions  $I_{i,j}$ ,  $1 \leq i < j \leq k$ , are invariant under all the Reidemeister moves of Figure 2.*

**PROOF.** Immediate.

The functions  $I_{i,j}$  are not independent. For instance, for  $k = 3$  let us define the numbers  $i_{1,2}, i_{2,3}, i_{3,1}$  as the sums

$$\sum_{b_1, b_2, b_3 = -\infty}^{\infty} b_3 I_{1,2}(b_1, b_2, b_3), \quad \sum_{b_1, b_2, b_3 = -\infty}^{\infty} b_1 I_{2,3}(b_1, b_2, b_3),$$

$$\sum_{b_1, b_2, b_3 = -\infty}^{\infty} -b_2 I_{1,3}(b_1, b_2, b_3),$$

respectively.

**PROPOSITION 1.** *All three numbers  $i_{1,2}, i_{2,3}, i_{3,1}$  coincide.*

**PROOF.** Indeed, for an unlinked ornament all three numbers are equal to 0, and any forbidden move from Figure 3 simultaneously increases or decreases by 1 all three numbers.

This number  $i_{1,2}$  is called the *index* of the 3-ornament  $\varphi$  and denoted by  $i(\varphi)$ .

More generally, for any  $k \geq 3$  and any  $k$ -ornament  $\varphi$ , define the *index*  $i(\varphi)$  of  $\varphi$ , as the number

$$\frac{1}{3} \sum_{1 \leq i < j \leq k} \sum_{b_1, \dots, b_k = -\infty}^{\infty} (b_1 + \dots + b_{i-1} - b_{i+1} - \dots$$

$$- b_{j-1} + b_{j+1} + \dots + b_k) I_{i,j}(b_1, \dots, b_k)$$

This number  $i(\varphi)$  is always an integer: again, any elementary surgery of Figure 3 decreases or increases the previous double sum by 3.

In a similar way, given a regular  $k$ -ornament, for any  $k$  integer nonnegative exponents  $\beta_1, \dots, \beta_k$  we can define the corresponding momenta

$$M_{i,j}(\beta_1, \dots, \beta_k) = \sum_{b_1, \dots, b_k = -\infty}^{\infty} b_1^{\beta_1} \cdots b_k^{\beta_k} I_{i,j}(b_1, \dots, b_k).$$

It is natural to call the function  $M_{i,j}$  the Laplace transform of  $I_{i,j}$ .

Since all the functions  $I_{i,j}$  are finite, they can be reconstructed from their Laplace transforms.

Here are some other relations on the indices  $I_{i,j}$  and their momenta.

**PROPOSITION 2.** For any  $1 \leq i < j \leq k$  and any two values  $b_i$  and  $b_j$ , the sum

$$\sum_{b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{j-1}, b_{j+1}, \dots, b_k = -\infty}^{\infty} I_{i,j}(b_1, \dots, b_k)$$

equals 0.

**PROOF.** The curves other than the  $i$ th and  $j$ th ones actually do not participate in these sums; after they are removed the statement becomes trivial.

Here is an equivalent reformulation of this proposition in terms of momenta.

**PROPOSITION 2'.** If  $\beta_l = 0$  for all  $l$  other than  $i$  or  $j$ , then  $M_{i,j}(\beta_1, \dots, \beta_k) = 0$ .

**REMARK.** The construction of the invariants  $I_{i,j}$  and  $M_{i,j}$  can be immediately extended to that of invariants which distinguish maps of collections of  $(n-1)$ -dimensional manifolds in  $\mathbb{R}^n$ , no  $n+1$  of which intersect at the same point; the corresponding functions  $I$  and  $M$  in this case have  $n$  lower indices.

**REMARK.** I expect that there are many other elementary invariants of ornaments, and the spectral sequence of §4 can be considered as a regular method of guessing them: for instance, I guessed the invariants  $I_{i,j}$  and  $M_{i,j}$  after calculating the terms  $E^{-i \cdot i}$  of the sequence with  $i = 2, 3, 4$ .

For a generalization of these index-type invariants see §§3, 4 in [Mx].

**1.5. Examples. A.** The simplest picture of the nontrivial ornament (see Figure 4) has 16 nonequivalent realizations, depending on the orientation and ordering (coloring) of circles. All of them can be distinguished by the functions  $I_{i,j}$ . The Fenn-Taylor invariants split these 16 ornaments into two groups, with 8 ornaments in each (and are constant on any of these two groups): indeed, all ornaments in any of these groups are equivalent as ornaments on a sphere.

**B.** For the ornament in Figure 5, all invariants  $I_{i,j}$  vanish. However, this ornament is nontrivial because so is its Fenn-Taylor invariant.

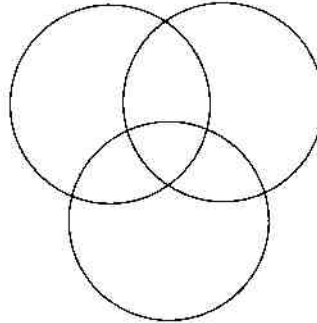


FIGURE 4.

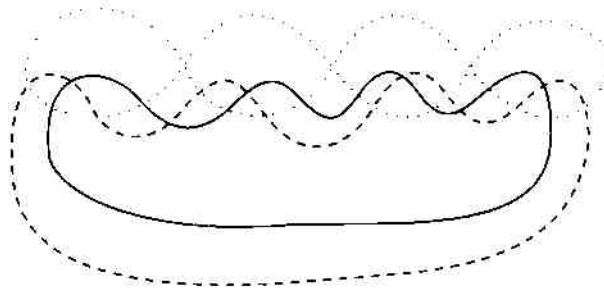


FIGURE 5. An ornament which annihilates all index-type invariants

C. For the ornament in Figure 5 of the work [Mx], both Fenn-Taylor invariant and invariants  $I_{i,j}$  vanish. (This example is due to Merkov, who also proved the nontriviality of this ornament by using a refinement of the Fenn-Taylor invariant, see [Mx].) It seems also likely that the link obtained from this ornament by the construction of 1.3 is homotopy nontrivial, which also proves the nontriviality of this ornament.

## §2. Elementary definition of finite-order invariants

**DEFINITION AND NOTATION.** A *quasiornament* is any  $C^\infty$ -smooth map  $C_k \rightarrow \mathbb{R}^2$ . The space of all  $k$ -quasiornaments is denoted by  $\kappa_k$ . The *discriminant*  $\Sigma \subset \kappa_k$  is the space of all quasiornaments that are not ornaments, i.e., have triple points.

The discriminant is a singular subvariety in  $\kappa_k$ . Its regular points are the quasiornaments having only one forbidden triple point such that the three local branches at the triple point are smooth and pairwise nontangent; singular points of  $\Sigma$  correspond to quasiornaments with several singularities or with more complicated singularities. A natural stratification of the discriminant is provided by the classification of these singularities.

