

The absolute of finitely generated groups: I. Commutative (semi)groups*

A. M. Vershik ^{$\alpha, \beta, \gamma, \dagger$} and A. V. Malyutin ^{$\alpha, \beta, \ddagger$}

^{α} *St. Petersburg Department of Steklov Institute of Mathematics,*

^{β} *St. Petersburg State University,*

^{γ} *Institute for Information Transmission Problems, Moscow*

Abstract

We give a complete description of the absolute of commutative finitely generated groups and semigroups. The absolute (previously called the exit boundary) is a further elaboration of the notion of the boundary of a random walk on a group (the Poisson–Furstenberg boundary); namely, the absolute of a (semi)group is the set of all ergodic probability measures on the compactum of all infinite trajectories of a simple random walk which has the same so-called cotransition probability as the simple random walk. Related notions have been discussed in the probability literature: Martin boundary, entrance and exit boundaries (Dynkin), central measures on path spaces of graphs (see [10]). The main result of this paper, which is a far-reaching generalization of de Finetti’s theorem, is as follows: the absolute of a commutative semigroup coincides with the set of central measures corresponding to (nonstationary) Markov chains with independent identically distributed increments. Topologically, the absolute is (in the main case) a closed disk of finite dimension.

1 Introduction

The problem of describing the set of all Borel measures satisfying some invariance condition is typical for several areas of mathematics (probability, dynamical systems, graph theory, representation theory, etc.). Its most general setting presumes the existence of some equivalence relation on a Borel space (for instance, the orbit partition for a group action) and a 2-cocycle on this equivalence relation, and the problem is to describe all probability measures for which this cocycle is the Radon–Nikodym cocycle.

*Partially supported by the RFBR grant 17-01-00433.

^{\dagger} e-mail: avershik@gmail.com

^{\ddagger} e-mail: malyutin@pdmi.ras.ru

If the equivalence classes are countable, the equivalence relation itself is hyperfinite (i. e., is a monotone limit of finite equivalence relations), and the cocycle is identically 1, then the problem reduces to describing the so-called central measures (see below) on the space of infinite paths of a graded graph (Bratteli diagram). In a certain sense, the notion of centrality coincides with the notion of invariance, namely, if we introduce the transformation of paths called the adic shift, then the centrality of a measure coincides with its invariance under this shift. It follows, in particular, that the set of central measures is a Choquet simplex.

The set of all ergodic central measures for a given equivalence relation, endowed with the weak topology of the space of all Borel measures, is called the absolute (for more details on the setting and the history of the problem, see [8, 9]).

The special case of this problem considered in this paper is that of finding the *absolute* for random walks on groups, semigroups, and for dynamic graphs. We develop an approach and present a solution of the problem for an important special case, namely, for random walks on countable commutative groups and semigroups. This case has important special properties as compared with the general case; the details are discussed below.

Note that the definition of absolute resembles the definition of boundaries in potential theory or the theory of random walks (the Poisson–Furstenberg (PF) boundary, Martin boundary, etc.; see, e. g., [4]). And indeed, the absolute is a generalization, or, better to say, a refinement of the PF boundary; more exactly, it can be nontrivial even if the PF boundary is trivial, i. e., consists of a single point; hence the absolute provides additional information on random walks on groups.

The foundations of the theory of absolute were laid in [8, 9, 10]. In [12], a description of the absolute for the case of free groups and homogeneous trees is obtained. This paper deals with the opposite class of groups, that of commutative groups and semigroups. In a paper in preparation, we will consider the next case: the absolute of nilpotent groups and, in particular, Heisenberg groups; this case seems to be much more complicated and interesting.

Let us discuss in more detail what is meant by a description of the absolute. The absolute is defined as a collection of measures. Measures on the compactum of infinite paths admit a direct description in terms of their values on finite paths (i. e., on cylinder sets corresponding to finite paths). For central measures, there is a more concise description in terms of functions on the set of vertices of the dynamic graph. With this approach, the absolute corresponds to classes of proportional minimal nonnegative harmonic functions on the dynamic graph. Another form of describing the absolute, in terms of the transition probabilities of a Markov chain, appears since the random process corresponding to a central measure is Markovian. It is this description that is most convenient for our constructions.

The key result of this paper is Theorem 3.1, which says that in the case of a commutative semigroup (with an arbitrary system of generators), the set of ergodic central measures (i. e., the absolute) coincides with the set of central measures that give rise to Markov chains with independent identically distributed

(i.i.d.) increments. The transition probabilities of a chain with i.i.d. increments are the same for all vertices of the graph, they depend only on the generators assigned to edges.

Theorem 3.1, on the one hand, generalizes de Finetti's theorem and, on the other hand, is related to known results on harmonic functions on commutative groups (see [1, 3], and also [14, pp. 311–312], and references therein). De Finetti's theorem follows from Theorem 3.1 if we consider the case of a free semigroup. Harmonic functions are related to the absolute as follows: in the case of a group, there is a natural bijection between the main part of the absolute (for the definition, see Section 2) and the space of classes of proportional minimal positive eigenfunctions of the Laplace operator.¹ (For this reason, in the case of a group, the main part of the absolute is also called the *Laplace part* of the absolute or simply the *Laplace absolute*. We will discuss this in more detail in [13].)

The paper is organized as follows. Section 2 contains the basic definitions. Theorem 3.1 is proved in Section 3, where we also present Theorem 3.2, which gives equations for describing the absolute. Theorems 3.1 and 3.2 allow us to describe the absolute of a commutative semigroup given a set of defining relations. The absolute is described as the set of solutions of a system of equations in a Euclidean space. In the same section, we give a series of examples of such a description. In Section 4, we use Theorems 3.1 and 3.2 to derive theorems on the topological structure of the absolute of commutative groups and semigroups. In this case, the absolute is compact; moreover, for groups and cancellative semigroups, it is a closed disk of finite dimension. The main technical difficulty in the proof of these theorems is to describe the degenerate part of the absolute. A technical result solving this difficulty is placed in a separate Section 5. In Section 6, we discuss the relation of the absolute of a commutative group to multiplicative semigroup characters.

The authors are grateful to V. Kaimanovich and S. Podkorytov for valuable comments and discussions.

2 Necessary definitions

First, we will give another definition of the absolute of a graph, which does not involve the group-theoretic terminology. By a *graph* we mean a locally finite directed graph with a distinguished vertex. Loops and multiple edges are allowed. A *path* in a graph is a (finite or infinite) sequence of alternating vertices and edges of the form

$$v_0, e_1, v_1, e_2, \dots, e_n, v_n,$$

¹Hereafter, given a semigroup G with a fixed finite system of generators S , by the Laplace operator (Laplacian) we mean the operator on the space of functions on G that sends a function f to the function L_f defined by the formula

$$L_f(g) := \frac{1}{|S|} \sum_{s \in S} f(gs).$$

where e_k is an edge leading from the vertex v_{k-1} to the vertex v_k (both vertices and edges may be repeated). We will consider graphs in which each path is a part of an infinite one.

Let Γ be a graph of the above form. Denote by \mathcal{P}_Γ the set of all infinite paths in Γ starting at the distinguished vertex. This set is compact in the weak topology. We consider Borel probability measures on this space. Given such a measure, by the measure of a finite path R (starting at the distinguished vertex) we mean the measure of the cylinder of all infinite paths that begin with R . A measure ν on \mathcal{P}_Γ is called *central* if it has the following property: for every vertex v of Γ and every positive integer n , the measure ν takes the same value at all paths of length n that lead from the distinguished vertex to v . (For more details, see the definition of centrality in [13].) The set of central measures is a convex compactum, which is a simplex (see [10]) in the compactum of all measures on \mathcal{P}_Γ . A central measure is called *ergodic* (or *regular*) if it is an extreme point of this simplex.

The *absolute* of a graph is the set of all ergodic central measures on the compactum of infinite paths starting at the distinguished vertex. (More details are given in [9, 13].) The absolute of a finitely generated semigroup with a fixed finite system of generators is the absolute of the corresponding Cayley graph. In this definition, there is a subtlety related to noncancellative semigroups. Recall that a semigroup G is called *cancellative* if there are no elements a, b, c in G such that $a \neq b$, but $ac = bc$ and/or $ca = cb$. In the case of the Cayley graph of a cancellative semigroup, the choice of the distinguished vertex does not affect the absolute; but in the case of a noncancellative semigroup G , we will assume that in G there is an identity element (or it has been added), and it is this element that is chosen as the distinguished vertex. The absolute of a semigroup G with a system of generators S is denoted by $\mathcal{A}_S(G)$.

A measure ν on the compactum of paths is called *nondegenerate* if the probability of every finite path is nonzero. The *main part* of the absolute is its subset consisting of the nondegenerate measures. The set of degenerate ergodic central measures will be called the *degenerate part* of the absolute.

A *branching graph* is a graph in which the set of paths leading from the distinguished vertex to every vertex v is nonempty (in this case, one says that v is *reachable* from the distinguished vertex) and all these paths have the same length. On the set of vertices of a branching graph there is a natural grading by the distance to the distinguished vertex. It turns out that in the theory of absolute, graphs of this special form are most general in the following sense. To a graph Γ with distinguished vertex v_0 we canonically associate the corresponding *dynamic graph* $D_{v_0}(\Gamma)$, which is a branching graph constructed in the following way. The n th level of $D_{v_0}(\Gamma)$ is a copy of the set of vertices of Γ connected with v_0 by paths of length n . There are exactly k edges leading from a vertex v_1 to a vertex v_2 in $D_{v_0}(\Gamma)$ if and only if the level of v_2 is greater by one than the level of v_1 , and there are exactly k edges leading from the vertex w_1 of Γ corresponding to v_1 to the vertex w_2 of Γ corresponding to v_2 . Every branching graph is isomorphic to its dynamic graph. The spaces of paths starting at the

distinguished point coincide for a graph and its dynamic graph. The absolute of a graph coincides with the absolute of its dynamic graph.

The construction of a branching graph has a counterpart at the level of groups and semigroups. A *branching monoid* is a monoid (semigroup with identity element) whose Cayley graph (with respect to some system of generators) is a branching graph. A semigroup is called a *branching semigroup* if it is a branching monoid or can be obtained from a branching monoid by removing the identity element. The following property is characteristic for branching semigroups: if G is a branching semigroup with respect to a system of generators S , then for every element of G , all words over the alphabet S representing this element have the same length (in other words, relations in this case identify only words of equal length). In a branching semigroup there is a canonical set of generators, which consists exactly of all irreducible elements of the semigroup. Systems of generators build from this canonical set (a system of generators may contain repeated elements, i. e., include elements with multiplicities) will be called *admissible*. The Cayley graph of a branching monoid is a branching graph only for an admissible system of generators. To a semigroup G with a fixed system of generators S we canonically associate a branching monoid $D_S(G)$ defined as follows: the system of generators of $D_S(G)$ is a copy of S , and the set of relations is the subset of the full set of relations for (G, S) consisting of the relations that identify words of equal length.

3 The absolute of commutative groups and semigroups

As one can easily see, the random process corresponding to a central measure (on an arbitrary graph of the form described above) is Markovian. For Markov chains on the Cayley graph of a semigroup, we introduce the notion of independent identically distributed increments: a Markov chain is said to have *independent identically distributed increments* if its transition probabilities at all edges marked by the same generator are equal.² For commutative semigroups, the following key theorem holds, which is a far-reaching generalization of de Finetti's theorem.

3.1. Theorem. *For every finitely generated commutative semigroup with an arbitrary finite system of generators, the set of ergodic central measures (i. e., the absolute) coincides with the set of central measures that give rise to Markov chains with independent identically distributed increments.*

Thus the absolute is in a bijective correspondence with the set of measures on the set of generators that determine Markov chains with the above centrality

²In the paper "Markov processes in asymptotic combinatorics and their transfers" in preparation, by the first author, a more general notion of *transfer* is introduced; it is a transformation on the path space of a graph with the meaning of a shift of increments. The notion of a Markov chain with independent identically distributed increments can be rephrased as that of a Bernoulli transfer.

property. An explicit condition that distinguishes these measures is given in Theorem 3.2.

Proof. For brevity, measures that give rise to Markov chains with independent identically distributed increments will be called *measures with i.i.d. increments*. Let P be a finite path (starting at the distinguished point) in the Cayley graph of the given semigroup (with respect to the given system of generators). The left translation in the semigroup determines a homeomorphism between the subcompactum \mathcal{P}_P of infinite paths that begin with P and the compactum of all infinite paths (starting at the distinguished vertex), as well as an isomorphism ϕ_* between the spaces of measures on these compacta.³ If ν is an ergodic central measure and $\nu(\mathcal{P}_P) > 0$, denote by ν_P the corresponding conditional measure on \mathcal{P}_P . Then the measure $\phi_*(\nu_P)$ is also central (since $a = b$ implies $ca = cb$). As one can easily see, since the semigroup is commutative, the central measure ν dominates the (finite central) measure $\nu(\mathcal{P}_P) \cdot \phi_*(\nu_P)$. By ergodicity, it follows that $\nu = \phi_*(\nu_P)$. This proves that ν is a measure with i.i.d. increments. On the other hand, if all ergodic central measures have i.i.d. increments, then every central measure with i.i.d. increments is ergodic, since noncoinciding measures with i.i.d. increments are mutually singular (the problem reduces to the mutual singularity of noncoinciding Bernoulli measures). \square

3.1 Explicit computation of the absolute

Theorem 3.1 provides a recipe for describing the absolute, and below we carry out its computation.

For an arbitrary semigroup G with a fixed finite system of generators S , the set $\mathcal{S}_S(G)$ of measures with i.i.d. increments can be identified in a natural way with the simplex Δ_S of probability distributions on S : to a distribution μ on S we associate the measure in $\mathcal{S}_S(G)$ for which the probability of the increment by $s \in S$ is equal to $\mu(s)$. A distribution on S for which the corresponding measure in $\mathcal{S}_S(G)$ is central will be called *precentral*. By Theorem 3.1, in the case of a commutative semigroup, the absolute coincides with the intersection of the $(S - 1)$ -dimensional simplex $\mathcal{S}_S(G) \cong \Delta_S$ (in general, it is not convex in the space of measures on the compactum of paths) with the infinite-dimensional simplex $\Sigma_S(G)$ of central measures:

$$\mathcal{A}_S(G) = \Sigma_S(G) \cap \mathcal{S}_S(G).$$

Thus Theorem 3.1 reduces the problem of describing the absolute $\mathcal{A}_S(G)$ of a commutative semigroup to the problem of describing the set $\sigma_S(G)$ of precentral

³Examples of noncancellative semigroups are interesting, but they are partly beyond the context we are interested in. In particular, the Cayley graph of such a semigroup is inhomogeneous: a pair of edges marked by s_1 and s_2 may have the same initial vertices and the same final vertices at one segment of the graph, and the same initial vertices but different final vertices at another segment of the graph. In the context of this proof, it is worth mentioning that in the case of a noncancellative semigroup, the tail filtrations (for the theory of filtrations, see [11]) on the subcompactum \mathcal{P}_P and on the compactum of all paths are not necessarily isomorphic.

distributions in Δ_S . The (pre)centrality condition splits into a set of necessary conditions for pairs of finite paths of equal length leading to the same vertex of the Cayley graph. The following proposition immediately follows from the definition of centrality.

3.2. Theorem. *A probability distribution $\mu = \{\mu(s); s \in S\}$ on a finite system of generators S of a commutative semigroup G is precentral if and only if for every pair of vectors $(m_s)_{s \in S}$ and $(n_s)_{s \in S}$ from \mathbb{N}_0^S such that*

$$\sum_{s \in S} m_s = \sum_{s \in S} n_s \quad \text{in } \mathbb{N}_0 \quad \text{and} \quad \sum_{s \in S} m_s \cdot s = \sum_{s \in S} n_s \cdot s \quad \text{in } G, \quad (1)$$

the following equation holds:

$$\prod_{s \in S} (\mu(s))^{m_s} = \prod_{s \in S} (\mu(s))^{n_s}. \quad (2)$$

In other words, the set $\sigma_S(G)$ of precentral distributions corresponding to the absolute coincides with the set of distributions on S that are solutions of equations (2) for all coefficients satisfying (1).

Theorem 3.2 allows one to obtain a description of the absolute of a commutative semigroup from the set of defining relations. Conditions (2) are called the *centrality equations* and are of the main interest in the study of the topology of the absolute. When describing the absolute in the context of Theorem 3.2, it is convenient to take into account the following considerations.

1. The pairs of vectors satisfying condition (1) for given G and S form a semigroup (we denote it by $\mathcal{R}_S^c(G)$; this semigroup describes the relations in the branching monoid $D_S(G)$). To verify the precentrality, it suffices to verify condition (2) for an arbitrary set of generators of the semigroup $\mathcal{R}_S^c(G)$.

2. The pairs of vectors from \mathbb{N}_0^S consisting of two equal⁴ vectors form a subsemigroup in $\mathcal{R}_S^c(G)$ (we denote it by R_0). Equations (2) corresponding to elements from R_0 are trivial, i. e., they are identities, so in order to verify the precentrality, it suffices to take an arbitrary set of vectors from $\mathcal{R}_S^c(G)$ that yields a generating set being combined with R_0 . In other words, in order to describe the absolute, it suffices to take the system of equations (2) for a set of noncommutative relations that is defining for the branching monoid $D_S(G)$ modulo the commutativity relations.

Examples. 1. The absolute of the commutative semigroup freely generated by a set of generators S is represented by the simplex Δ_S , since vectors $(m_s)_{s \in S}$ and $(n_s)_{s \in S}$ from \mathbb{N}_0^S satisfy condition (1) only if they coincide, and in this case equation (2) becomes an identity. At the level of branching monoids, this fact manifests itself as the absence of noncommutative relations.

2. Let $G = \mathbb{Z}$ and $S = \{+1, -1\}$. In this case, as in Example 1, vectors $(m_s)_{s \in S}$ and $(n_s)_{s \in S}$ satisfy condition (1) only if they coincide, so the

⁴The same vector in \mathbb{N}_0^S can represent different paths of equal length leading to the same vertex.

absolute $\mathcal{A}_{\{+1,-1\}}(\mathbb{Z})$ is homeomorphic to the one-dimensional simplex. At the level of branching monoids, the explanation is that the branching monoid for $(\mathbb{Z}, \{+1, -1\})$ is the free monoid with two generators.

3. Let $G = \mathbb{Z}^2$ and $S = \{(+1, 0), (-1, 0), (0, +1), (0, -1)\}$. The semigroup $\mathcal{R}_S^c(G)$ is generated by the subsemigroup R_0 and the pair $((1, 1, 0, 0), (0, 0, 1, 1))$. This pair gives rise to the equation $x_1x_2 = x_3x_4$ in \mathbb{R}^4 . The absolute is represented by the intersection of the set of solutions of this equation with the simplex

$$\left\{x \in \mathbb{R}^4 : \sum x_i = 1, x_i \geq 0\right\}.$$

It is homeomorphic to the closed disk of dimension 2 (see Theorem 4.1).

4. Let $G = \mathbb{Z}^d$ and S be the standard symmetric system of generators. This is a generalization of the previous example. In this case, the system of equations is as follows:

$$x_1x_2 = x_3x_4 = \cdots = x_{2d-1}x_{2d}.$$

5. For $G = \mathbb{Z}$ and $S = \{0, +6, -1\}$, the centrality relation takes the form $x_1^7 = x_2x_3^6$.

6. Let G be the commutative semigroup with three generators a, b, c and the additional noncommutative relation $a + b = a + c$. Then G is a branching semigroup, so the set $\sigma_S(G)$ is determined by the equation $\mu(a)\mu(b) = \mu(a)\mu(c)$ corresponding to the relation $a + b = a + c$. The absolute is homeomorphic to the tripod \perp .

7. If G is the commutative semigroup with generators a, b, c and the additional noncommutative relations $a + b = 2c$ and $a + c = 2b$, then the absolute is disconnected, it consists of two points.

3.3. Proposition. *The absolute of the quotient semigroup of a commutative semigroup by a finite subgroup coincides with the absolute of the semigroup.*

Proof. Assume that there is a semigroup epimorphism $G_1 \rightarrow G_2$ and we consider the system of generators S in G_2 inherited from G_1 . If a pair of vectors $(m_s)_{s \in S}$ and $(n_s)_{s \in S}$ from \mathbb{N}_0^S satisfies condition (1) for G_1 , then it also satisfies condition (1) for G_2 , since we deal with a homomorphism. Conversely, if an epimorphism $G_1 \rightarrow G_2$ corresponds to taking the quotient by a finite subgroup and a pair of vectors $(m_s)_{s \in S}$ and $(n_s)_{s \in S}$ satisfies condition (1) for G_2 , then there is a positive integer k such that the pair of vectors $(k \cdot m_s)_{s \in S}$ and $(k \cdot n_s)_{s \in S}$ satisfies condition (1) for G_1 . In view of Theorem 3.2, it follows that the sets $\sigma_S(G_1)$ and $\sigma_S(G_2)$ of precentral distributions are described by equivalent systems of equations, and Theorem 3.1 implies the desired assertion. \square

Comments. 1. The idea of shifting used in the proof of Theorem 3.1 appeared in the probability literature of the 1960s in connection with harmonic functions; see [3, Theorem 5], [7, Lemma 1], and also [14, Lemma 25.2].

2. Already for nilpotent groups, Theorem 3.1 does not hold for an arbitrary ergodic central measure.

4 The topology of the absolute of commutative (semi)groups

As we have already mentioned, the set of all central measures is a simplex, and the absolute is its Choquet boundary. A meaningful question is what topology is induced on the absolute by the weak topology in the space of measures on the compactum of paths. In this section, we study the topology of the absolute of commutative (semi)groups. From Theorems 3.1 and 3.2 we derive (the proofs are given below) the following theorems on the topological structure of the absolute. The most important result is as follows: in the case of groups and cancellative semigroups, the main part of the absolute is the interior of a disk, and the degenerate part is the boundary of this disk.

4.1. Theorem (on the topology of the absolute of commutative groups). *The absolute of a finitely generated commutative group with respect to any finite system of semigroup generators is homeomorphic to the closed disk of dimension equal to the rank of the group. The main part of the absolute corresponds to the interior of the disk.*

Theorem 4.1 extends to cancellative semigroups.

4.2. Theorem (generalization for cancellative commutative semigroups). *The absolute of a finitely generated cancellative commutative semigroup G (with respect to any finite system of generators S) is homeomorphic to the closed disk whose dimension either coincides with the rank of the group of fractions⁵ of G , or is one less than this rank if G is a branching semigroup and S is an admissible system of generators. The main part of the absolute corresponds to the interior of the disk.*

In the case of a noncancellative commutative semigroup, the absolute can have a more complex structure (see Example 4 above). However, it is still compact, and the main part has the same form.

4.3. Theorem (generalization for noncancellative commutative semigroups). *The absolute of an arbitrary finitely generated commutative semigroup (with respect to any finite system of generators) is compact⁶, and its main part is homeomorphic to the open disk whose dimension is determined by the rule described in Theorem 4.2.*

Proof of Theorems 4.1–4.3. Let G be a commutative semigroup with a finite system of generators S . Theorems 3.1 and 3.2 reduce the problem of describing the absolute $\mathcal{A}_S(G)$ to that of describing the set $\sigma_S(G)$ of solutions of equations (2) in the simplex Δ_S of probability distributions on S .

⁵The *group of fractions* of a semigroup G is the group with the same generators as that of G in which the relations are all corollaries of the relations in G .

⁶In the literature, a simplex whose Choquet boundary is closed is called a *Bauer simplex*.

A proof that the absolute is compact. For arbitrary pair of vectors $(m_s)_{s \in S}$ and $(n_s)_{s \in S}$ from \mathbb{N}_0^S , the subset of solutions of equation (2) in Δ_S is compact, since Δ_S is compact and the expressions in both sides of (2) are continuous on Δ_S as functions of μ . Therefore, the set $\sigma_S(G)$ (and hence the absolute $\mathcal{A}_S(G)$) is compact as an intersection of compact subsets.

The centrality equations. If vectors $(m_s)_{s \in S}$ and $(n_s)_{s \in S}$ from \mathbb{N}_0^S satisfy the condition $\sum_{s \in S} m_s \cdot s = \sum_{s \in S} n_s \cdot s$ in G , then the vector $(m_s - n_s)_{s \in S}$ from \mathbb{Z}^S will be called a *reduced relation vector* for (G, S) . If, in addition, the equation $\sum_{s \in S} m_s = \sum_{s \in S} n_s$ holds, then the reduced relation vector $(m_s - n_s)_{s \in S}$ will be called *central*, equation (2) will be called a *centrality equation*, and the equation (in the variables $\mu(s)$)

$$\prod_{s \in S} (\mu(s))^{m_s - \min\{m_s, n_s\}} = \prod_{s \in S} (\mu(s))^{n_s - \min\{m_s, n_s\}} \quad (3)$$

will be called a *reduced centrality equation*.⁷ Denote by $\bar{\sigma}_S(G)$ the set of distributions from Δ_S that are solutions of all reduced centrality equations of the pair (G, S) . Then the following holds.

(i) *The set $\bar{\sigma}_S(G)$ is contained in $\sigma_S(G)$, since $\sigma_S(G)$ coincides with the set of solutions of the centrality equations lying in Δ_S , and every centrality equation (2) can be obtained from the corresponding reduced equation (3) by multiplying both sides of the latter by $\prod_{s \in S} (\mu(s))^{\min\{m_s, n_s\}}$.*

(ii) *If G is a cancellative semigroup, then $\bar{\sigma}_S(G) = \sigma_S(G)$, since in such a semigroup every reduced centrality equation is, obviously, a centrality equation, so $\sigma_S(G)$ is contained in $\bar{\sigma}_S(G)$ (and $\bar{\sigma}_S(G)$ is contained in $\sigma_S(G)$ by (i)).*

(iii) *The intersection $\sigma_S(G) \cap \text{int}(\Delta_S)$ coincides with the intersection of the sets $\bar{\sigma}_S(G)$ and $\text{int}(\Delta_S)$, both in the case of a cancellative semigroup and in the general case, since in $\text{int}(\Delta_S)$ the condition $\mu(s) > 0$ holds for all $s \in S$, and a centrality equation in $\text{int}(\Delta_S)$ has the same set of solutions as the corresponding reduced centrality equation.*

Now observe that the set $\bar{\mathcal{R}}_S(G)$ of reduced relation vectors for (G, S) is, obviously, a subgroup in \mathbb{Z}^S , and the subset $\bar{\mathcal{R}}_S^c(G)$ of all central reduced relation vectors either coincides with $\bar{\mathcal{R}}_S(G)$, or is a subgroup in $\bar{\mathcal{R}}_S(G)$ of corank 1.⁸

It follows that the sets $\bar{\mathcal{R}}_S^c(G)$ and $\bar{\sigma}_S(G)$ satisfy the conditions of Proposition 5.1 (in the notation of this proposition, $\bar{\sigma}_S(G) = \Lambda_{\bar{\mathcal{R}}_S^c(G)}$), which implies that the set $\bar{\sigma}_S(G)$ is homeomorphic to the closed disk of dimension $|S| - 1 - \text{rank}(\bar{\mathcal{R}}_S^c(G))$ (hereafter, rank stands for the rank of a commutative group), and, moreover, the interior of the disk $\bar{\sigma}_S(G)$ lies in $\text{int}(\Delta_S)$, while the boundary of $\bar{\sigma}_S(G)$ lies in the boundary $\partial\Delta_S$.

To complete the proof of Theorems 4.1–4.3, it remains to observe that under the bijection $\mathcal{A}_S(G) \cong \sigma_S(G)$, the main part of the absolute is represented by the intersection $\sigma_S(G) \cap \text{int}(\Delta_S)$, the rank of the group of fractions of the

⁷In the general case of a noncancellative semigroup, a reduced centrality equation is not necessarily a centrality equation.

⁸If the group of fractions of a semigroup is torsion-free, then the subgroups $\bar{\mathcal{R}}_S(G)$ and $\bar{\mathcal{R}}_S^c(G)$ are linear subspaces in \mathbb{Z}^S .

semigroup G is equal to $|S| - \text{rank}(\bar{\mathcal{R}}_S(G))$, and the rank of the group $\bar{\mathcal{R}}_S(G)$ coincides with the rank of $\bar{\mathcal{R}}_S^c(G)$ if and only if (these groups coincide, so) (G, S) is a branching semigroup with an admissible system of generators. \square

5 A proposition on linear spaces

Let S be a finite set, Δ_S be the simplex of probability distributions on S , \mathbb{R}^S be the space of real functions on S , $V_0 \subset \mathbb{R}^S$ be the subset of functions with values summing to zero. We identify Δ_S with the subset of nonnegative functions in \mathbb{R}^S with values summing to one. Given a vector $\kappa = (k_s)_{s \in S}$ from V_0 , denote by λ_κ the subset in Δ_S consisting of all distributions μ satisfying the condition

$$\prod_{s \in S: k_s > 0} (\mu(s))^{k_s} = \prod_{r \in S: k_r < 0} (\mu(r))^{|k_r|}. \quad (4)$$

Given a subset⁹ K in V_0 , put

$$\Lambda_K := \bigcap_{\kappa \in K} \lambda_\kappa.$$

By V_K we denote the linear hull of K , and $\dim(V)$ stands for the dimension of a space V . We also denote by $\text{int}(M)$ and ∂M the interior and the boundary of a multidimensional polyhedron M (irrespective of the ambient space).

The purpose of this section is to prove the following proposition.

5.1. Proposition. *If a subset K of a hyperplane V_0 is a linear subspace or a semigroup of vectors with integer coordinates, then the set Λ_K is homeomorphic to the closed disk of dimension $|S| - 1 - \dim(V_K)$; moreover, the interior of Λ_K lies in $\text{int}(\Delta_S)$, and the boundary of Λ_K lies in $\partial\Delta_S$.*

We split the proof of Proposition 5.1 into a series of claims.

5.2. Claim. *For every subset K in V_0 , the intersection of the set Λ_K with the interior $\text{int}(\Delta_S)$ of the simplex Δ_S is homeomorphic to the open disk of dimension $|S| - 1 - \dim(V_K)$.*

Proof. Observe that the maps

$$\exp: V_0 \rightarrow \text{int}(\Delta_S) \quad \text{and} \quad \log: \text{int}(\Delta_S) \rightarrow V_0$$

given by the formulas

$$\exp((v_s)_{s \in S}) = \left(\frac{e^{v_s}}{\sum_{s \in S} e^{v_s}} \right)_{s \in S} \quad \text{and}$$

⁹If we extend the above definitions to subsets K not in V_0 , then part of the assertions stated below remains valid. However, here we do not consider generalizations; we are mainly interested in the simplex Δ_S and the space V_0 , as the vector space associated with the affine hull of Δ_S ; these spaces are embedded into the auxiliary space \mathbb{R}^S only for convenience.

$$\log((p_s)_{s \in S}) = \left(\log p_s - \sum_{s \in S} \log p_s \right)_{s \in S}$$

are mutually inverse diffeomorphisms between V_0 and $\text{int}(\Delta_S)$. Taking the logarithms of both sides of (4), we see that *the maps exp and log provide a diffeomorphism between the set $\Lambda_K \cap \text{int}(\Delta_S)$ and the orthogonal complement to V_K in V_0* . It remains to observe that this complement has dimension

$$\dim(V_0) - \dim(V_K) = |S| - 1 - \dim(V_K). \quad \square$$

5.3. Claim. *For every subset K in V_0 , the set Λ_K is compact.*

Proof. For every vector $\kappa = (k_s)_{s \in S}$, the set λ_κ is compact, since Δ_S is compact and the expressions in both sides of (4) are continuous on Δ_S as functions of μ . Therefore, Λ_K is compact as an intersection of compact sets. \square

5.4. Claim. *If vectors κ and κ' in V_0 are collinear, then $\lambda_\kappa = \lambda_{\kappa'}$.*

Proof. Follows from the definition of λ_κ for κ and κ' from V_0 . \square

5.5. Claim. *If K is a subgroup in V_0 that contains only vectors with integer coordinates, then $\Lambda_K = \Lambda_{V_K}$.*

Proof. First, observe that if K is a group of vectors with integer coordinates, then in V_K every vector with rational coordinates is proportional to a vector from K . In view of Claim 5.4, it follows that every distribution μ from Λ_K belongs also to λ_κ if κ is a vector in V_K with rational coordinates.

Second, it is clear that for every fixed distribution $\mu \in \Delta_S$, both sides of (4), regarded as functions of a vector $(k_s)_{s \in S}$, are continuous on each connected component of each stratum of the form

$$\mathbb{R}_m^S := \{(k_s)_{s \in S} \in \mathbb{R}^S : \text{card}\{s \in S : k_s = 0\} = m\}.$$

Finally, for every $m \in \mathbb{N}_0$, the vectors with rational coordinates are dense in the stratum $V_K \cap \mathbb{R}_m^S$ (this follows from the fact that the intersection of two linear subspaces spanned by vectors with integer coordinates is also spanned by vectors with integer coordinates; this fact becomes obvious if one regards the intersection as the orthogonal complement to the sum of the orthogonal complements to the original subspaces and observes that the property of being spanned by vectors with integer coordinates is preserved under taking orthogonal complements and sums). \square

5.6. Claim. *If an affine line L in \mathbb{R}^S intersects the simplex Δ_S by an interval I with endpoints a and b , then the set λ_{a-b} intersects L at a single point, and this point lies in the interior of I .*

Proof. For the vector $\kappa = a - b$, condition (4) takes the form

$$\prod_{s \in S: a_s - b_s > 0} (\mu(s))^{a_s - b_s} = \prod_{r \in S: b_r - a_r > 0} (\mu(r))^{b_r - a_r}. \quad (5)$$

As μ moves along the interval I from a to b , the function $F_L(\mu)$ given by the left-hand side of (5) decreases, while the function $F_R(\mu)$ given by the right-hand side of (5) increases. Since I is cut out of Δ_S by a straight line, there is $q \in S$ with $a_q = 0$ and $b_q > 0$ and there is $r \in S$ with $a_r > 0$ and $b_r = 0$ such that $F_L(b) = 0$ and $F_R(a) = 0$. This immediately implies the desired assertion. \square

5.7. Claim. *If K is a linear subspace in V_0 , then taking the quotient $\rho: \mathbb{R}^S \rightarrow \mathbb{R}^S/K$ yields a homeomorphism between the set Λ_K and the convex polyhedron $\rho(\Delta_S)$; moreover, the intersection $\Lambda_K \cap \partial\Delta_S$ is homeomorphic to the boundary $\partial(\rho(\Delta_S))$.*

We split the proof of Claim 5.7 into several parts.

5.8. Claim. *Under the conditions of Claim 5.7, the restriction of ρ to Λ_K is injective.*

Proof. Aiming at a contradiction, assume that in Λ_K there are distinct points x and y with $\rho(x) = \rho(y)$. Then the vector $x - y$ lies in K , so $\{x, y\} \subset \Lambda_K \subset \lambda_{x-y}$, which contradicts Claim 5.6 if we choose L to be the affine line passing through the points x and y (see also Claim 5.4). \square

5.9. Claim. *Under the conditions of Claim 5.7, the map ρ sends the set $\Lambda_K \cap \partial\Delta_S$ to the boundary $\partial(\rho(\Delta_S))$ of the polyhedron $\rho(\Delta_S)$.*

Proof. Aiming at a contradiction, assume that in $\Lambda_K \cap \partial\Delta_S$ there is a point b with $\rho(b)$ in $\text{int}(\rho(\Delta_S))$. Observe that the interior of the polyhedron $\rho(\Delta_S)$ is covered by the interior of the polyhedron Δ_S (and, moreover, $\rho(\text{int}(\Delta_S)) = \text{int}(\rho(\Delta_S))$). Therefore, in $\text{int}(\Delta_S)$ there is a point y with $\rho(y) = \rho(b)$. Then the vector $b - y$ lies in K . By Claim 5.6, the point b does not lie in λ_{b-y} , since it is an endpoint of the interval cut out of Δ_S by the straight line passing through b and y (see also Claim 5.4). This contradicts the assumption that b lies in Λ_K . \square

5.10. Claim. *Under the conditions of Claim 5.7, the image $Q := \rho(\Lambda_K \cap \text{int}(\Delta_S))$ coincides with the interior $\text{int}(\rho(\Delta_S))$ of the polyhedron $\rho(\Delta_S)$.*

Proof. We have proved that the restriction of ρ to Λ_K is injective (Claim 5.8), and the set $\Lambda_K \cap \text{int}(\Delta_S)$ is homeomorphic to the open disk of dimension $|S| - 1 - \dim(K)$ (see Claim 5.2), which coincides with the dimension of the polyhedron $\rho(\Delta_S)$. As is well known, *the image of a continuous embedding of a Euclidean space into itself is open* (Brouwer's invariance of domain theorem). It follows that the set Q is contained in $\text{int}(\rho(\Delta_S))$ and open in $\rho(\Delta_S)$. Therefore, if Q did not cover the domain $\text{int}(\rho(\Delta_S))$, then there would be a point x in this domain that does not belong to Q but belongs to the closure of Q . Since Λ_K is compact, this would mean that in $\Lambda_K \cap \partial\Delta_S$ there is a point b with $\rho(b) = x$. However, this contradicts Claim 5.9. \square

Completing the proof of Proposition 5.7. Since Λ_K is compact (see Claim 5.3), it follows from Claim 5.10 that $\rho(\Lambda_K) = \rho(\Delta_S)$. Since the restriction of ρ to Λ_K

is injective (see Claim 5.8), it follows that ρ yields a bijection between Λ_K and $\rho(\Delta_S)$. It remains to observe that, as one can easily see, a continuous bijection of a metric compact space is a homeomorphism. Thus ρ yields a homeomorphism between Λ_K and $\rho(\Delta_S)$, and, in view of Claim 5.10, a homeomorphism between the spheres $\Lambda_K \cap \partial\Delta_S$ and $\partial(\rho(\Delta_S))$. \square

Completing the proof of Proposition 5.1. Claim 5.5 reduces the situation to the case of a linear subspace. In this case, the existence of a homeomorphism follows from Claim 5.7, and the refinement on the dimension, from Claim 5.2. \square

Remark. If a subset K of the space V_0 consists of vectors with integer coordinates but is not a subgroup, then the set Λ_K is not necessarily homeomorphic to a disk. For instance, for $S = \{1, 2, 3\}$, we have

$$\Lambda_{\{(1,1,-2),(1,-2,1)\}} = \{(1/3, 1/3, 1/3), (1, 0, 0)\}.$$

6 Characters

The theory of the absolute of commutative groups and semigroups has an interesting reformulation in terms of characters of semigroups. Here by a *character* we mean a homomorphism to the multiplicative semigroup of nonnegative real numbers. For a general theory of characters, see [2, 5, 6].

Let G be an arbitrary commutative semigroup with a finite system of generators S and $D_S(G)$ be the branching monoid for the pair (G, S) . Every central measure ν of the pair (G, S) gives rise to the functional f_ν on the monoid $D_S(G)$ whose value at an element of $D_S(G)$ is equal to the measure of a path leading to this element. Clearly, under this correspondence, functions corresponding to central measures with i.i.d. increments are exactly characters of the monoid $D_S(G)$ whose values on the generators from S sum to 1 (such characters will be called *normalized*, or *probability characters*). In these terms, Theorem 3.1 takes the following form.

6.1. Corollary. *In a commutative semigroup G with an arbitrary finite system of generators S , the above correspondence $\nu \mapsto f_\nu$ is a bijection of the set of ergodic central measures (i. e., the absolute) onto the set of normalized \mathbb{R}_{0+} -characters on the branching monoid $D_S(G)$.*

Proof. If ν is ergodic, then the functional f_ν is a normalized \mathbb{R}_{0+} -character, since the increments are i.i.d. (Theorem 3.1). Conversely, if a functional f on $D_S(G)$ is a normalized \mathbb{R}_{0+} -character, then the restriction of f to S gives a precentral distribution, so $f = f_\nu$ for the ergodic central measure ν corresponding to this distribution. \square

A character is called *trivial* if it vanishes at all elements of the semigroup except the identity element. As one can easily see, the set of nontrivial \mathbb{R}_{0+} -characters forms a fiber bundle over the set of normalized \mathbb{R}_{0+} -characters

with fiber $(0, +\infty)$. Thus the problem of describing the absolute of commutative semigroups is essentially equivalent to the problem of describing the set of \mathbb{R}_{0+} -characters of commutative branching monoids. Correspondingly, assertions about the absolute of commutative semigroups can be translated into assertions about characters of branching semigroups. For instance, Theorem 4.2 gives the following.

6.2. Corollary. *For a commutative cancellative branching semigroup, the space of nontrivial \mathbb{R}_{0+} -characters endowed with the weak topology is homeomorphic to the direct product of a closed disk of certain dimension and an open interval.*

In the case of commutative and nilpotent groups, there is a correspondence between the \mathbb{R}_+ -characters of the group and the main part of the absolute. This correspondence is known, it has been studied from the point of view of eigenfunctions of Laplacians. We discuss it in more detail in [13].

References

- [1] Choquet, G., Deny, J.: Sur l'équation de convolution $\mu = \mu * \sigma$. C. R. Acad. Sci. Paris **250**, 799–801 (1960)
- [2] Clifford, A. H., Preston, G. B.: The Algebraic Theory of Semigroups. Vol. I. Math. Surveys **7**, Amer. Math. Soc., Providence, RI (1961)
- [3] Doob, J. L., Snell, J. L., Williamson, R. E.: Application of boundary theory to sums of independent random variables. In: Contributions to Probability and Statistics, pp. 182–197. Stanford Univ. Press, Stanford, Calif. (1960)
- [4] Kaimanovich, V. A., Vershik, A. M.: Random walks on discrete groups, boundary and entropy. Ann. Probab. **11**(3), 457–490 (1983)
- [5] Lesokhin, M. M.: Characters of commutative semigroups. I. Izv. Vysš. Učebn. Zaved. Matematika **8**(99), 67–74 (1970)
- [6] Lesokhin, M. M.: Characters of commutative semigroups. II. Izv. Vysš. Učebn. Zaved. Matematika **2**(105), 71–77 (1971)
- [7] Molchanov, S. A.: Martin boundary for the direct product of Markov chains. Theory Probability Appl. **12**(2), 307–310 (1967)
- [8] Vershik, A. M.: Intrinsic metric on graded graphs, standardness, and invariant measures. J. Math. Sci. (N.Y.) **200**(6), 677–681 (2014)
- [9] Vershik, A. M.: The problem of describing central measures on the path spaces of graded graphs. Funct. Anal. Appl. **48**(4), 256–271 (2014)
- [10] Vershik, A. M.: Equipped graded graphs, projective limits of simplices, and their boundaries. J. Math. Sci. (N.Y.) **209**(6), 860–873 (2015)

- [11] Vershik, A. M.: The theory of filtrations of subalgebras, standardness, and independence. *Russian Math. Surveys* **72**(2), 257–333 (2017)
- [12] Vershik, A. M., Malyutin, A. V.: Phase transition in the exit boundary problem for random walks on groups. *Funct. Anal. Appl.* **49**(2), 86–96 (2015)
- [13] Vershik, A. M., Malyutin, A. V.: The absolute of finitely generated groups: II. The Laplace part and degenerate part. *Funct. Anal. Appl.* **52**(3), (2018)
- [14] Woess, W.: *Random Walks on Infinite Graphs and Groups*. Cambridge Tracts in Math. **138**. Cambridge Univ. Press, Cambridge (2000)