ISSN 1064-5624, Doklady Mathematics, 2012, Vol. 86, No. 1, pp. 450–453. © Pleiades Publishing, Ltd., 2012.
Original Russian Text © A.M. Vershik, N.I. Nessonov, 2012, published in Doklady Akademii Nauk, 2012, Vol. 445, No. 1, pp. 9–12.

MATHEMATICS

Stable States and Representations of the Infinite Symmetric Group[1]

A. M. Vershik and N. I. Nessonov

Presented by Academician Yu. V. Prokhorov December 27, 2011

Received March 11, 2012

DOI: 10.1134/S1064562412040072

1. INTRODUCTION

The problem studied in this paper can be briefly formulated as follows: what class of representations of countable groups is a natural extension of the class of representations with finite traces, i.e., representations determined by finite characters? Recall that a character of a group G is a positive definite central (i.e., such that $\chi(gh) = \chi(hg)$ for all $g, h \in G$) function on $e \in G$ whose value at the group identity is equal to one. According to the GNS construction, an indecomposable character (which cannot be written as a nontrivial convex combination of other characters) determines a factor representation of finite type I_n, $n < \infty$, or II_1. Conversely, a factor representation of finite type uniquely determines a character. For many (though not all) countable groups, the set of all factor representations of type II_1 has the structure of a standard Borel space. At the same time, for countable groups that have no Abelian subgroup of finite index, the set of classes of pairwise nonequivalent irreducible representations is not tame. In particular, it has no standard Borel structure. Hence for these groups, there is no reasonable classification of irreducible representations. Moreover, there exist representations that have several different decompositions into irreducible components. Such groups are usually called wild. The infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$, which consists of all finite permutations of the set of integers $\mathbb{N}$, is a typical example of a wild group. However, its factor representations of type II_1 are completely classified in [1, 2, 6] and are used for constructing harmonic analysis. A similar situation holds for the infinite-dimensional unitary group $\mathbb{U}(\infty)$. At the same time, important classes of countable groups, in particular, the infinite general linear group over a finite field $GL(\infty, \mathbb{F}_q)$, have too few finite characters for constructing a nontrivial harmonic analysis. For instance, the characters of $GL(\infty, \mathbb{F}_q)$ do not separate points. These considerations lead to the need for an extension of the class of representations of finite type, i.e., for a generalization of the notion of a character.

But the class of all representations of type II_∞ is too wide: e.g., for the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$, it is as wild (and in the same sense) as the set of classes of irreducible representations. Therefore, a putative extension must include some reasonable restrictions on representations of type II_∞. An attempt to construct such an extension was made in [2], where a class of representations determined by semifinite traces on the group algebra was introduced. But this class turned out to be too narrow, it does not even cover all representations of $\mathfrak{S}_{\mathbb{N}}$ of type II_∞ associated with admissible representations of $\mathfrak{S}_{\mathbb{N}} \times \mathfrak{S}_{\mathbb{N}}$.

In this paper, we introduce the class of stable representations, which is a natural extension of the class of representations of finite type. It turns out that stable representations of $\mathfrak{S}_{\mathbb{N}}$ are of type I or II. We also give a complete classification of stable factor representations up to quasiequivalence. At the same time, we obtain an answer to the question posed by the first author [3] in connection with Olshanski's [4] theory of admissible representations of the group $\mathfrak{S}_{\mathbb{N}} \times \mathfrak{S}_{\mathbb{N}}$, that of identifying the components of an admissible representation. Namely, we prove that the set of stable factor representation coincides with the class of representations that can be obtained as the restrictions of admissible irreducible representations of $\mathfrak{S}_{\mathbb{N}} \times \mathfrak{S}_{\mathbb{N}}$ to the left and right components ($\mathfrak{S}_{\mathbb{N}} \times e$ and $e \times \mathfrak{S}_{\mathbb{N}}$, respectively).

2. BASIC NOTIONS

We introduce a new notion of stability for a positive definite function on a group and the corresponding representation. Typical groups for which this notion is

[1] The article was translated by the authors.

St. Petersburg Department of the Steklov Mathematical Institute of the Russian Academy of Sciences, St. Petersburg, Russia

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, Kharkov, Ukraine

meaningful and amenable to study are inductive limits of compact groups.

2.1. Topology on Groups of Automorphisms of a Group

Let G be a countable group and $\mathbf{C}^*(G)$ be its group $\mathbf{C}^*$-algebra. We identify G with its natural image in $\mathbf{C}^*(G)$, and positive functionals on $\mathbf{C}^*(G)$ with the corresponding positive definite functions on G. A positive functional is called a state if it is equal to one at the group identity. Let $\mathrm{Aut}G$ be the group of all automorphisms of G. An element $g \in G$ determines an inner automorphism $\mathrm{Ad}g \in \mathrm{Aut}G$: $\mathrm{Ad}\,g(x) = gxg^{-1}$, $x \in G$. The group $\mathrm{Int}G$ of inner automorphisms is a normal subgroup in $\mathrm{Aut}G$.

We endow $\mathrm{Aut}G$ with the strong topology, in which a base of neighborhoods of the identity automorphism consists of the sets

$$\mathcal{U}_g = \{\theta \in \mathrm{Aut}G\colon \theta(g) = g\}, \quad g \in G. \qquad (1)$$

Now we introduce the strong topology on $\mathrm{Aut}G$ for an arbitrary locally compact group. Let $\mathrm{Aut}\mathbf{C}^*(G)$ be the group of all automorphisms of the algebra $\mathbf{C}^*(G)$. We identify $\mathrm{Aut}G$ and $\mathrm{Int}G$ with the corresponding subgroups in $\mathrm{Aut}\mathbf{C}^*(G)$. A base of the strong topology on $\mathrm{Aut}\mathbf{C}^*(G)$ is determined by the neighborhoods $\mathcal{U}_{a,\epsilon}$, $\epsilon > 0$ of the identity automorphism, where

$$\mathcal{U}_{a,\epsilon} = \{\theta \in \mathrm{Aut}\mathbf{C}^*(G)\colon \|\theta(a) - a\| < \epsilon\}, \quad a \in \mathbf{C}^*(G). \qquad (2)$$

Then $\mathrm{Aut}G$ is a closed subgroup in $\mathrm{Aut}\mathbf{C}^*(G)$. Denote by $\overline{\mathrm{Int}}G$ the completion of $\mathrm{Int}G$.

Let $G = \lim G_i$ be the inductive limit of a sequence of locally compact groups $\{G_i\}_{i \in \mathbb{N}}$, where G_i is a closed subgroup in G_{i+1} for all i. A base of the strong topology on $\mathrm{Aut}G$ is determined by the neighborhoods $U_{n,a,\epsilon}$, $a \in \mathbf{C}^*(G_n)$, where

$$\mathcal{U}_{n,a,\epsilon} = \{\theta \in \mathrm{Int}G\colon \theta(G_n) = G_n, \ \|\theta(a) - a\|_{\mathbf{C}^*(G_n)} < \epsilon\}. \qquad (3)$$

1. Our basic example is as follows. Let $\mathbb{N}$ be the set of positive integers. A bijection $s : \mathbb{N} \to \mathbb{N}$ is called finite if the set $\{i \in \mathbb{N} | \, s(i) \neq i\}$ is finite. We define $\mathfrak{S}_{\mathbb{N}}$ as the group of all finite bijections $\mathbb{N} \to \mathbb{N}$ and set $\mathfrak{S}_n = \{s \in \mathfrak{S}_{\mathbb{N}} | \, s(i) = i$ for all $i > n\}$. Denote by $\mathfrak{S}_{\mathbb{N}\setminus n}$ the subgroup in $\mathfrak{S}_{\mathbb{N}}$ consisting of the elements leaving the numbers 1, 2, ..., n fixed ($n < \infty$).

If $\overline{\mathfrak{S}}_{\mathbb{N}}$ is the group of all bijections of $\mathbb{N}$, then $\mathfrak{S}_{\mathbb{N}} \subset \overline{\mathfrak{S}}_{\mathbb{N}}$, and for every $s \in \overline{\mathfrak{S}}_{\mathbb{N}}$, the map $\mathfrak{S}_{\mathbb{N}} \ni x \mapsto sxs^{-1} \in \mathfrak{S}_{\mathbb{N}}$ is an automorphism $\mathrm{Ad}s$ of $\mathfrak{S}_{\mathbb{N}}$, which can be naturally extended to an automorphism of the group $\mathbf{C}^*$-algebra $\mathbf{C}^*(\mathfrak{S}_{\mathbb{N}})$ of $\mathfrak{S}_{\mathbb{N}}$. One can easily check that $\mathrm{Ad}\overline{\mathfrak{S}}_{\mathbb{N}}$ coincides with $\mathrm{Aut}\mathfrak{S}_{\mathbb{N}}$ and is the closure of $\mathrm{Int}\mathfrak{S}_{\mathbb{N}}$ in each of the topologies (1), (2), and (3), which are equivalent. In particular, $\mathfrak{S}_{\mathbb{N}}$ is a dense normal subgroup in $\overline{\mathfrak{S}}_{\mathbb{N}}$.

2. Consider the infinite-dimensional unitary group $U(\infty)$, which is the inductive limit $U(\infty) = \lim U(n)$ of the finite-dimensional unitary groups with respect to the natural embeddings. If $\mathrm{Ad} \in \mathcal{U}_{n,a,\epsilon}$, $\overset{\geq}{g} \in U(\infty)$, then $g = g(n) \cdot g_\infty(n)$, where $g(n) \in U(n)$ and $g_\infty(n) \cdot u = u \cdot g_\infty(n)$ for all $u \in U(n)$. It is not difficult to show that $\min_{z \in \mathbb{C}: |z| = 1} \|g(n) - zI_n\| \to 0$ as $\epsilon \to 0$ and $\mathrm{Int}U(\infty) \neq \overline{\mathrm{Int}}\,U(\infty)$.

2.2. Characters and Representations of $\overline{\mathrm{Int}}G$

Let G be a countable group, and let Π be the biregular representation[2] of the group $G \times G$. Then it follows from (2) that the map $\mathrm{Int}G \ni \mathrm{Ad}g \overset{\mathcal{A}}{\mapsto} \Pi((g, g)) \in \mathbb{U}(l^2)$ can be extended by continuity to $\overline{\mathrm{Int}}G$. This continuity is preserved if we replace the biregular representation of G by a representation corresponding to a character.

On the other hand, in the next section we will describe a construction of a unique, up to unitary equivalence, representation of $G \times G$, which plays the role of the biregular representation, for an arbitrary positive definite function on G. But the corresponding representation of $\mathrm{Int}G$ is, in general, no longer continuous. Hence it cannot be extended by continuity to $\overline{\mathrm{Int}}G$. However, there is a class of states on G, containing finite characters, for which this continuity persists. This is exactly the class of stable states.

2.3. The Canonical Construction of Representations of $G \times G$ *and* $\mathrm{Int}G$

Let π be the representation of a group G corresponding to a state φ and M be the w^*-algebra generated by the operators from $\pi(G)$. Let M' be the commutant of M, M_*^+ be the set of weakly continuous positive functionals on M, $\tilde{\varphi} \in M_*^+$ be a faithful state, and θ be the representation of M corresponding to $\tilde{\varphi}$ in a Hilbert space $\tilde{H}$ with a bicyclic vector $\tilde{\xi}$. Since $\tilde{\varphi}$ is faithful, the map $M \overset{\theta}{\mapsto} \theta(M) = \tilde{M}$ is an isomorphism (see [5]).

Denote by $\tilde{\mathcal{J}}$ the antilinear isometry from the polar decomposition of the closure of the map $\tilde{M}\tilde{\xi} \ni m\tilde{\xi} \overset{\tilde{S}}{\mapsto} m^*\tilde{\xi} \in \tilde{M}\tilde{\xi}\, m \in \tilde{M}$. Set $\tilde{\pi} = \tilde{\theta} \circ \pi$. Since $\tilde{\mathcal{J}}\tilde{M}\tilde{\mathcal{J}}^{-1} = \tilde{M}'$, the map

[2] Given by the formula $\Pi(g, h)\eta = \eta(g^{-1}xh)$, $\eta \in l^2(G)$.

$$G \times G \ni (g, h) \mapsto \tilde{\Pi}((g, h)) = \tilde{\pi}(g)\tilde{\mathfrak{J}}\tilde{\pi}(h)\tilde{\mathfrak{J}}^{-1}$$

is a representation.

Since $\tilde{\mathfrak{J}} m \tilde{\mathfrak{J}}^{-1} = m^*$ for all m from the center of the algebra $\tilde{M}$ (see [5]), a representation $\mathcal{A}_\pi$ of the group $\mathrm{Int}G$ is well defined by analogy with Section 2.2: $\mathrm{Int}G \ni \mathrm{Ad}g \overset{\mathcal{A}_\pi}{\mapsto} \tilde{\Pi}((g, g))$.

2.4. Stable States and Representations; Admissible Representations

Let G be a locally compact group, $\mathfrak{A} = \mathbf{C}^*(G)$, and $\mathfrak{A}^*$ be the dual space to $\mathfrak{A}$. Given a state $\varphi \in \mathfrak{A}^*$, denote by o_φ the map $\mathrm{Int}G \ni \theta \overset{o_\varphi}{\mapsto} \varphi \circ \theta \in \mathfrak{A}^*$.

Definition 1. A state φ is called stable if the map o_φ is continuous in the strong topology on $\mathrm{Int}G$ (see (2)) and in the norm topology of the dual space on $\mathfrak{A}^*$.

Remark 1. For every $\psi \in \mathfrak{A}^*$, the map o_ψ is continuous in the weak topology of the dual space on $\mathfrak{A}^*$.

The representation of G corresponding to a stable state will also be called stable.

To define a stable positive definite function on an inductive limit $G = \lim\limits_n G_n$ of locally compact groups, consider the cone C_G^+ of positive definite functions on G with the topology defined by the metric $\rho(\psi, \varphi) = \sup\limits_n \|\psi - \varphi\|_n$, $\varphi, \psi \in C_G^+$, where $\|\cdot\|_n$ stands for the norm of the dual space $\mathbf{C}^*(G_n)^*$.

Definition 2. A positive definite function φ on G is called stable if the map $\mathrm{Int}G \ni \theta \overset{o_\varphi}{\mapsto} \varphi \circ \theta \in C_G^+$ is continuous in the topology on $\mathrm{Int}G$ defined according to (3).

Remark 2. If G is a free group, then the topology (1) on $\mathrm{Aut}G$ is discrete. Hence all positive definite functions on G are stable.

Now we introduce the notion of an admissible representation of $G \times G$, which extends the corresponding notion from [4] to the case of an arbitrary group.

Definition 3. Let G be a locally compact group or an inductive limit of locally compact groups. A unitary representation Π of the group $G \times G$ in a Hilbert space H is called admissible if the map $\mathrm{Int}G \ni \mathrm{Ad}g \mapsto \Pi((g, g)) \in \mathbb{U}(H)$ is continuous in the strong topology on $\mathrm{Int}G$ (see (2), (3)) and in the strong operator topology on $\mathbb{U}(H)$.

Theorem 1. *If the representation $\tilde{\Pi}$ from Section* 2.3 *corresponds to a stable representation π of G, then $\tilde{\Pi}$ is an admissible representation of the group $G \times G$. The restriction of $\tilde{\Pi}$ to $\mathfrak{S}_{\mathbb{N}} \times$ e is quasi-equivalent to π, the center $\mathcal{C}$ of the algebra $\tilde{\Pi}(\mathfrak{S}_{\mathbb{N}} \times \mathfrak{S}_{\mathbb{N}})''$ coincides with the center of $\tilde{M}$, and the components of the decomposition of $\tilde{\Pi}$ with respect to $\mathcal{C}$ are irreducible representations.*

3. STABLE REPRESENTATIONS OF THE GROUP $\mathfrak{S}_{\mathbb{N}}$ AND THEIR CLASSIFICATION

In this section, we formulate results on the classification of stable representations of $\mathfrak{S}_{\mathbb{N}}$ up to quasi-equivalence.

With a stable factor representation π of the group $\mathfrak{S}_{\mathbb{N}}$ we associate an invariant $\chi^{\pi a}$, called an asymptotic character. Let $M = \pi(\mathfrak{S}_{\mathbb{N}})''$.

Proposition 1. *Let π be a stable factor representation of $\mathfrak{S}_{\mathbb{N}}$. For every $g \in \mathfrak{S}_{\mathbb{N}}$ there exists a sequence $\{\sigma_n^g\}_{n \in \mathbb{N}} \subset \mathfrak{S}_{\mathbb{N}}$ such that $\sigma_n^g g(\sigma_n^g)^{-1} \in \mathfrak{S}_{\mathbb{N}\backslash n}$ and $\sigma_{n+1}^g(\sigma_n^g)^{-1} \in \mathfrak{S}_{\mathbb{N}\backslash n}$. If $\psi \in M_*^+$, then the limit*

$$\lim_{n \to \infty} \psi(\pi(\sigma_n^g g(\sigma_n^g)^{-1})) \cdot \psi(I)^{-1} = \chi^{\pi a}(g)$$

exists and does not depend on ψ and $\{\sigma_n^g\}_{n \in \mathbb{N}}$. The function $\chi^{\pi a}$ is an indecomposable character of $\mathfrak{S}_{\mathbb{N}}$ (see [4]). If $\tilde{\pi}$ is quasi-equivalent to π, then $\chi^{\pi a} = \chi^{\tilde{\pi} a}$.

Theorem 2. *Fix a representation π of $\mathfrak{S}_{\mathbb{N}}$, and let $M_*^+(n) = \{\phi \in M_*^+ : \phi(\pi(s)x) = \phi(x\pi(s))$ for all $\forall x \in M$, $s \in \mathfrak{S}_{\mathbb{N}\backslash n}\}$. The following conditions are equivalent:*

(i) *the representation π is stable;*

(ii) *the union $\bigcup\limits_n M_*^+(n)$ is dense in the norm topology of the dual space on M_*^+.*

We define the central depth $\mathrm{cd}(\pi)$ of a representation π as $\min\{n\colon M_*^+(n) \neq 0\}$. Clearly, $\mathrm{cd}(\pi)$ is a quasi-equivalence invariant.

Theorem 3. *Let π be a stable factor representation, $n = \mathrm{cd}(\pi)$, and $\psi \in M_*^+(n)$. If E is the support*[3] *of ψ, then $E \in \pi(\mathfrak{S}_n\mathfrak{S}_{\mathbb{N}\backslash n})'$ and $E\pi(s)E = 0$ for all $s \notin \mathfrak{S}_n\mathfrak{S}_{\mathbb{N}\backslash n}$. In particular, the representation π_E of the group $\mathfrak{S}_n\mathfrak{S}_{\mathbb{N}\backslash n}$ determined by the operators $E\pi(s)E$ has the form $T_\lambda \otimes \pi_{\alpha\beta}$, where $\lambda \vdash n$ is a Young diagram, T_λ is the corresponding irreducible representation of $\mathfrak{S}_n$, and $\pi_{\alpha\beta}$ is the representation of $\mathfrak{S}_{\mathbb{N}\backslash n}$ corresponding to a Thoma character $\chi_{\alpha\beta}$. Hence π is quasi-equivalent to the representation $\Pi_{\alpha\beta}^\lambda = \mathrm{Ind}_{\mathfrak{S}_n\mathfrak{S}_{\mathbb{N}\backslash n}}^{\mathfrak{S}_{\mathbb{N}}} T_\lambda \otimes \pi_{\alpha\beta}$.*

The next result describes the dependence of $\mathrm{cd}(\pi)$ on the Thoma parameters α, β of the asymptotic character $\chi^{\pi a}$.

[3] That is, E is the smallest projection from $\pi(\mathfrak{S}_{\mathbb{N}})''$ such that $\psi(I - E) = 0$.

Theorem 4. *Let* $n \geq 1$, *and let* λ, T_λ, $\pi_{\alpha\beta}$ *be as in Theorem* 3. *Then* $\Pi^\lambda_{\alpha\beta}$ *is a f.r. of type* II_∞ *if* $\sum \alpha_i + \sum \beta_i = 1$, *and a f.r. of type* II_1 *if* $\sum \alpha_i + \sum \beta_i < 1$.

Theorem 5. *Let* $\sum \alpha_i + \sum \beta_i = 1$. *The representations* $\Pi^\lambda_{\alpha\beta}$ *and* $\Pi^{\tilde{\lambda}}_{\tilde{\alpha}\tilde{\beta}}$ *are quasi-equivalent if and only if* $\alpha_i = \tilde{\alpha}_i$, $\beta_i = \tilde{\beta}_i$, *and* $\lambda = \tilde{\lambda}$.

Now we describe a bijection between the set of quasi-equivalence classes of stable factor representations and the set "partially central" states on $\mathfrak{S}_{\mathbb{N}}$.

Theorem 6. *Let* π *be a stable factor representation of* $\mathfrak{S}_{\mathbb{N}}$ *and* $n = \mathrm{cd}(\pi)$. *If a state f on* $\mathfrak{S}_{\mathbb{N}}$ *determines a representation quasi-equivalent to* π *and satisfies the condition* $f(tst^{-1}) = f(s)$ *for all* $s \in \mathfrak{S}_{\mathbb{N}}$ *and* $t \in \mathfrak{S}_n\mathfrak{S}_{\mathbb{N}\setminus n}$, *then*

$$f(s) = \begin{cases} \chi_\lambda(r)\chi_{\alpha\beta}(t), & \text{if } r \in \mathfrak{S}_n,\ t \in \mathfrak{S}_{\mathbb{N}\setminus n} \text{ and } s = rt; \\ 0, \ \text{if } s \neq \mathfrak{S}_n \cdot \mathfrak{S}_{\mathbb{N}\setminus n}, \end{cases}$$

where $\lambda \vdash n$ *is a diagram*, χ_λ *is the normalized character of the corresponding irreducible representation of* $\mathfrak{S}_n$, *and* α, β *are the Thoma parameters of the asymptotic character* $\chi^{\pi a}$ (*see Proposition* 1).

ACKNOWLEDGMENTS

This work was partially supported by the Russian Foundation for Basic Research (project no. 10-01-90411-Ukr-a) and partially supported by the National Academy of Sciences of Ukraine Grant 2010–2011-284.

REFERENCES

1. E. Thoma, Math. Z. **85** (1), 40–61 (1964).
2. A. M. Vershik and S. V. Kerov, Funct. Anal. Appl. **15**, 246–255 (1981).
3. A. M. Vershik, Notices Am. Math. Soc. **54**, 391–393 (2007).
4. G. I. Ol'shanskii, Leningr. Math. J. **1**, 983–1014 (1990).
5. M. Takesaki, *Theory of Operator Algebras* (Springer-Verlag, Berlin, 2005), Vol. 2.
6. A. Yu. Okun'kov, Funct. Anal. Appl. **28**, 100–107 (1994).