Linearly rigid metric spaces and the embedding problem.

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Abstract

We consider the problem of isometric embedding of the metric spaces to the Banach spaces; and introduce and study the remarkable class of so called linearly rigid metric spaces: these are the spaces that admit a unique, up to isometry, linearly dense isometric embedding into a Banach space. The first nontrivial example of such a space was given by R. Holmes; he proved that the universal Urysohn space has this property. We give a criterion of linear rigidity of a metric space, which allows us to give a simple proof of the linear rigidity of the Urysohn space and some other metric spaces. The various properties of linearly rigid spaces and related spaces are considered.

Introduction

The goal of this paper is to describe the class of complete separable metric (=Polish) spaces which have the following property: there is a unique (up to isometry) isometric embedding of this metric space (X, ρ) in a Banach space such that the affine span of the image of X is dense (in which case we say that X is *linearly dense*). We call such metric spaces *linearly rigid spaces*.

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The first nontrivial example of a linearly rigid space was Urysohn's universal space; this was proved by R.Holmes [9]. Remember that the Urysohn space is the unique (up to isometry) Polish space which is universal (in the class of all Polish spaces) and ultra-homogeneous (= any isometry between finite subspaces extends to an isometry of the whole space). It was discovered by P.S.Urysohn in his last paper [17], which was published after his tragic death. The criterion of linear rigidity which we give in this paper is a weakening of well-known criterion of Urysohnness of the metric space, so the linear rigidity of Urysohn space is an evident corollary of our theorem.

In this connection, in the first section we consider the general statement of the problem of isometric embedding of the metric spaces into Banach spaces. There exist several distinguish functorial embeddings of an arbitrary metric space into a Banach space. The first one is well-known Hausdorf-Kuratowski (HK): this is embedding into the Banach space of bounded continuous functions: $x \to \rho(x,.)$. Our point is to define isometric embedding using corresponding *norms and seminorms compatible with the metric* on the free space over metric space, or, in another words, - seminorms in the space $V(X)$ of all finite affine combinations of elements of X. This method was used in [1], but it goes back to the idea of 40-th of the free group over metric or topological spaces. The most important compatible norm is the (KR)-norm, which based on the classical Kantorovich metric on the space of measures on the initial metric space; it was defined by L.V.Kantorovich in 1942 ([13]) in the framework of Kantorovich-Monge transportation problem. The Kantorovich-Rubinstein (KR) norm which is simply extension of Kantorovich metric to the space $V(X)$ (more exactly $_0(X)$ -see below) was defined in [12] for compact spaces.

A remarkable fact is that (KR)-norm is the *maximal norm compatible with the given metric, so a linearly rigid space is a space for which (KR)-norm is the unique compatible norm.*

The original definition of (KR)-norm is direct - as solution of transportation problem; the main observation by Kantorovich was that the dual definition of the norm used the conjugate space to the space V_0 which is the space of Lipschitz functions (with Lipschitz norm) on the metric space. Thus the completion of the space V_0 under that norm is predual to the Banach space Lip of Lipschitz function and is called sometimes as free Lipschitz space (see also [5, 10, 21]). It also worth mentioning that geometry of the sphere in the (KR) norm is nontrivial even for finite metric space and is related to the *geometry of root polytopes of Lie algebras of series* (A), and other in-

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teresting combinatorial and geometrical questions. This geometry is directly concerned with the problem of the embedding of the finite metric spaces to the Banach space; the authors do not know if this question had been discussed systematically in full generality. Because of maximality of KR-norm all other compatible norms and seminorms can be defined using some subspaces of the space of Lipschitz functions. We define the wider class of such norms article; the main example of this class is what we called the double-point - (dp)-norm; is used in the proof of the main theorem.

In the second section we prove the main result — a criterion of linear rigidity in terms of distances. Namely, we prove that the characteristic property of that spaces is roughly speaking the following: any extremal (as the point of unit sphere) Lipschitz function of norm 1 is approximated by functions $\rho(x, \cdot) + \text{const}$. The characteristic property of the Urysohn space is stronger: one does not need extremality of Lipschitz functions and there are some natural restrictions to the choice of constant. We prove in particular that the metric space is linearly rigid space *if only two norms -maximal (KR) and double-point (dp) are coincide.* We discuss some properties of the linearly rigid spaces, for example, we prove that such space must have infinite diameter; another property - the unit sphere of it is exclusively degenerated in a sense.

In the third section we give several examples of linearly rigid spaces and first of all obtain the Holmes's result about linear rigidity of Urysohn space using apply our criterion and compare it with the criterion of Urysohnness. The integral and rational (with the distances more than 1) universal spaces are also linearly rigid. We discuss also the notions which are very close to rigidity. One of them is the notion of almost universal space which has an approximation property that is stronger than for linearly rigid spaces, but weaker than for the Urysohn space; Another one is the notion of weak rigidity of the metric spaces which corresponds to the the coincidence of HK and KR norms which is weaker than linear rigidity.

We formulate several questions appeared on the way. The geometry of those Banach spaces E_X (and its unit spheres) which are corresponded to linearly rigid metric spaces (X, ρ) are very intriguing. The most important concrete question: to define axiomatically the Banach space $E_{\mathbb{U}}$ which corresponds to Urysohn space $\mathbb{U}$, and can be called as Urysohn-Holmes-Kantorovich-Banach spaces or shortly - Urysohn Banach space. This is very interesting universal Banach space with a huge group of the linear isometries; it must be considered from various points of view.

1 Isometric embeddings of metric spaces into Banach spaces.

1.1 Compatible norms and seminorms.

Let (X, ρ) be a complete separable metric (=Polish) space[1]. Consider the free vector space $V = \mathbb{R}(X)$ over the space X, and the free affine space $V_0 = \mathbb{R}_0(X)$ generated by the space X (as a set) over the field of real numbers:

$$V(X) = \mathbb{R}(X) = \left\{ \sum a_x \cdot \delta_x, \ x \in X, a_x \in \mathbb{R} \right\}$$

$$\supset V_0(X) = \mathbb{R}_0(X) = \left\{ \sum a_x \cdot \delta_x, \ x \in X, a_x \in \mathbb{R}, \sum a_x = 0 \right\}$$

(all sums are finite). The space $V_0(X)$ is a hyperplane in $V(X)$. We omit the mention of the space (and also of the metric, see below) if no ambiguity is possible; we will mostly consider only the space V_0. Another interpretation of the space $\mathbb{R}(X)$ (respectively, $\mathbb{R}_0(X)$) is that this is the *space of real measures* with finite support (respectively, the space of measures with finite support and total mass equal to zero: $\sum_x a_x = 0$). Now we introduce the class of *seminorms* on V_0 *compatible with the metric* ρ. For brevity, denote by $e_{x,y} = \delta_x - \delta_y$ the elementary signed measure corresponding to an ordered pair (x, y).

Definition 1. *We say that a seminorm $\|\cdot\|$ on the space $V_0(X)$ is compatible with the metric ρ on the space X if $\|e_{x,y}\| = \rho(x, y)$ for all pairs $x, y \in X$.*

Each compatible seminorm on V_0, can be extended to a seminorm on the space V by setting $\|\delta_x\| = 0$ for some point $x \in X$, and *vise versa*, the restriction of the compatible seminorm on V is a compatible seminorm on V_0; it is more convenient to consider compatible seminorms only on the space V_0.

The rays $\{ce_{x,y} \in V_0, \ c > 0\}$ going through elementary signed measures will be called *fundamental rays* (the set of fundamental rays does not depend on the metric). If the metric is fixed, then a norm compatible with this metric determines a unique vector of unit norm on each fundamental ray;

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[1]The main definitions, assertions and proofs of the paper are valid for nonseparable complete metric spaces

let us call these vectors (elementary signed measures) *fundamental vertices corresponding to a given metric*. They are given by the formula $(x \neq y)$:

$$\frac{e_{x,y}}{\rho(x,y)} \equiv \bar{e}_{x,y}.$$

Thus the set of seminorms compatible with a given metric ρ is the set of seminorms for which the fundamental vertices corresponding to this metric are of norm one.

The following useful elementary lemma describes all possible metrics on a space X in the geometrical terms of $V_0(X)$.

Lemma 1. *Let X be a set. Consider the linear space $V_0(X)$ and specify some points $c(x,y) \cdot e_{x,y}$ on the fundamental rays $\mathbb{R}_+ \cdot e_{x,y}$, where the function $c(x,y)$ is defined for all pairs (x,y), $x \neq y$, positive, and symmetric: $c(x,y) = c(y,x)$. This set of points is the set of fundamental vertices of some metric on X if and only if no point lies in the relative interior of the convex hull of a set consisting of finitely many other fundamental vertices and the zero.*

Proof. Consider the function defined by the formulas $\rho(x,y) = c(x,y)^{-1}$, $x \neq y$, and $\rho(x,x) = 0$. Let us check that the triangle inequality for this function is equivalent to the property of convex hulls mentioned in the lemma. At first, if the triangle inequality does not hold, say $\rho(c,a) > \rho(c,b) + \rho(b,a)$, then

$$\bar{e}_{c,a} = \frac{\rho(c,b)}{\rho(c,a)} \cdot \bar{e}_{c,b} + \frac{\rho(b,a)}{\rho(c,a)} \cdot \bar{e}_{b,a}.$$

The sum of coefficients $\frac{\rho(c,b)}{\rho(c,a)} + \frac{\rho(b,a)}{\rho(c,a)}$ is less then 1, hence $\bar{e}_{c,a}$ lies in a relative interior of the triangle with vertices $\bar{e}_{c,b}$, $\bar{e}_{b,a}$ and 0.

Next, assume that ρ is a metric. We need to prove that the sum $\lambda = \sum \lambda_i$ of the coefficients in the linear representation

$$\bar{e}_{x,y} = \sum_{i=1}^{n-1} \lambda_i \bar{e}_{x_i,y_i}, \ \lambda_i \geq 0$$

is at least 1. Integrate the function $\rho(\cdot,y)$ with respect to measures in both sides. Due to triangle inequality,

$$\int \rho(\cdot,y) d\bar{e}_{x_i,y_i} = \frac{\rho(x_i,y) - \rho(y_i,y)}{\rho(x_i,y_i)} \leq 1,$$

so we get $1 \leq \sum \lambda_i$. □

Let $\hat{V}_0$ and $\hat{V}$ be the quotients of the spaces V_0 and V over the kernel $K = \{v : \|v\| = 0\}$ of the seminorm and let $\bar{V}$ and $\bar{V}_0$ be the completions of the spaces $\hat{V}_0$ and $\hat{V}$ with respect to the norm.

Proposition 1. *Every compatible seminorm $\|\cdot\|$ on the space $V_0(X)$ defines an isometric embedding of the space (X, ρ) into the Banach space $(\bar{V}, \|\cdot\|)$. Every isometric embedding of the metric space (X, ρ) into a Banach space E corresponds to a compatible seminorm on $V_0(X)$.*

Indeed, obviously, the metric space (X, ρ) has a canonical isometric embedding into $\bar{V}$, and, conversely, it is easy to see that if there exists an isometric embedding of the space (X, ρ) into some Banach space E, then the space V_0 is also imbedded to the space E linearly and the restriction of the norm onto V_0 defines a compatible seminorm (not norm in general!) on V_0.

Question 1. *Let us call metric space solid if each compatible seminorm is a norm. What metric spaces are solid? For example, what finite metric spaces are solid? A more concrete question: what is the minimal dimension of the Banach space into which a given finite metric space can be isometrically embedded?*

A similar (but different) question is studied in [16].

Of course it is enough to consider the case when the closure of the affine hull of the image of X in E or the closure of the image of space V is dense in E. We will say that in this case the isometric embedding of X into a Banach space E is *linearly dense*. Thus, our problem is *to characterize the metric spaces for which there is only one compatible norm or, equivalently, there is a unique, up to isometry linearly dense embedding into a Banach space.*

1.2 Examples, functorial embeddings

We will start with several important examples of isometric embeddings and compatible norms.

1.2.1 Hausdorf-Kuratowski embedding

The following is a well-known isometric embedding of an arbitrary metric space into a Banach space:

Definition 2. *Define a map from the metric space (X, ρ) into the Banach space $\bar{C}(X)$ of all bounded continuous functions on X, endowed with the sup-norm, as follows:*

$$X \ni x \mapsto \rho(x, .), V_0 \ni e_{x,y} \mapsto \rho(x, .) - \rho(y, .).$$

We call it the Hausdorf-Kuratowski (HK) embedding.

It is evident that this embedding is an isometry. In general, it is not a linearly dense embedding, because the image of the space $V(X)$ consists of very special Lipschitz functions. It is difficult to describe exactly the closed subspace of $\bar{C}(X)$ that is the closed linear hull of the image under this embedding. At the same time, the corresponding compatible norm is given explicitly: for $v = \sum_k c_k \delta_{x_k}, \sum_k c_k = 0$;

$$\left\| \sum_k c_k \delta_{x_k} \right\| = \sup_z \left| \sum_k c_k \rho(z, x_k) \right|$$

1.2.2 Compatible norms which are defined by the class of Lipschitz functions.

Let us give another example of a class of embeddings, which will play an important role. Choose the class of Lipschitz functions $\mathcal{L} \subset \mathrm{Lip}(X)$ which contains for each pair of points $x, y \in X$ the Lipschitz function $f_{x,y}(\cdot)$ such that $f_{x,y}(y) - f_{x,y}(x) = \rho(x, y)$

Definition 3. *For every element $v = \sum_k c_k \delta_{z_k} \in V_0$ set*

$$N_{\mathcal{L}}(v) = \sup_{f \in \mathcal{L}} | \sum c_k f(z_k)|.$$

Then $N_{\mathcal{L}}$ is a seminorm on V_0. We call it the $\mathcal{L}$-seminorm.

By definition of the $N_{\mathcal{L}}$ we have $N_{\mathcal{L}}(e_{x,y}) = \rho(x, y)$, so this seminorm is compatible with the metric ρ. Note that any compatible seminorm can be obtained in a similar way: by choosing a subset of Lipschitz functions $\mathcal{L}$.

Let us consider the following important specific example of seminorms (at fact, they are actually norms) which is defined in this way. Let $\mathcal{L}$ be the following class of 1-Lipschitz functions:

$$\mathcal{L} = \{\phi_{x,y}(\cdot) = \frac{\rho(y, \cdot) - \rho(x, \cdot)}{2}, x, y \in X; x \neq y\}.$$

Thus, the corresponding compatible norm is the following:

$$\|v\| = \sup_{x,y} |\sum_k c_k \phi_{x,y}(z_k)|,$$

or in the notation for continuous measures μ:

$$\|\mu\| = \sup_{x,y} |\int \phi_{x,y}(z) d\mu(z)|.$$

Definition 4. *We will call this norm on $V_0(X)$ double-point norm and denote it as $\|\mu\|_{dp}$.*

We will heavily use double-point norm in the proof of the main result.

The choice of functions $\phi_{x,y}$ in the definition above of double-point norm can be extended, for example, as follows: one may define

$$\phi_{x,y} = \theta(x,y) \cdot \rho(x,\cdot) - (1 - \theta(x,y)) \cdot \rho(y,\cdot)$$

for any function $\theta : X^2 \to (0,1)$ such that $\inf_{x,y} \theta(x,y) > 0$ and $\sup_{x,y} \theta(x,y) = 1$.

1.2.3 Kantorovich embedding and maximal compatible norm

Now we consider the most important compatible norm: it appeared as consequence of the classical notion of Kantorovich (transport) metric ([11]) on the set of Borel measures on the compact metric spaces later called as Kantorovich-Rubinstein norm [12].

The shortest way to define it is the following:

Definition 5. *The Kantorovich-Rubinstein norm is the norm of $N_{\mathcal{L}}$-type where the set $N_{\mathcal{L}}$ is the set of all Lipschitz functions with Lipschitz norm one; more directly: let $v = \sum_k c_k \delta_{z_k} \in V_0$ then*

$$\|v\| = \sup_u |\sum_k u(z_k) c_k|,$$

where u run over all Lipschitz functions over norm 1:

$$\|u\| = \sup_{x \neq y} \frac{|u(x) - u(y)|}{\rho(x,y)}.$$

We can define this norm not only on V_0 but on the space of all Borel measures μ on the metric space (X, ρ) with compact support:

$$\|\mu\| = \sup_{\|u\|=1} |\int_X u(z)d\mu(z)|.$$

Immediately from the definition and from the remark above we have the following theorem.

Theorem 1. *Let (X, ρ) is a Polish space; then the KR-norm is the maximal compatible norm; we will denote it as $\|.\|_{max}$.*

In the geometrical terms, this means that the unit ball in KR-norm is closed convex hull of the set of fundamental vertices $\bar{e}_{x,y}$ (see above); the unit ball with respect to every seminorm compatible with the metric ρ contains the unit ball in KR-norm.

It is not difficult to prove this theorem directly. Recall that initial definition of Kantorovich-Rubinstein norm was different and the previous theorem is the duality definition of the norm with comparison to original, in that setting the definition above is the duality theorem due to L.Kantorovich ([11]). As we mentioned the conjugate space to the V_0 with KR-norm is the Banach space of Lipschitz functions.

Denote the Banach space that is the completion of the space $V_0(X)$ with respect to the KR-norm by $E_{X,\rho}$. Sometimes this space called "free Lipschitz space" ([5]. It is easy to check that *The correspondence*

$$(X, \rho) \mapsto E_{X,\rho}$$

is a functor from the category of metric spaces (with Lipschitz maps as morphisms) to the category of Banach spaces (with linear bounded maps as morphisms).

Recall the initial definition was

Definition 6. *Let (X, ρ) be a Polish space. For the Borel measure $\mu = \mu_+ - \mu_-$, where μ_+ and μ_- are the Borel probability measures the Kantorovich-Rubinstein norm is defined as*

$$\|\mu\| = \inf_{\psi} \int_X \int_X \rho(x, y)d\psi(x, y)$$

where ψ runs over set $\Psi = \Psi(\mu_+, \mu_-)$ of all Borel probability measures on the product $X \times X$, with the marginal projection onto the first (second) factor equal to $\mu_+(\mu_-)$.

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The equality between two definitions of the KR-norm is precisely duality theorem in linear programming.

1.2.4 Comments

1.We have seen that the norm from Definition 6 defines a metric on the simplex of probability Borel measures: $\|\mu\| \equiv k_\rho(\mu_+, \mu_-)$, and just this metric was initially defined by L.V.Kantorovich [13].[2] The same is true for all other compatible norms: each of which defines the metric on the affine simplex

$$V_0^+(X) = \{v = \sum_k c_k \delta_{z_k} \in V_0(X) : c_k > 0, \sum_k c_k = 1\}$$

of probability measures by formula: $k_{\|.\|}(v_1, v_2) = \|v_1 - v_2\|, v_1, v_2 \in V_0^+$.

More generally we can define the notion of *compatible metric* on $V_0^+(X)$:

Definition 7. *A metric on the simplex $V_0^+(X)$, that is convex as a function on affine set $V_0^+(X) \times V_0^+(X)$, and which has the property $k_\rho(\delta_x, \delta_y) = \rho(x, y)$ is called compatible with the metric space (X, ρ).*

There are many compatible metrics which do not come as above from the compatible norms. F.e. the L_p-analogs, $p > 1$ of the Kantorovich metric are not generated by compatible norm. The compatible metrics are very popular now in the theory of transportation problems - see [20].

2.Our examples of compatible norms e.g. Hausdorf-Kuratowski, double-point norm, maximal (Kantorovich-Rubinstein) norms etc. are functorial norms (with respect to isometries as morphisms)in the natural sense.

3.In opposite to existence of the maximal compatible norm there is no *the least compatible norm*; moreover it happened that for given compatible norm the infimum of the norms which which are less that given norm, is *seminorm*, but not norm.

4.The unit ball in KR norm for finite metric space with all distances equal to 1 is nothing more than generalization of the root polytopes of the Lie algebras of series A. More generally, unit ball for the general metric also could be considered as a polytope in Cartan subalgebra.

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[2]There are many authors who rediscovered this metric later; unfortunately some of them did not mention the initial paper by L.V.Kantorovich (see the survey [19]).

2   Main theorem: criterion of linear rigidity

2.1   Criterion of Linear rigidity

Let (X, ρ) be a metric space. We denote the Banach space of the individual Lipschitz functions on the metric space (X, ρ) as $\text{Lip}(X, \rho)$ or $\text{Lip}(X)$ and denote quotient Banach space $\text{PLip}(X) = \text{Lip}(X)/\{\text{const}\}$ with norm: $\sup_{x \neq y} \frac{|u(x) - u(y)|}{\rho(x,y)}$. The image of function $u \in Lip(X)$ under projection $\pi : \text{Lip}(X) \to \text{PLip}(X)$ we denote as $\hat{u}$.

Definition 8. *1)The Lipschitz function of type $\phi_x(\cdot) = \rho(x, \cdot)$ for some point $x \in X$ is called a distance function,*

2)The Lipschitz function u is called admissible if

$$|u(x) - u(y)| \leq \rho(x, y) \leq u(x) + u(y).$$

3) Let $F \subset X$ (in particular $F = X$) and $u(\cdot)$ be an individual Lipschitz function on metric space F (equipped with the induced metric) of norm 1. We say that the function u is representable (respectively, ϵ-representable for $\epsilon > 0$) in X if there exists a point $x \in X$ such that $u(z) = \rho(x, z)$ (correspondingly $|u(z) - \rho(x, z)| < \epsilon$) for all $z \in F$.

4)We say that u is additively representable (ϵ-representable) if there exists a constant a in$\mathbb{R}$ such that the function $u(\cdot) - a$ is representable (ϵ-representable).

Now we formulate the criterion of linearly rigidity of a metric space.

Recall that for finite metric space F a 1-Lipschitz function f on F is called *extremal*, if it is an extreme point of the unit ball of Lipschitz functions (factored by the constants). In other words, if functions $f \pm g$ are both 1-Lipschitz, then g must be a constant function.

Theorem 2. *The following assertions are equivalent:*

1)The complete metric space (X, ρ) is linearly rigid; that is, all norms on the space $V_0(X)$ compatible with metric coincide;

2)$\| \cdot \|_{max} = \| \cdot \|_{dp}$; — i.e. two norms, the Kantorovich-Rubinstein norm and double-point norm, (see item 1.2.2-3) coincide;

*3)(criterion) For each finite subset $F \subset X$ (with the induced metric) and each $\epsilon > 0$, any **extremal** 1-Lipschitz function u on the space F is additively ϵ-representable.*

4) The weak* closure of the convex hull of the set of distance functions is the unit ball in the space $\mathrm{PLip}(X)$.[3]

Remember that weak*-topology on the space is defined by duality between the spaces $V_0(X)$ and $\mathrm{PLip}(X)$.

Proof. 1) $\Longrightarrow$ 2) is trivial;

2) $\Longrightarrow$ 3): Let (X, ρ) be a metric space, for which maximal and double-point norms coincide. Our goal is to prove that for any given finite $F \subset X$ and any extremal function f on the set F there exists a point $x \in X$ and a constant a such that $\sup_{y \in F} |\rho(x, y) - f(y) - a| < \epsilon$.

If F contains 1 or 2 elements, we may choose x equal to one of these elements, so let $F = \{x_1, x_2, \ldots, x_n\}$, $n \geq 3$.

Let us define a directed graph on F for $n > 2$ as follows: draw an edge $x_i \to x_j$ if $f(x_i) - f(x_j) = \rho(x_i, x_j)$. Note that the constructed graph regarded as an undirected graph is connected. Indeed, if it is not connected, then for some disjoint nonempty sets F_1, F_2 such that $F = F_1 \cup F_2$ there are no edges between x_i and x_j for $x_i \in F_1$, $x_j \in F_2$ Then for small positive ϵ the functions $f \pm \epsilon \chi_{F_1}$ are 1-Lip too, this contradicts extremality of f. (Here χ_{F_1} takes value 1 on F_1 and 0 on F_2). Let us define an element $\mu \in V_0(F)$ as the sum

$$\mu := \sum e_{a,b}$$

(here and up to end of the proof the summation is taken by all edges $a \to b$ of F). A linear functional $\nu \to \int f d\nu$ attains its supremum on each function $e_{a,b}$ for any edge $a \to b$. Hence it attains its maximum also on the sum of these measures, i.e. on μ. So

$$\|\mu\|_{max} = \int f d\mu = \sum \rho(a, b).$$

Then also

$$\|\mu\|_{dp} = \sum \rho(a, b)$$

Hence for any $\epsilon > 0$ there exist $x, y \in X$ such that the function $\phi_{x,y}$ satisfies the following inequality:

$$\left| \sum \rho(a, b) - \sum (\phi_{x,y}(a) - \phi_{x,y}(b)) \right| < \epsilon.$$

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[3]The last formulation was suggested by one of the reviewer of the paper.

We have

$$\rho(a,b) - (\phi_{x,y}(a) - \phi_{x,y}(b)) = \left(\rho(a,b) - \rho(x,a) + \rho(x,b)\right)/2 + \left(\rho(a,b) + \rho(y,a) - \rho(y,b)\right)/2.$$

Both summands are nonnegative and so both are less than ϵ for any edge $a \to b$. It means that the function $g(\cdot) = \rho(x,\cdot) - f(\cdot)$ is almost constant on F (since $g(a) - g(b)$ is small for any edge and the graph is connected). So, x satisfies the necessary conditions.

3)$\Longrightarrow$ 1). We prove that if 3) holds for any finite F, then for every signed measure $\mu \in V_0(F)$ and every norm $\|\cdot\|$ on $V_0(X)$ compatible with the metric, one has $\|\mu\| = \|\mu\|_K$.

Recall that the unit ball of $\|\cdot\|_K$-norm is the closed convex hull of the points $\bar{e}_{a,b}$; every finitely-supported measure μ such that $\|\mu\|_K = 1$ is a convex combination of some $\bar{e}_{a_k,b_k}$. Applying this to the measure $\mu/\|\mu\|_K$ we get

$$\mu = \sum_{k=1}^{N} \alpha_k \bar{e}_{a_k,b_k}, \quad a_k, b_k \in F, \ \alpha_k \geq 0, \quad \|\mu\|_K = \sum \alpha_k.$$

The points $\bar{e}_{a_k,b_k}$ lie on some face of the unit ball of the space E_F. We may assume without loss of generality that it is a face of codimension 1. The corresponding supporting plane is determined by some linear functional of norm 1, i.e., some 1-Lipschitz function f on F. Then for every k, $f(\bar{e}_{a_k,b_k}) = 1$, that is $f(a_k) - f(b_k) = \rho(a_k, b_k)$. f is an extremal Lipschitz function on F.

First consider the case when all the a_k are equal: $a_k = a$.

By assumption, there is a point $c \in X$ such that $\rho(c,a) \geq \rho(c,b_k) + \rho(a,b_k) - \varepsilon$.

We have

$$\|\mu\| = \left\| \sum \alpha_k \frac{(\delta_a - \delta_c) + (\delta_c - \delta_{b_k})}{\rho(a,b_k)} \right\|$$

$$\geq \sum \alpha_k \cdot \frac{\rho(a,c)}{\rho(a,b_k)} - \sum \alpha_k \cdot \frac{\rho(c,b_k)}{\rho(a,b_k)} \geq \sum \alpha_k \left(1 - \frac{\varepsilon}{\min_k \rho(a,b_k)} \right).$$

Letting $\varepsilon \to 0$, we obtain

$$\|\mu\| = \sum \alpha_k = \|\mu\|_K.$$

So, for any measure of the type

$$\mu = \sum \alpha_k \bar{e}_{a,b_k}, \ \alpha_k \geq 0$$

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we get $\|\mu\| = \|\mu\|_K = \sum \alpha_k$ (it is easy to check that $\|\mu\|_K = \sum \alpha_k$ for any such measure).

Now consider the general case. Find a point d such that $\rho(d, a_k) \geq \rho(d, b_k) + \rho(a_k, b_k) - \varepsilon$. Then we obtain

$$
\begin{aligned}
\|\mu\| &= \left\| \sum \alpha_k \frac{(\delta_{a_k} - \delta_d) + (\delta_d - \delta_{b_k})}{\rho(a_k, b_k)} \right\| \\
&\geq \left\| \sum \alpha_k \frac{\delta_{a_k} - \delta_d}{\rho(a_k, b_k)} \right\| - \sum \alpha_k \left\| \frac{\delta_d - \delta_{b_k}}{\rho(a_k, b_k)} \right\| \\
&\geq \sum \alpha_k \frac{\rho(a_k, d)}{\rho(a_k, b_k)} - \sum \alpha_k \frac{\rho(d, b_k)}{\rho(a_k, b_k)} \geq \sum \alpha_k + o(1)
\end{aligned}
$$

(in the second inequality we use the case which we have considered at first). This completes the proof in this general case too.

3) $\implies$ 4) and 4) $\implies$ 3).

It suffices to prove that any *extremal* Lipschitz function lies in the weak*-closure of the set distance functions (which consists of extremal Lipschitz functions itself). Take the Lipschitz function $u \in PLip(X)$ which is an extremal on the unit ball of the space. It lies in the weak*-closure of a set L iff any weak*-open neighborhood $W \ni u$ intersects L. But such a neighborhood by definition of weak topology is defined by a finite subset of the space X, so we just get the criterion in formulation 3). $\qquad \square$

Remark that it is possible to give a shorter proof of 1) $\Leftrightarrow$ 3) directly by using the Krein-Milman theorem; but the proof above is elementary and reduces the general case to the case of a finite metric space and did not use infinite constructions. We will compare our criterion with the characteristic property of the Urysohn space in section 3.

Remark. We see that the coincidence of the double-point norm with the Kantorovich norm implies coincidence of all compatible norms. But instead of double-point norm we can choose any norm which was defined in 1.3.1. The proof is essentially the same. So there are many ways to define a norm, compatible with the metric, the coincidence of which with Kantorovich norm implies the linear rigidity.

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2.2 Properties of linearly rigid spaces

2.2.1 Unboundedness of linearly rigid spaces.

The following theorem shows that a linearly rigid space cannot have a finite diameter if it has more than two points.

Theorem 3. *A linearly rigid metric space X containing more than two points is of infinite diameter and, in particular, noncompact.*

Proof. Assume the contrary. Without loss of generality we assume that the space X is complete. Fix a point $a \in X$, denote by r_a the supremum of the distances $\rho(a, x)$ over $x \in X$, and choose a sequence of points (x_n) such that $\rho(a, x_n) \geq r_a - 1/n$. Then pick a countable dense subset $\{y_n\}$ of X, and define a sequence (z_n) by setting $z_{2n} = x_n$ and $z_{2n+1} = y_n$. Consider the points $\bar{e}_{a,z_k}$, $k = 1, \dots, N$. They lie on the same face of the unit ball of the space E_{X_N}, where $X_N = \{a, z_1, z_2, \dots, z_N\}$. Applying Theorem 2, we may find for each N a point c_N such that

$$\rho(a, c_N) \geq \rho(a, z_k) + \rho(z_k, c_N) - 1/N, \qquad k \leq N.$$

In particular, $\rho(a, c_N) \geq \rho(a, x_k) + \rho(x_k, c_N) - 1/N$, $2k \leq N$. Hence $\rho(a, c_N) \to r_a$ and therefore $\rho(x_k, c_N) \to 0$ as $k, N \to \infty$, so that the sequences (x_k), (c_k) are both Cauchy and have a common limit a'. The point a' satisfies the equalities $\rho(a, x) + \rho(x, a') = \rho(a, a')$ for all $x \in X$ (this is why we used the countable dense set $\{y_n\}$ in the definition of our sequence (z_n)).

Such a construction may be done for any point $a \in X$; note that for any $a, b \in X$ such that $a \neq b \neq a'$ (such a, b do exist if X has more than two points) we have $2\rho(a, a') = \rho(a, b) + \rho(a, b') + \rho(a', b) + \rho(a', b') = 2\rho(b, b')$, whence $\rho(a, a') \equiv D < \infty$. It also follows that $\rho(a, b) = \rho(a', b')$.

Without loss of generality, $\rho(a, b') = \rho(b, a') \geq \rho(a, b) = \rho(a', b') = 1$. Let $A = \{a, b, a', b'\}$. Define a function φ by the formulas $\varphi(a) = \varphi(a') = 1$, $\varphi(b) = \varphi(b') = 0$.

Such a function is 1-Lipschitz on $\{a, b, a', b'\}$; the corresponding face contains the points $\bar{e}_{a,b}$, $\bar{e}_{a',b'}$. Hence there exists a point c such that

$$\rho(c, a') \geq \rho(c, b') + 1/2, \qquad \rho(c, a) \geq \rho(c, b) + 1/2$$

We have

$$\rho(a, a') = \rho(a, c) + \rho(c, a') \geq \rho(c, b') + \rho(c, b) + 1 = D + 1.$$

The obtained contradiction proves the theorem. □

2.2.2 How to construct inductively a linearly rigid space

Let (X, ρ) is linearly rigid metric space and $E_{X,\rho}$ corresponding Banach space. The properties of the unit sphere of the space $E_{X,\rho}$ are very peculiar and can be used for the recursive construction of the metric space and corresponding Banach space. We give a draft of the inductive construction; it is based on the following finite-dimensional

Theorem 4 (Piercing theorem). *Let (Y_1, r_1) be an arbitrary finite metric space, $\epsilon > 0$, and Γ be a face of the unit ball of the space E_{Y_1, r_1} (e.g. $V_0(Y_1)$ with compatible norm). Then the space (Y_1, r_1) can be isometrically embedded into a finite metric space (Y_2, r_2) so that there exists a face Δ of the unit ball of the space E_{Y_2, r_2} containing Γ and two vectors $\bar{e}_{z_1, z_2}$ and $\bar{e}_{u_1, u_2}$ such that the line segment connecting them intersects the face Δ at an interior point.*

The proof is direct. Note that if an interior point of a face is of norm one, then all points of the face are also of norm one. Enumerating the sequences of faces of the root polytopes already constructed and "piercing" them by new line segments, we obtain a sequence of finite metric spaces for which all faces of all root polytopes are rigid; hence the completion of the constructed countable space will be linearly rigid. Note that such a degeneracy of the unit sphere is typical for universal constructions (cf. the Poulsen simplex - [15]). We expect the the geometry of the unite sphere of the spaces $E_{X,\rho}$ for linearly rigid metric space (X, ρ) is very unusual and interesting.

3 Examples of linearly rigid spaces and related problems

Up to now we haven't provided any example of linearly rigid space. A trivial example of linearly rigid spaces is one- and two- point spaces. R. Holmes in [9] had discovered that universal Urysohn space has this property: each isometric embedding of it to the universal space $C([0, 1])$ generates as a linear hull the isometric Banach spaces, consequently, in our terminology this means that Urysohn space is linearly rigid, and this was the first nontrivial example of such a space. We will deduce this result as well as other examples as easy consequence of our criterion.

3.1 Criteria of Urysohnness and linear rigidity of the Urysohn space.

In order to prove the linear rigidity of the Urysohn universal space we recall its characterization. The following criteria is a treatment of the different characterization from the original papers of Urysohn [17] and subsequent papers [13, 18, 7].

Theorem 5. *A Polish space (X, ρ) is isometric to the universal Urysohn space if and only if for every $\epsilon > 0$, any every finite subset $F \subset X$, every admissible Lipschitz function u on F is ϵ-representable in the space X. (see Definition 8.4)*

From the point of view of functional analysis we can reformulate this condition in the following words: the set of distance function is weakly* dense in the unit ball of the space $Lip(X)$.

Theorem 6 (R. Holmes [9]). *The Urysohn space $\mathbb{U}$ is linearly rigid.*

Proof. It suffices to compare the assumptions of the criterion of universality above and linear rigidity criterion from the previous section: the assumptions of the latter require additive ϵ-representability of extremal Lipschitz functions, while the universality criterion requires ϵ-representability of all positive Lipschitz functions.

In other words, according to item 4) of Theorem 2, the linear rigidity is equivalent to weak* density of convex hull of the set distance function in unit ball of $PLip(X)$ while the universality is equivalent to the density of the set of distance functions in the in the unit ball of $Lip(X)$, which is a much stronger condition. □

It is natural to call the Banach space $E_{\mathbb{U}}$ (completion of the space $V_0(\mathbb{U})$ with respect to the unique compatible norm) as "Urysohn Banach space"[4]. The geometry of this space seems to be very interesting. First of all $E_{\mathbb{U}}$ as Banach space is universal, which means that each separable Banach space can be linearly isometrically embedded to it. As Professor V.Pestov pointed out this follows from a strong theorem of Godefroy and Kalton [6], which states that if some separable Banach space F has an isometric embedding into a Banach space B, then it also has a linear isometric embedding into B.

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[4]or Urysohn-Holms-Kantorovich Banach space; in the case of coincidences of it with Gurariy space -see Question 2 below - also add name Gurariy.

However, $E_{\mathbb{U},\rho}$ is not a homogeneous universal as Banach space -it is known that there is no separable Banach space, in which each linear isometry between any two finite dimensional isometric linear subspaces can be extended to a global isometry of the space ([14, 8, 15]). The *Gurariy space* ([8]) has ϵ-version of this property.

Question 2. *Is the Urysohn Banach space $E_{\mathbb{U}}$ isometrically isomorphic to the Gurariy space?*

3.2   The further examples

3.2.1   Rational discrete universal metric spaces are linearly rigid

Let us discuss other examples of linearly rigid universal spaces.

Let us consider the countable metric space denoted by $\mathbb{Q}\mathbb{U}_{\geq 1}$. It is a universal and ultra-homogeneous space in the class of countable metric spaces with rational distances not smaller than one. Such a space can be constructed in exactly the same way as the Urysohn space.

Theorem 7. *The space $\mathbb{Q}\mathbb{U}_{\geq 1}$ is linearly rigid.*

Indeed, the assumptions of the criterion of linear rigidity are obviously satisfied.

This example, as well as the next one, is of interest because it is an example of a discrete countable linearly rigid space. Thus the corresponding Banach space $E_{\mathbb{Q}\mathbb{U}_{\geq 1}}$ has a basis. It is not known whether the space $E_{\mathbb{U}}$ has a basis.

Definition 9. *Let us call metric space (X, ρ)almost universal if the set of distance functions is weakly* dense in the unit ball of the space $\mathrm{PLip}(X)$.*

This notion is in between universal Urysohn and linearly rigid spaces but does not coincide with any of them. It is easy to prove

Proposition 2. *The space $\mathbb{Q}\mathbb{U}_{\geq 1}$ is almost universal.*

In the same time integral universal space is linearly rigid (see next item), but not almost universal.

Question 3. *To describe all almost universal metric spaces.*

3.2.2 Integral universal metric space is linearly rigid

The following example is of special interest also for another reason. Consider the space $\mathbb{Z}\mathbb{U}$, the universal and ultra-homogeneous space in the class of metric spaces with integer distances between points. Let us show that it is also linearly rigid. For this, let us check the condition of the criterion of linear rigidity. Fix extremal ray $L = \{\lambda f\}; \lambda > 0$ of the set of Lipschitz functions on the finite subset of $X/$ Note that the differences of the coordinates of every vector from the ray L are integers; hence on this ray there is a vector with integer coordinates, which is realized as the vector of distances from some point $x \in X$ to L.

Let us introduce a graph structure on this countable space by assuming that pairs of points at distance one are neighbors. This graph has remarkable properties: it is universal but not homogeneous (as a graph), its group of isomorphisms coincides with the group of isometries of this space regarded as a metric space. As follows from [3, 4], there exists an isometry that acts transitively on this space.

Question 4. *To characterize this graph using universality of it as the metric space.*

3.3 Weakly linearly rigid spaces

We have proved that coincidence of KR and DP-norms leads to linear rigidity. Now let us compare another two compatible norms: the (KR)-norm and (HK)-norm.

Definition 10. *A metric space for which the maximal (or KR-) and HK-norms coincide is called a weakly linear rigid space (WLR-space).*

If the space (X, ρ) is linearly rigid, then all compatible norms must coincide with the maximal norm of $\sum c_k \delta_{x_k}$, which is

$$\| \sum c_k \delta_{x_k} \|_{HK} = \| \sum c_k \delta_{x_k} \|_{KR} = \sup_{F \in \mathrm{Lip}_1(X)} \sum c_k F(x_k),$$

where the supremum is taken by all 1-Lipschitz functions on X (or, equivalently, on the set of $\{x_k\}$). The equality above means that every extremal Lipschitz function is almost realized for a function $\pm\rho(x, \cdot)$ for some $x \in X$.

The criterion of linear rigidity says that any extremal Lipschitz function on X is almost realized as a function $\rho(x, \cdot)$. The only difference is the absence of $\pm$. Recall that if we take a supremum not by the set of functions $\pm\rho(x, \cdot)$, but by the set of functions $\frac{1}{2}(\rho(x, \cdot) - \rho(y, \cdot))$, then the coincidence of the corresponding norm with the maximal (KR) norm implies linear rigidity. But this difference is quite essential. There are some spaces which are weakly linearly rigid but not linearly rigid. We do not have any less or more complete description of such spaces. So only some examples follow.

1. Any metric space on three points gives a WLR-space (but it is never linearly rigid, as any finite metric space on more than two points, see theorem 3).

2. Let us define a family of WLR-spaces on four points. Let points be A, B, C, D; let a, b, c be arbitrary positive numbers and define: $\rho(D, A) = a$, $\rho(D, B) = b$, $\rho(D, c) = c$, $\rho(A, B) = a+b$, $\rho(B, C) = b+c$, $\rho(A, C) = a+c$. It is easy to see that any extremal Lipschitz function is realized as $\pm\rho(X, \cdot)$ for some $X \in \{A, B, C, D\}$.

Question 5. *Does there exist the finite WLR metric space with more than four points?*

It looks more likely that there are no such spaces at least with sufficiently many points.

3. In the same time there are an infinite WLR-space which is not be linearly rigid either. Consider the Urysohn space $\mathbb{U}$ and fix a point $a \in \mathbb{U}$. Add a point a' to the space $\mathbb{U}$ and define the distances $\rho(a', x) = \rho(a, x) + 1$ for any $x \in X$ (in particular, $\rho(a, a') = 1$). This new space $U' = \mathbb{U} \cup \{a'\}$ is WLR, but not linearly rigid.

3.4 Extremality and the properties of the Banach spaces $E_{X,\rho}$

The set of all possible distance matrices (semimetrics) is a convex weakly (i.e. in pointwise topology) closed cone (see [18]). If the distance matrix of a Polish space X, which corresponds to some dense sequence in X, belongs to extremal ray of this cone, we say that X is *extremal*. This property does not depend on the choice of dense sequence, so the definition is correct. This notion is interesting even for finite metric spaces (see.[2]).

Question 6. *To describe extremal finite metric spaces with n points, to estimate exact number of such spaces or asymptotics on n.*

Note that the universal real Urysohn space $\mathbb{U}$ *is extremal* metric space ([18]). It follows from the genericity of $\mathbb{U}$ that the distance matrices of everywhere dense systems of points of extremal metric spaces form an everywhere dense G_δ-set in the space of distance matrices. The integer space $\mathbb{ZU}$ is also extremal; the extremality of both spaces follows from a result of Avis [2], which states that every finite metric space with commensurable distances (i.e. such that the ratio of any two distances is rational number) can be embedded into a finite extremal metric space, and hence the assumptions of the criterion of linear rigidity are satisfied. Using the criterion of linear rigidity given above, and the procedure of the previous section one can build an example of a not extremal linearly rigid metric space.

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