THE BEHAVIOR OF THE LAPLACE TRANSFORM OF THE INVARIANT MEASURE ON THE HYPERSPHERE OF HIGH DIMENSION

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To my friend Dima Arnold

Abstract

We consider the sequence of the hyperspheres $M_{n,r}$, i.e., the homogeneous transitive spaces of the Cartan subgroup $SDiag(n, \mathbb{R})$ of the group $SL(n, \mathbb{R})$, $n = 1, 2, \dots$, and study the normalized limit of the corresponding sequence of invariant measures m_n on these spaces. In the case of compact groups and homogeneous spaces, e.g., for the classical pairs $(SO(n), S^{n-1})$, $n = 1, 2, \dots$, the limit of the corresponding measures is the classical infinite-dimensional Gaussian measure: this is the well-known Maxwell–Poincaré lemma. Simultaneously, this Gaussian measure is a unique (up to a scalar) invariant measure with respect to the action of the infinite orthogonal group $O(\infty)$. This coincidence means the asymptotic equivalence between the grand and small canonical ensembles for the series of pairs $(SO(n), S^{n-1})$. Our main result shows that for noncompact groups, e.g., for the case $(SDiag(n, \mathbb{R}), M_{n,r})$ (see definitions below), the situation is completely different: the limit of the measures m_n does not exist in the literal sense, and we show that only the normalized logarithmic limit of the Laplace transform of these measures does exist. At the same time, there exists a measure invariant with respect to the continuous analog of the Cartan subgroup of the group $GL(\infty)$: this is the so-called infinite-dimensional Lebesgue measure (see [1]). This difference is an evidence of the nonequivalence between the grand and small canonical ensembles for the noncompact case.

1 Setting of the problem

Consider the series of classical Lie groups G_n and the corresponding homogeneous spaces M_n equipped with the G_n-invariant measures m_n. Does there exist a natural (weak) limit of such sequences of measures, as a measure defined in some infinite-dimensional space?

There are at least two specifications of the question. First, we can try to find a measure that is invariant under the action of an infinite-dimensional analog of our groups. The second approach is to find a direct limit of the finite-dimensional measures. These two approaches coincide for the classical case of the orthogonal group $SO(n)$ and the spheres S^{n-1} as homogeneous spaces: in both cases, we obtain the standard Gaussian measure. In the first approach, it is obtained as the unique (up to a scalar) ergodic measure invariant under the infinite-dimensional orthogonal group $SO(\infty)$ (the Schoenberg theorem, see [1, 2]); and in the second approach, it appears from the Maxwell–Poincaré lemma (MP-lemma) on the weak limit of the sequence of normalized Lebesgue measures on the $(n-1)$-dimensional spheres of radius $c\sqrt{n}$ (for a detailed discussion of this case, see [1]).

We consider a generalization of this scheme for noncompact groups. The first nontrivial example is given by the case of the diagonal (Cartan) subgroup $SDiag(n, \mathbb{R})$ of the group $SL(n\mathbb{R})$. More exactly, in the paper [1] we considered the group $SDiag_+(n, \mathbb{R})$ of positive diagonal matrices with determinant 1 and a homogeneous space of this group, the manifold

$$M_{n,r} = \{\{y_k\} : \prod_{k=1}^{n} y_k = r^n, \quad y_k > 0;\, k = 1, \dots, n\}$$

sometimes called the "hypersphere of radius r," as a homogeneous transitive space (here $r = r_n$ depends on n). One of the infinite-dimensional analogs of the group $SL(n\mathbb{R})$ is the group of all bounded linear operators on the space of Schwartz distributions on a manifold T. Then the continuous analog of the diagonal (Cartan) group can be defined as the group of multiplicators $M_a\xi(\cdot) = a(\cdot)\xi(\cdot)$, where ξ is a Schwartz distribution on the manifold T and a runs over tame functions on T. Consider the multiplicative abelian group $\mathcal{M}$ of *positive functions* $a(\cdot)$ satisfying the condition

$$\int_{t\in T} \ln a(t)dt = 0;$$

this condition is the logarithmic analog of the condition $\det A = 1$ on diagonal matrices.[1] Thus $\mathcal{M}$ is the continuous analog of the group $SDiag_+(n, \mathbb{R})$. It was proved in [4] that there exists a one-parameter family $\mathcal{L}_\theta$, $\theta > 0$, of $\mathcal{M}$-invariant measures on the Schwartz space $S(T)$, i.e., the space of Schwartz

[1] It is useful to consider also the wider group $\mathcal{M} = \{a : \int_T \ln a(t)dt < \infty\}$, which is the analog of the group $GDiag_+(n, \mathbb{R})$.

distributions on the manifold T; this family is unique up to a scalar; the measure $\mathcal{L}_1$ is the infinite-dimensional Lebesgue measure.

The Laplace transform of the measures $\mathcal{L}_\theta$, $\theta > 0$, is the functional

$$\Psi_\theta(f) \equiv \int_{\xi \in S(T)} e^{-<f,\xi>} d\mathcal{L}_\theta(\xi) = \exp\{-\theta \int \ln f(t)dt\},$$

where f is a positive tame function on the manifold T (see [1]). Note that we can regard the above formulas for the Laplace transform as a definition of the measures $\mathcal{L}_\theta$; a constructive definition and a detailed list of properties of the infinite-dimensional Lebesgue measure $\mathcal{L}_1$ can be found in [1]. Note that the measure $\mathcal{L}_1$ (as well as $\mathcal{L}_\theta$) can be regarded as the law of the "infinitely divisible process with σ-finite measure" generated by the semigroup of σ-finite measures on $[0, \infty)$ with density

$$\frac{x^{\theta-1}}{\Gamma(\theta)}, \quad \theta > 0.$$

This explains the above formula for the Laplace transform of the measure $\mathcal{L}_1$.

In this paper we will directly calculate the asymptotics of the Laplace transform of the invariant measures on the manifolds $M_{n,r}$ and see that it gives another answer than in the first method: the limit of the Laplace transforms of the measures m_n does not coincide with the Laplace transform of the measures $\mathcal{L}_\theta$. Thus we have, in a sense, the nonequivalence of two ensembles: the Lebesgue measure $\mathcal{L}_1$ is a natural measure on the grand canonical ensemble and has the Laplace transform written above; on the other hand, the limit of the measures on the small ensembles does not literally exist. Nevertheless, it is interesting to understand the behavior of the asymptotics of the Laplace transform of the invariant measure on the hypersphere, the function L, which has very intriguing properties. We briefly discuss this subject at the end of the paper.

2 The Laplace transform of the invariant measures on the hyperspheres $M_{n,r}$

We want to find the asymptotic properties of the invariant measure on the positive Cartan subgroup $SDiag_+(n, \mathbb{R})$ (the group of positive diagonal real matrices) of the group $SL(n,\mathbb{R})$ as n tends to infinity. More exactly, we want to find the *Laplace transform $D_n(\cdot)$ of the invariant σ-finite measure on the hypersphere*

$$M_{n,r} = \{\{y_k\} : y_k > 0,\ k = 1,\dots,n; \prod_{k=1}^{n} y_k = r^n\}$$

(in general, the radius r of the hypersphere depends on n). We use the Laplace transform instead of the Fourier transform applied in the case of the MP-lemma, because the measure m_n is not finite but σ-finite. The group $SDiag_+$ acts transitively and freely on this hypersphere. So we want to investigate the following integral over the invariant measure $m_n(\cdot)$ on $M_{n,r}$:

$$D_n(f) = \int_{M_{n,r}} \exp\{-\sum_{k=1}^{n} f_k \cdot y_k\} dm_n(y).$$

By definition, the functional $D_n(f)$, regarded as a function of a vector $f = (f_1,\dots,f_n)$, *is the Laplace transform* of the measure m_n on the hypersphere $M_{n,r}$. The measure $m_n(\cdot)$ is the Haar measure on the group $SDiag_+$. Since the Haar measure is defined up to a constant, we can choose a normalization later, but here the choice of the coordinates y is fixed by the normalization of the Haar measure. Our goal is to find the asymptotic properties of the measure $m_n(\cdot)$; for this we use the Laplace transform.

In this section we will calculate the asymptotics of the above integral. Let us change the variables: $y_k \mapsto \frac{\rho_n(f) y_k}{f_k}$, where $\rho_n(f) = (\prod_{k=1}^{n} f_k)^{\frac{1}{n}}$. Then we obtain

$$D_n(f) = \int_{M_{n,r}} \exp\{-\rho_n(f) \sum_{k=1}^{n} y_k\} dm_n(y).$$

Let $y_k = e^{x_k}$, $k = 1,\dots,n$. Then we have

$$D_n(f) = \int_{P_n} \exp\{-\rho_n(f) \sum_{k=1}^{n} \exp x_k\} \prod_k dx_k,$$

where $P_n = \{(x_1,\dots,x_n) : \sum_k x_k = n \ln r_n\}$.

Finally, letting $x_k \mapsto x_k - \ln r_n$, we obtain the following expression:

$$D_n(f) = \int_{\mathbb{R}^n} \exp\{-\rho_n(f) r_n \sum_{k=1}^{n} e^{x_k}\} \delta_0(\sum_{k=1}^{n} x_k) \prod_{k=1}^{n} dx_k$$

$$= \int_{H_n} \exp\{-\rho_n(f) r_n \sum_{k=1}^{n} e^{x_k}\} dx,$$

where the integration is over the hyperplane

$$H_n = \{(x_1, \dots, x_n) : \sum_k x_k = 0\}$$

and $\rho_n(f) = (\prod_{k=1}^n f_k)^{\frac{1}{n}}$.

Note that, keeping in the mind the continuous limit of the whole picture, we can regard the vector $f = (f_1, \dots, f_n)$ as the values of a function $f(\cdot)$ at some points t_k of the domain: $f_k = f(t_k)$, $k = 1, \dots, n$. Then the functional $\rho(f)$ turns into the functional $\exp \int \ln f(t) dm(t)$, which is defined on the class of positive functions f of a continuous variable t with $\int \ln f(t) dt < \infty$ (for details, see [1]). In any case, the functional D_n depends only on the geometrical average $\rho(f)$ of the coordinates of the vector f.

Now recall the definition of the function F_n on the positive half-line $\mathbb{R}_+$ which is sometimes called the Mellin–Barnes function (see, e.g., [7]):

$$F_n(\lambda) = \int_{H_n} \exp\{-\lambda \sum_{k=1}^{n} \exp x_k\} dx,$$

where $H_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{k=1}^n x_k = 0\}$.

We have

$$D_n(f) = F(\rho_n(f) r_n),$$

and this formula *reduces our problem to the calculation of the asymptotics of the function* $F_n(\cdot)$ as n tends to infinity. The role of the multiplier ρ will be discussed later. In some sense, we obtain an asymptotic problem about a function of one variable ("radius") instead of a function of many variables; the explanation lies in the high symmetry of the initial problem under the group $SDiag(n, \mathbb{R}_+)$. We observed the same effect in the problem about the asymptotics of the uniform measures on the spheres, which also reduces to the asymptotics of the Bessel functions $J_n(\cdot)$ of one variable (see [2, 1]). In the case under consideration, the function F_n plays the same role as the Bessel function in the compact case. But, as we will see, the answer here will be very different.

As is well-known, the function F_n satisfies the differential equation

$$(1 + \lambda \frac{d}{d\lambda})^{n-1} \frac{dF_n}{d\lambda} = F_n(\lambda)$$

and is related to the hypergeometric functions. We will not use these properties of F_n.

3 Calculation of the asymptotics of the function F_n

First of all, we will find the integral

$$\int_0^\infty F_n(e^{-\frac{t}{n}}) \cdot e^{-ts} dt$$

$$= \int (n) \int e^{-e^{-\frac{t}{n}} \cdot [e^{x_1} + e^{x_2} + \cdots + e^{x_n}]} \cdot e^{-ts} dx_1 \ldots dx_{n-1} dt$$

$$= \int (n) \int \exp[-\sum_{k=1}^{n} e^{(x_k - \frac{t}{n})}] \cdot e^{-ts} dx_1 \ldots dx_{n-1} dt,$$

where the integration is over

$$H_n \times \mathbb{R} = \{(x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1} : \sum_{k=1}^{n} x_k = 0,\, t > 0\}.$$

Let us make the change of variables $(x_1, \ldots, x_{n-1}, t) \mapsto (x_1, \ldots, x_n)$, where $x_n = t - x_1 - \cdots - x_{n-1}$, so that $t = \sum_{k=1}^n x_k$, and let

$$y_k = \exp(x_k - \frac{t}{n}), \quad k = 1, \ldots, n.$$

The integral under consideration equals

$$\int \cdots (n) \cdots \int e^{-\sum_{k=1}^n y_k} \prod_{k=1}^{n} (y_k)^s \frac{dy_k}{y_k}$$

$$= \prod_{k=1}^{n} [\int_0^\infty e^{-y_k} y_k^{s-1}] = \Gamma(s)^n.$$

Thus

$$\int_0^\infty F_n(e^{-\frac{t}{n}}) \cdot e^{-ts} dt = \Gamma(s)^n.$$

Now make the change of variables $e^{-\frac{t}{n}} = \lambda$; then we have

$$\int_0^\infty nF_n(\lambda)\lambda^{ns-1}d\lambda = \Gamma(s)^n.$$

So the function $F_n(\cdot)$ is the inverse Mellin transform of the nth power $\Gamma(s)^n$ *of the gamma function* (up to the multiplier n, which we will use later). Thus the functions $F_n(\lambda)$ and $\Gamma(s)^n$ form a so-called "Mellin pair."

For example, for $n = 1$ the Mellin pair is the pair of functions $(\exp s, \Gamma(s))$.

Now, in order to obtain an expression for the function $F_n(\cdot)$ and to find its asymptotics, we need to use the standard calculation of the contour integral

$$F_n(\lambda) = \frac{1}{2\pi in}\int_{\gamma-i\infty}^{\gamma+i\infty}[\Gamma(s)]^n\lambda^{-ns}ds$$

and apply the saddle point method. These are the Laplace transforms of a sequence of measures on the orbit of $S_+Diag(n,\mathbb{R})$. We want to study the asymptotics of this sequence as $n \to \infty$.

Let us rewrite the last formula in the "real" form and put $s = \gamma + it$:

$$F_n(\lambda) = \frac{1}{2\pi in}\int_{-\infty}^{+\infty}[\Gamma(\gamma+it)]^n\lambda^{-n(\gamma+it)}dt.$$

Now we are ready to apply the saddle point method to the calculation of the asymptotics of the function F_n and, first of all, to find a saddle point.

4 Saddle point method for F_n

It turns out that a suitable saddle point is the point $(\gamma, 0) \in \mathbb{C}$ where γ is the root of the following equation:

$$\ln\lambda = \frac{\Gamma'(\gamma)}{\Gamma(\gamma)},$$

or

$$\lambda = \exp\left\{\frac{\Gamma'(\gamma)}{\Gamma(\gamma)}\right\}.$$

Recall that the fraction

$$\frac{\Gamma'(\gamma)}{\Gamma(\gamma)},$$

or the logarithmic derivative of the gamma function, is the well-known Euler's ψ-function, and its properties are well-known (see, e.g., [6, 5]).

In particular, the ψ-function is strictly monotonic on the half-line $(0, \infty)$ and maps it to the whole line $(-\infty, +\infty)$, so for all $\lambda > 0$ there exists a value $\gamma = \gamma(\lambda)$ that satisfies the equation above. Thus $\gamma = \gamma(\lambda)$, regarded as a function of λ, is the inverse to the ψ-function. We will consider this function later.

Rewrite our integral in the form

$$F_n(\lambda) = \frac{1}{2\pi i} \frac{\Gamma(\gamma)^n}{\lambda^{n\gamma}} \int_{-\infty}^{+\infty} \left\{ \frac{\Gamma(\gamma + it)}{\Gamma(\gamma)} \right\}^n \cdot \lambda^{int} dt.$$

Now expand the fraction in the integrand into a power series in t at a neighborhood of the point γ:

$$\frac{\Gamma(\gamma + it)}{\Gamma(\gamma)} \cdot \lambda^{-it}$$

$$= (1 + \frac{\Gamma'(\gamma)}{\Gamma(\gamma)} it - -\frac{\Gamma''(\gamma)}{\Gamma(\gamma)} \cdot t^2/2 + \ldots)(1 - it \ln \lambda - t^2/2(\ln \lambda)^2 + \ldots).$$

Our choice of γ leads to the vanishing of the coefficient of the first power of t, so that we have

$$= 1 - \frac{t^2}{2} \left\{ \frac{\Gamma''(\gamma)}{\Gamma(\gamma)} - (\frac{\Gamma'(\gamma)}{\Gamma(\gamma)})^2 \right\} + \ldots.$$

Denote

$$\sigma = \frac{\Gamma''(\gamma)\Gamma(\gamma) - \Gamma'(\gamma)^2}{\Gamma(\gamma)^2}.$$

Then

$$F_n(\lambda) = \frac{1}{2\pi n} \left\{ \frac{\Gamma(\gamma)}{\lambda^\gamma} \right\}^n \int_{-\infty}^{+\infty} \left\{ 1 - \frac{\sigma t^2}{2} + o(t^2) \right\}^n dt,$$

and, applying the arguments usual for the saddle point method (check that $(\gamma, 0)$ is indeed a saddle point with respect to the imaginary and real axes), we obtain the asymptotics

$$F_n(\lambda) \approx \frac{1}{2\pi n} \left(\frac{\Gamma(\gamma)}{\lambda^\gamma} \right)^n \int_{-\infty}^{+\infty} \exp\{-\frac{\sigma n t^2}{2}\} dt = \frac{L(\lambda)^n}{\sqrt{2\pi n \sigma^{-1}}},$$

where

$$L(\lambda) = \frac{\Gamma(\gamma)}{\lambda^\gamma}, \quad \lambda = \exp\left\{ \frac{\Gamma'(\gamma)}{\Gamma(\gamma)} \right\}.$$

5 The function L and its behavior at zero and at infinity

Let us begin with the following theorem-definition:

Theorem 1. *Define a function L by the formula*

$$L(\lambda) \equiv \lim_n \frac{\ln F_n(\lambda)}{n}.$$

Then

$$L(\lambda) = \frac{\Gamma(\gamma(\lambda))}{\lambda^{\gamma}},$$

where γ and λ satisfy the equation

$$\lambda = \exp\left\{\frac{\Gamma'(\gamma)}{\Gamma(\gamma)}\right\}.$$

It seems that the function L is a very interesting object; it looks like the free energy in statistical mechanics; the author does not know if it has been studied in the literature. This function plays a role analogous to the role of generators in the theory of semigroups under the convolution of measures, or, more exactly, a role analogous to the role of the Fourier transform of the generator.

Now we can look at the behavior of the function L, which is the main goal of our calculations.

According to Theorem 1, the function L can be regarded as a function not only of λ, but also of γ:

$$L(\lambda) = \frac{\Gamma(\gamma(\lambda))}{\lambda^{\gamma(\lambda)}}.$$

Consequently, first of all we need to clarify the interrelation between the variables λ and γ given by the above equation for γ, which can be rewritten as

$$\lambda = \exp\psi(\gamma).$$

Theorem 2. *The asymptotic correspondence between the variables λ and γ at infinity is*

$$\lambda = \gamma + o(1);$$

and the asymptotic correspondence between λ and γ at zero is

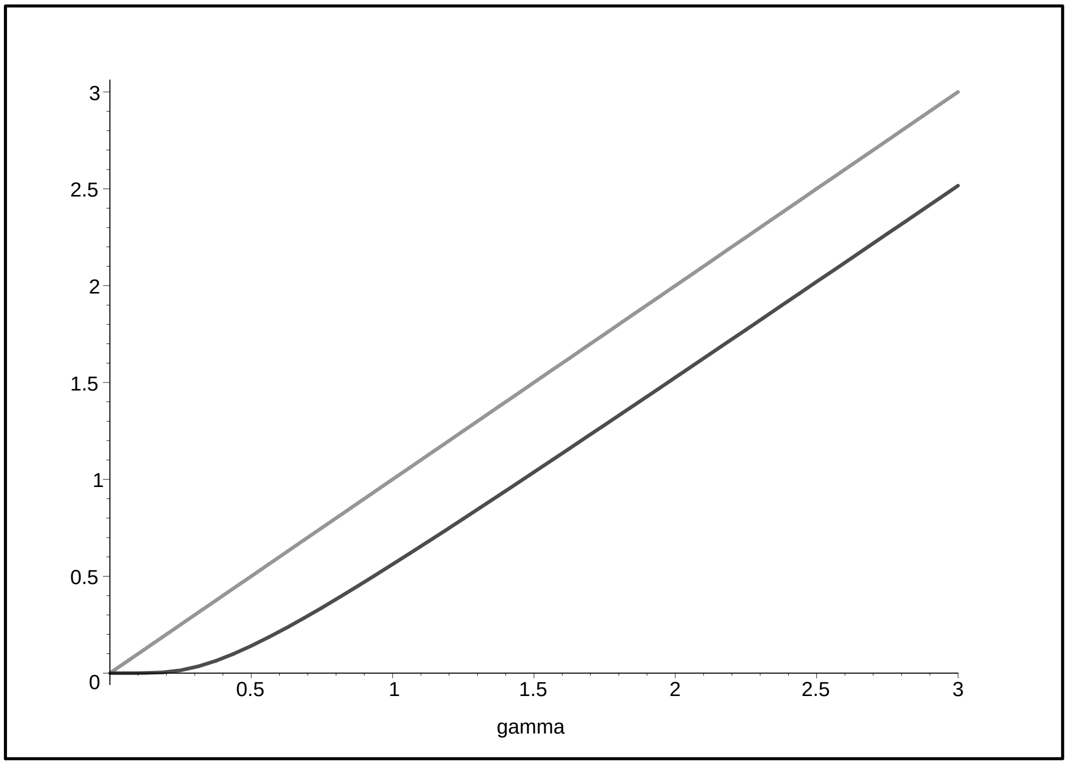
$$\lambda(\gamma) = e^{\boldsymbol{C} - \frac{1}{\gamma}}(1 + o(1)),$$

or, inversely,

$$\gamma \equiv \gamma(\lambda) = \frac{1}{|\ln \lambda| + \boldsymbol{C} + o(\lambda)}, \quad \lambda \sim 0;$$

here $\boldsymbol{C} = 0.577...$ *is Euler's constant.*

For the graph of the function $\lambda = \lambda(\gamma)$, see the figure below.



Proof. We will use the following relations for the ψ-function (see [5, 8.365]):

$$\lim_{z\to\infty}(\psi(z + y) - \ln z) = 0, \quad \psi(z + 1) = \psi(z) + \frac{1}{z}.$$

It follows that

$$\psi(z) = \ln z + 1/z + o(1/z) \quad \text{as} \quad z \to \infty.$$

Since $\lambda = \exp\psi(\gamma)$, we have

$$\lambda = \exp\{\ln\gamma + o(1)\} = k_\gamma \cdot \gamma = \gamma + o(\gamma^{-1});$$

here $k_\gamma \to 1$ as $\gamma \to \infty$.

Now we use the second relation at a neighborhood of 0; then for $z \sim 0$ we have

$$\psi(z) = \psi(1+z) - \frac{1}{z} = \mathbf{C} + o(z) - \frac{1}{z}.$$

Consequently, λ as a function of γ satisfies

$$\lambda \equiv \lambda(\gamma) = \exp\{\mathbf{C}\} \cdot \exp\{-\frac{1}{\gamma}\}(1 + O(\gamma)).$$

From this it is easy to obtain an expression for γ as a function of λ, which was done in the formulation of the theorem. $\square$

As we have seen, the asymptotic correspondences between λ and γ at zero and at infinity are very different: at infinity, they are almost equal, while the correspondence at a neighborhood of zero is very nontrivial: all jets of λ as a function of γ vanish.

We consider the behavior of the function L at zero and at infinity, and we are also interested in the value 1. One can verify that L monotonically decreases on $(0, \infty)$ from ∞ at zero to 0 at infinity.

As to the asymptotics at 0, our formulas and the properties of the functions $\Gamma(\cdot)$ and $\psi(\cdot)$ imply

Theorem 3. *The behavior of the function L at zero is described by the formulas*

$$\exp L(\lambda) = \frac{\boldsymbol{C}}{\lambda}(1 + o(\lambda)),$$

or

$$L(\lambda) = \ln\boldsymbol{C} - \ln\lambda + o(\lambda),$$

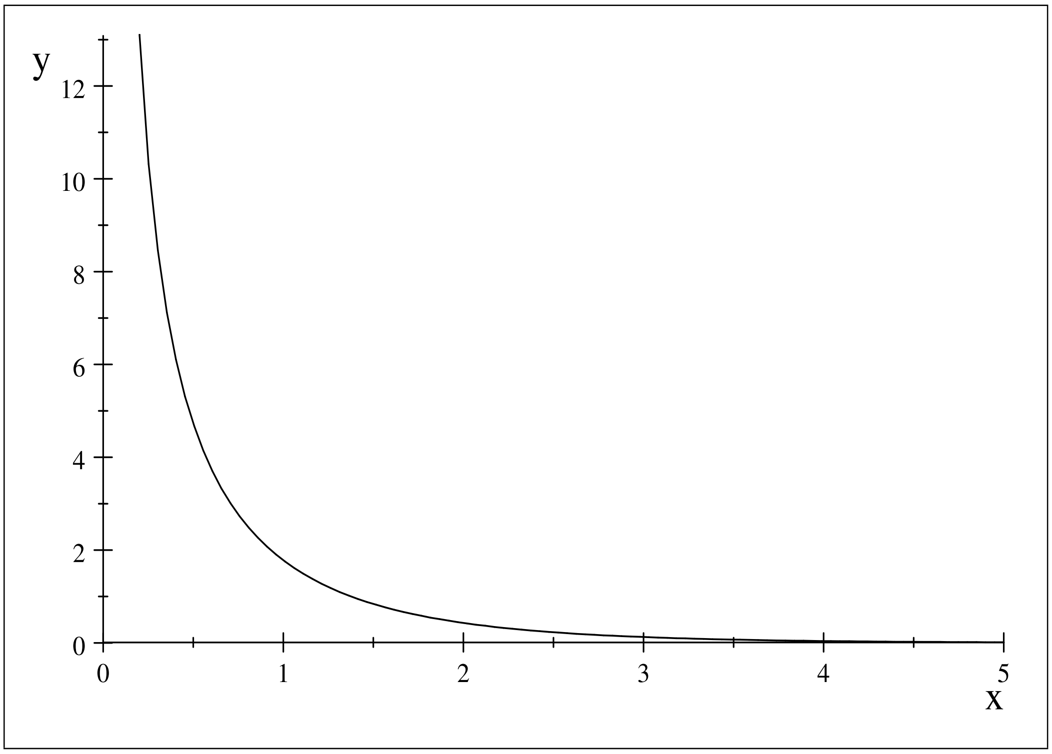
and

$$\lim_{\lambda\to 0} \lambda \cdot e^{L(\lambda)} = \boldsymbol{C} > 0.$$

The behavior of the function L at infinity is described by the formula

$$L(\lambda) \cong \exp\{-\lambda\}, \quad \lambda \gg 0.$$

For the graph of the function $L = L(\lambda)$, see the figure below.



Proof. In order to find the asymptotics of $L(\cdot)$ at infinity, we apply the Stirling formula for the gamma function, $\Gamma(z) = z^{z-\frac{1}{2}}e^{-z}\sqrt{(}2\pi)(1+o(1)$, to the formula for L, and use the relation $\lambda = \gamma + o(\lambda)$ at infinity. Then we have

$$L(\lambda) = \frac{\Gamma(\gamma(\lambda))}{\lambda^{\gamma(\lambda)}} \approx \frac{\Gamma(\lambda)}{\lambda^{\lambda}} \approx \frac{\sqrt{2\pi}e^{-\lambda}}{\sqrt{\lambda}}.$$

The asymptotic formula for the function L at zero follows from the fact that the gamma function has a simple pole at zero with residue 1 and Theorem 3:

$$L(\lambda) = -\ln\lambda + \mathbf{C} + o(\lambda).$$

In order to find the value 1 of the function L, it is more convenient to consider the function $\bar{L}(\gamma) = L(\lambda(\gamma))$. Then

$$\bar{L}(\gamma) = \frac{\Gamma(\gamma)}{\lambda(\gamma)^{\gamma}}.$$

Denote by γ_{cr} the value of γ for which $(\bar{L})(\gamma_{cr}) = 1$ (the critical value). Using the expression for $\lambda = \exp\psi(\gamma)$, we see that if $\bar{L} = 1$ (or $\gamma = \gamma_{cr}$), then

$$\Gamma(\gamma_{cr}) = \exp\gamma_{cr}\psi(\gamma_{cr}).$$

In the logarithmic scale, we have the following equation for this critical value:

$$\Gamma(\gamma_{cr})\ln\Gamma_{cr}(\gamma_{cr}) - \gamma_{cr}\Gamma'(\gamma_{cr}) = 0.$$

It turns out that there is a unique such γ_{cr}, and approximately

$$\gamma_{cr} \sim 1,38\ldots;$$

the corresponding value of λ (recall that $\lambda = \exp\{\frac{\Gamma'(\gamma)}{\Gamma(\gamma)}\}$) is

$$\lambda_{cr} \sim 0,95\ldots.$$

$\square$

Now we can summarize the obtained information on the function L, in order to find the asymptotics of the functions $F_n(\cdot)$ in n. The behavior of L in a neighborhood of the critical point plays a special role.

Theorem 4. *For each $\epsilon > 0$, the values of the function $F_n(\lambda)$ exponentially tend to infinity for all positive $\lambda < \lambda_{cr} - \epsilon$, and exponentially tend to zero for $\lambda > \lambda_{cr} + \epsilon$ as n tends to infinity; there exists a sequence $\{\lambda_n\}$ that tends to $\lambda_c r$ such that*

$$\lim_n F_n(\lambda_n) = 1.$$

6 Discussion

We have calculated the asymptotic behavior of the Laplace transforms $D_n(f)$ of the invariant measures m_n on the hyperspheres M_{n,r_n}, which is determined by the asymptotics of the function F_n. The answer is given in terms of the distinguished function L, and this analysis shows that there is no weak convergence of these measures m_n (in the sense of the convergence of Laplace transforms), and, consequently, we cannot say that it coincides with the Laplace transform of the infinite-dimensional Lebesgue measure $\mathcal{L}_1$.

Recall that the formula for the Laplace transform of the measure m_n,

$$D_n(f) = F(\rho_n(f)r_n),$$

involves the multiplier $\rho_n(f)$ which is a characteristic of the normalization of the hypersphere M_{n,r_n}. The choice of this parameter has a geometric meaning: this is the growth of the radius r_n of the hypersphere M_{n,r_n}.

We can conclude our analysis with the following words: *it is impossible to choose a normalization of the radius r_n of the hyperspheres that would lead to a nondegenerate limit distribution of the invariant measures on the hyperspheres.* For comparison, in the compact case (Euclidean spheres) we can choose the normalized radius equal to $c\sqrt{n}$ (the MP-lemma).

We can say that there is no equivalence between the grand canonical and microcanonical ensembles in the situation where the latter is not compact. We will discuss this phenomenon elsewhere.

At the same time, the analysis of the function L itself is very important, because it shows that the behavior of the Laplace transforms F_n in n asymptotically looks like that of the Laplace transforms of the powers (in the sense of convolution) of some distribution. The main question is: whether or not the function L is the generator of some semigroup of infinitely divisible distributions on the half-line.

In conclusion, we can say that there is still no suitable theory of divisibility for the Laplace transforms of σ-finite distributions, as well as no Lévy-Khintchine type formula for generators, and so on (at least, to the author's knowledge of the subject).

Our result also shows that although the groups $SDiag(n, \mathbb{R})$ in a natural sense tend (as n tends to infinity) to the group $\mathcal{M}$ of all symmetries of the Lebesgue measure $\mathcal{L}_1$, it may happen that the individual ergodic theorem (or the law of large numbers) for the increasing sequence of groups $\{SDiag(n, \mathbb{R}), n = 1, 2, \dots\}$, is not valid.

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