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Integral models of representations of the current groups of simple Lie groups

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Abstract. For the class of locally compact groups P that can be written as the semidirect product of a locally compact subgroup P_0 and a one-parameter group $\mathbb{R}_+^*$ of automorphisms of P_0, a new model of representations of the current groups P^X is constructed. The construction is applied to the maximal parabolic subgroups of all simple groups of rank 1. In the case of the groups $G = \mathrm{SO}(n,1)$ and $G = \mathrm{SU}(n,1)$, an extension is constructed of representations of the current groups of their maximal parabolic subgroups to representations of the current groups G^X. The key role in the construction is played by a certain σ-finite measure (the infinite-dimensional Lebesgue measure) in the space of distributions.

Keywords: current group, integral model, Fock representation, canonical representation, special representation, infinite-dimensional Lebesgue measure

Contents

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1. Introduction

1.1. The construction of an invariant multiplicative integral of representations, that is, an irreducible representation of the group $L^\infty(X;G) \equiv G^X$ of currents, bounded measurable functions on a measure space (X,m) with values in a semisimple Lie group G, was described in the early 1970s in [1] and [2]. Later it turned out that this construction can be embedded in a general scheme described several years earlier by Araki in terms of the Fock space (see [3]). However, as the author of [3] himself observes, this scheme was applied only to solvable and nilpotent Lie groups, and semisimple groups were not considered. Formally, the question is about a non-commutative analog of infinitely divisible measures, that is, semigroups of states on groups, and an analogue of the Lévy–Khinchin formulae, but of a very special form. The key point is to find non-trivial cohomologies of the group with values in an appropriate unitary representation. The construction suggested in [1] of an integral of representations for the group $\mathrm{SL}(2,\mathbb{R})$ implicitly contained such a cocycle. An explicit description of the cohomology for $\mathrm{SL}(2,\mathbb{R})$ and other simple Lie groups is given in [4].

The existence of an irreducible unitary representation π of G with $H^1(G;\pi) \neq 0$ is a sufficient (and in Araki's scheme, that is, for the Fock factorization, also necessary) condition for the existence of a multiplicative integral of representations. In turn, this condition means that the group G must not satisfy the Kazhdan property (T) [5] (see the book [6] and the bibliography therein), that is, the trivial representation must not be isolated in the space of unitary representations with the Fell topology. Indeed, as proved in [7], a representation π with non-zero group $H^1(G;\pi)$ must be 'glued' to the trivial representation (in the terminology of [7], it must be infinitesimal). Among the classical simple Lie groups G, only $\mathrm{SO}(n,1)$ with $n>1$ and $\mathrm{SU}(n,1)$ with $n \geqslant 1$ have such representations, and only these groups admit an invariant multiplicative integral of representations in the Fock model.

An analysis of the original papers [1], [2], [8]–[10] showed that there is an alternative approach to the description of Fock representations. At first it appeared as a mere result of the diagonalization of the representations considered in [1] for the group $\mathrm{SL}(2,\mathbb{R})$ with respect to the unipotent subgroup [11], [12]. A necessary consequence of this diagonalization was the definition of a remarkable σ-finite measure in the space of discrete measures on X. However, the true essence and the depth of the alternative description of the multiplicative integral of representations became clear only after a study in [13]–[16] of the general case of the groups $\mathrm{SO}(n,1)$ with $n>1$ and $\mathrm{SU}(n,1)$ with $n \geqslant 1$. In the present paper we summarize the results obtained in this series of papers; we regard it as a preparatory step towards a monograph devoted to representations of current groups. In contrast to the Fock model, the alternative model, which for certain reasons was called the *integral* model, essentially uses specific properties of simple groups of rank 1 (more precisely, of their maximal parabolic subgroups) and the invariance of a certain σ-finite measure with respect to the continual Cartan group. That is why it is not as general as the Fock model, and the isomorphism between the integral and Fock models is very involved. But first, it allows one to give a much simpler proof of the irreducibility and other properties of the representation, and second, it leads to a new explicit interpretation of the notion of continuous tensor product, which undoubtedly will be useful in the future.

1.2. A brief introduction to the Fock and the integral models.

1.2.1. Let us first describe the original (Fock) model in full generality. It is based on the construction of the exponential of a Hilbert space, which is a formalization, on the one hand, of the Fock space, and on the other hand, of the L^2 space with respect to a Gaussian measure (the Wiener–Itô space).

The exponential of a Hilbert space H is defined via the decomposition

$$\mathscr{H} \equiv \operatorname{EXP} H = \mathbb{C} \oplus H \oplus \frac{1}{\sqrt{2!}} S^2 H \oplus \frac{1}{\sqrt{3!}} S^2 H \oplus \cdots ,$$

where $S^n H$ is the nth symmetric tensor power of H, and one defines a map $\exp\colon H \to \operatorname{EXP} H$ by

$$\exp h = 1 \oplus h \oplus \frac{1}{\sqrt{2!}} h \otimes h \oplus \frac{1}{\sqrt{3!}} h \otimes h \otimes h \oplus \cdots \in \mathscr{H} \quad \text{for } h \in H.$$

The following relations hold:

$$\operatorname{EXP}(H_1 \oplus H_2) = \operatorname{EXP} H_1 \otimes \operatorname{EXP} H_2,$$
$$\langle \exp h_1, \exp h_2 \rangle = e^{(h_1,h_2)}, \qquad \exp(h_1 + h_2) = \exp h_1 \otimes \exp h_2.$$

The exponentials $\exp h$ form a total set in $\mathscr{H}$, that is, their linear span is dense in $\mathscr{H}$. Using these exponentials, one defines the whole structure of the Fock space: the decomposition into multi-particle subspaces, the creation and annihilation operators, and so on. Unitary operators that act in the space $\mathscr{H}$ and preserve its structure are said to be factorized. They are parameterized by triples (A, b, c), where A is a unitary operator on H, $b \in H$, and $c \in \mathbb{C}$ with $|c| = 1$, and they form the group $\mathscr{A} = \{(A, b, c) : A \in \operatorname{Unit}(H),\ b \in H,\ c \in \mathbb{C},\ |c| = 1\}$ with the multiplication law

$$(A_1, b_1, c_1)\,(A_2, b_2, c_2) = \bigl(A_1A_2, b_1 + A_1b_2, c_1c_2 \exp(i \operatorname{Im} \langle b_1, A_1b_2\rangle)\bigr).$$

The action of this group on exponentials is defined as follows:

$$(A, b, c)(\exp h) = c \exp(i \operatorname{Im}\langle b, h\rangle) \cdot \exp(Ah + b).$$

Thus, the group $\mathscr{A}$, which is sometimes called the Bogolyubov group, is a central extension of the group of isometric motions of the space $\mathscr{H}$ (that is, the semidirect product of the group $\operatorname{Unit}(H)$ and the group of translations P).

It is easy to see that a representation of the current group $G^X = L^\infty(X, G)$ determines a factorization in the representation space. If we assume that this is a Fock factorization, then the representation can be factored through the group of factorized operators (the Bogolyubov group). Under the assumption that the representation is invariant under the group of all measure-preserving transformations of the space X, we see immediately that this representation is parameterized by a unitary representation π of the group G itself on H and by a 1-cocycle $\beta\colon G \to H$ of G with values in H. For the representation of G^X to be irreducible it is sufficient that the representation π be irreducible and that the cocycle β not be cohomologous to zero (see [2], [12], [17], [18]). This general scheme does not use any specific features of the groups under consideration. The problem reduces to finding appropriate pairs π, β for a given group or proving that such pairs do not exist. It is this fact that leads to the above-mentioned answer for the simple Lie groups, since among these groups special representations with non-trivial $H^1(G;\pi)$ exist only for $\mathrm{SO}(n,1)$ with $n>1$ and $\mathrm{SU}(n,1)$ with $n\geqslant 1$.

For convenience the groups $\mathrm{SO}(n,1)$ *and* $\mathrm{SU}(n,1)$ *are replaced by their extensions* $\mathrm{O}(n,1)$ *and* $\mathrm{U}(n,1)$ *in all constructions.*

Note that the special representations of the groups $\mathrm{SO}(n,1)$ *and* $\mathrm{SU}(n,1)$, *as well as of their extensions* $\mathrm{O}(n,1)$ *and* $\mathrm{U}(n,1)$, *are trivial on their centres, so that they reduce to representations of the projectivizations of these groups. Accordingly, the Fock representations of the current groups* G^X *reduce to representations of the projectivizations of* G^X.

1.2.2. The integral model of representations of current groups, which we study in what follows for the groups $G=\mathrm{O}(n,1)$ with $n>1$ and $G=\mathrm{U}(n,1)$ with $n\geqslant 1$, is essentially based on the structure of the groups G. We do not use the existence of representations of the current groups and do not consider their cohomology groups, but on the contrary obtain all this information along the way. The following two fundamental facts about the groups $G=\mathrm{O}(n,1)$ with $n>1$ and $\mathrm{U}(n,1)$ with $n\geqslant 1$ are of importance for us.[1]

(A) The irreducible special representations of these groups remain irreducible when restricted to the maximal parabolic subgroup.

(B) The maximal parabolic subgroup P of each of these groups is the semidirect product of the multiplicative group $\mathbb{R}_+$ and a certain subgroup P_0 having a one-parameter family of irreducible unitary representations T_r, $r>0$, on which there is a transitive action of the group $\mathbb{R}_+$ of automorphisms; the family T_r, $r>0$, is a deformation of the trivial representation (corresponding to $r=0$).

The first fact reduces the problem of constructing a representation of the current group G^X to that of constructing a representation of the current group P^X, where P is the maximal parabolic subgroup of G. The latter problem can in turn be solved due to the second fact and, principally, to the existence of a remarkable σ-finite measure in the space of distributions, about which we will say several words at the end of the Introduction. The aim of our paper is to describe this solution in detail. The authors hope to give a more detailed treatment of the whole circle of problems related to representations of current groups in the book which is now under preparation.

The measure, which in [19]–[21] was called the infinite-dimensional Lebesgue measure and which is denoted by $\mathscr{L}$ in what follows, appeared in [11], [12] as the measure whose Laplace transform occurred naturally in the diagonalization of a representation of the group of $\mathrm{SL}(2,\mathbb{R})$-currents. Later, it was discovered that this measure, as well as the one-parameter family $\mathscr{L}_\theta$, $\theta>0$, in which it

[1] All the techniques used in what follows apply also to the two infinite-dimensional groups $\mathrm{O}(\infty,1)$ and $\mathrm{U}(\infty,1)$, and this enables us to construct the desired representation of the corresponding current groups.

is included (for $\theta = 1$), has a number of remarkable relations and properties. It is related to the Poisson–Dirichlet measure, the gamma process, and the Lévy processes corresponding to stable laws. The fundamental property of this measure is its invariance with respect to the continual analog of the Cartan group and its uniqueness under some natural assumptions; for more detail, see [21]. See also [22] for a detailed study of the quasi-invariance of the Lévy measure of the gamma process and its equivalence to $\mathscr{L}_\theta$.

In conclusion, we would like to note that up to now the possibilities of using other factorizations have not been studied at all from the viewpoint of constructing a multiplicative integral of representations. We mean, first, non-Fock type I factorizations whose existence was proved in [23] and, second, factorizations of types II and III. The latter appear in projective representations of current groups on the circle, that is, in Kac–Moody modules. These representations are totally different from those described above; they essentially rely on the positivity of the energy, the one-dimensionality of the base X, and the projectivity. It may well happen that bringing in non-Fock factorizations will extend the class of groups for which a multiplicative integral does exist, as well as the class of representations that can be obtained in this way. One should also remember that there may also exist non-unitary analogues of this theory, which also have not been studied.

1.3. Let us briefly describe the contents of the paper. In § 2 we describe the infinite-dimensional Lebesgue measure $\mathscr{L}$, which is the basis for the construction of integral models of representations of the groups G^X, and we compute some integrals with respect to this measure.

§ § 3 and 4 are devoted to constructing and studying integral models of representations of the current groups P^X for the class of locally compact groups P that can be written in the form $P = \mathbb{R}_+^* \leftthreetimes P_0$. This class includes, in particular, the maximal parabolic subgroups of simple Lie groups of rank 1, that is, of the groups $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$, and $\mathrm{Sp}(n, 1)$.

The elements $r \in \mathbb{R}_+^*$ induce the automorphisms $g \mapsto g^r$ of the subgroup P_0 and thus assign to every representation T of P_0 a one-parameter family of representations that act as $T_r(g) = T(g^r)$. We consider representations T of P_0 satisfying the following condition: the space H of T contains a vector h of norm $\|h\| = 1$ such that the estimate

$$\|T_r(g)h - h\| < c(g)\, r \quad \text{for every } g \in P_0$$

holds for sufficiently small r. It follows from this condition that the family of representations T_r is a deformation of the identity representation of P_0, that is, it tends to the identity representation in the Fell topology as $r \to 0$. We call such a representation canonical. We observe that this notion of a canonical representation is stronger than that introduced in other papers (see, for example, [1], [12], [24]).

The direct integral $\widetilde{T}$ of the representations T_r with respect to the multiplicative measure $d^*r = r^{-1}\, dr$, that is,

$$\int_0^\infty T_r\, d^*r,$$

can be naturally extended to a representation $\widetilde{T}$ of the whole group $P=\mathbb{R}_+^*\leftthreetimes P_0$: $(\widetilde{T}(r_0)f)(r)=f(r_0r)$ for $r_0\in\mathbb{R}_+^*$. The representation obtained has a non-trivial 1-cocycle.

The construction of the integral model of representation of the group P^X is similar to the construction of a representation $\widetilde{T}$ of the group P from a representation T of the group P_0. In this construction, P_0 is replaced by the current group P_0^X, and the representations T_r of P_0 are replaced by the representations $T_\xi(g(\cdot))=\bigotimes_{k=1}^\infty T_{r_k}(x_k)$ of P_0^X on countable tensor powers of the space H. Here ξ runs over the points of the cone

$$l_+^1(X)=\biggl\{\xi=\sum_{k=1}^\infty r_k\delta_{x_k}\biggm| r_k>0,\ \sum_k r_k<\infty,\ x_k\in X\biggr\},$$

on which the infinite-dimensional Lebesgue measure $\mathscr{L}$ is concentrated. To obtain the desired representation of the current group P^X we consider the direct integral of these representations of P_0^X with respect to the measure $\mathscr{L}$ and, using the properties of this measure, construct an extension of this representation of P_0^X to a representation of P^X. The representation of P^X thus obtained will be called the *integral model* and denoted by $\operatorname{INT}T$.

We prove that the representations obtained in this way are irreducible, and we establish their relation to Fock representations.

The subsequent sections are devoted to the integral models of representations of the current groups P^X, where P is the maximal parabolic subgroup of the Lie group $\mathrm{O}(n,1)$, $\mathrm{U}(n,1)$, or $\mathrm{Sp}(n,1)$ (§§ 5, 7, and 8, respectively). The case

$$P\subset\mathrm{SL}(2,\mathbb{R})\cong\mathrm{SU}(1,1)$$

is treated in a separate section (§ 6). These groups P can be written as the semidirect products $P=\mathbb{R}_+^*\leftthreetimes P_0$, and the subgroups P_0 have canonical representations. In each of these cases we give a description of the canonical representations of P_0 and thus, according to the general construction, a description of the corresponding integral models of representations of the current groups P^X.

The main problem here is to get representations of the current groups $\mathrm{O}(n,1)^X$ and $\mathrm{U}(n,1)^X$ as extensions of the integral models of representations $\operatorname{INT}T$ of the corresponding current groups P^X. To this end we consider canonical representations T of P_0 such that the associated special representations of P can be extended to representations of the corresponding simple Lie group. For them we explicitly construct an extension of the integral model $\operatorname{INT}T$ to a representation of the current group of the corresponding simple Lie group. The models obtained are compared with the Fock models of representations of these groups constructed in [12], [25]. Simultaneously, this construction leads to new models of the special representations of the groups $\mathrm{O}(n,1)$ and $\mathrm{U}(n,1)$, models which are of independent interest.

2. The measure $\mathscr{L}$ in the space of distributions

2.1. The definition of the measure $\mathscr{L}$. We consider an arbitrary manifold X with a fixed continuous non-negative finite Borel measure m. The construction of the integral models of representations of the current groups G^X is based on the existence, in the space $D(X)$ of Schwartz distributions on X, of a certain measure $\mathscr{L}$ which is an infinite-dimensional analogue of the Lebesgue measure. This measure appeared in [11], [12] and was investigated in the series of papers [19]–[21]. Here we only give the definition of $\mathscr{L}$ and present its main properties used in what follows.

With each finite partition of X into measurable sets,

$$\alpha: \quad X = \bigcup_{k=1}^{n} X_k, \qquad m(X_k) = \lambda_k, \quad k = 1, \dots, n,$$

we associate the cone $\mathscr{F}_\alpha = \mathbb{R}_+^n$ of piecewise constant positive functions of the form

$$f(x) = \sum_{k=1}^{n} f_k \chi_k(x), \qquad f_k > 0,$$

where χ_k is the characteristic function of X_k, and we denote by $\Phi_\alpha = (\mathbb{R}_+^n)'$ the dual cone in the space of distributions.

We define a measure $\mathscr{L}_\alpha$ on Φ_α by

$$d\mathscr{L}_\alpha(x_1, \dots, x_n) = \prod_{k=1}^{n} \frac{x_k^{\lambda_k - 1}\, dx_k}{\gamma(\lambda_k)}, \quad \text{where} \quad \lambda_k = m(X_k). \tag{2.1}$$

Let $D_+(X) \subset D(X)$ be the set (cone) of non-negative Schwartz distributions on X, and let $l_+^1(X) \subset D_+(X)$ be the subset (cone) of discrete finite (non-negative) measures on X, that is,

$$l_+^1(X) = \left\{ \xi = \sum_{k=1}^{\infty} r_k \delta_{x_k} \;\middle|\; r_k > 0,\ \sum_k r_k < \infty,\ x_k \in X \right\}.$$

There is a natural projection $D_+(X) \to \Phi_\alpha$.

Theorem-definition. *There is a σ-finite (infinite) measure $\mathscr{L}$ on the cone $D_+(X)$ that is finite on compact sets, concentrated on the cone $l_+^1(X)$, and such that for every partition α of the space X its projection on the subspace Φ_a has the form (2.1).*

This measure is uniquely determined by its Laplace transform

$$F(f) \equiv \int_{l_+^1(X)} \exp\Big(-\sum_k r_k f(x_k)\Big)\, d\mathscr{L}(\xi) = \exp\Big(-\int_X \log f(x)\, dm(x)\Big), \tag{2.2}$$

where f is an arbitrary non-negative measurable function on (X, m) which satisfies $\int_X \log f(x)\, dm(x) < \infty$.

Elements of $l_+^1(X)$ will be briefly denoted by $\xi = \{r_k, x_k\}_{k=1}^{\infty}$, or even just $\xi = \{r_k, x_k\}$ (sequences that differ only by the order of elements are regarded as identical).

Remark. The Laplace transform is well defined for a wide class of σ-finite measures $\mathscr{L}$ for which there are sufficiently many linear functionals with non-infinite distribution, and there is a uniqueness theorem for measures with a given Laplace transform.

As in the case of the classical Laplace transform, the formula (2.2) for the characteristic functional of the measure $\mathscr{L}$ makes sense for every complex-valued function $f(x)$ with positive real part and with $\int_X \log f(x)\,dm(x) < \infty$. Hereafter, log stands for the branch of the logarithm with $\log 1 = 0$ on the complex plane cut along the negative real axis.

The following characteristic definition of the measure $\mathscr{L}$ is important for our purposes. Let $A(X)$ be the group (with respect to multiplication) of all non-negative measurable functions $a(x)$ on X with convergent integral $\int_X \log a(x)\,dm(x) = c$, and let $A_0(X)$ be the subgroup of functions $a(x)$ with $c = 0$. With each function $a \in A(x)$ we associate the operator M_a that multiplies elements $\xi = \sum_k r_k \delta_{x_k}$ of the cone $l^1_+(X)$ by $a(x)$:

$$M_a \sum_k r_k \delta_{x_k} = \sum_k a(x_k) r_k \delta_{x_k}.$$

Theorem 2.1. *The measure $\mathscr{L}$ on the cone $l^1_+(X)$ is uniquely determined by the following two properties.*

1) *Projective invariance with respect to the group $\mathscr{M}$ of multipliers M_a: for every function $a \in A(X)$ the operator M_a multiplies the measure $\mathscr{L}$ by $\exp c$, that is,*

$$d\mathscr{L}(a(\cdot)\xi) = \exp\biggl(\int_X \log a(x)\,dm(x)\biggr)\,d\mathscr{L}(\xi). \tag{2.3}$$

In particular, the measure $\mathscr{L}$ is invariant with respect to the subgroup $\mathscr{M}_0$ of multipliers M_a with $a \in A_0(X)$.

2) *Invariance and ergodicity with respect to the group of all measure-preserving transformations of (X, m).*[2]

The fact that $\mathscr{L}$ satisfies these properties follows from the formula (2.2) for its Laplace transform. For the uniqueness, see [20], [21].

It follows from Theorem 2.1 that the measures $\mathscr{L}$ thus defined depend only on the one parameter $\theta = m(X)$, and that under convolution they form a multiplicative semigroup with respect to $\theta > 0$.

In the construction of integral models it suffices to consider only one of these measures, so in what follows we assume that $\theta = 1$, that is, m is a probability measure. It is natural to call $\mathscr{L}$ the infinite-dimensional Lebesgue measure, since it generalizes the invariance properties of finite-dimensional Lebesgue measure on the non-negative octant. The novelty of the infinite-dimensional case is that $\mathscr{L}$ is ergodic with respect to the group of multipliers. We will formulate this fact separately.

Theorem 2.2. *The group $\mathscr{M}$ of multipliers acts ergodically on the cone $l^1_+(X)$ equipped with the measure $\mathscr{L}$.*[3]

[2] As shown in [20], property 2) is a corollary of 1).

[3] In the finite-dimensional case this group is SDiag_+ (the positive part of the Cartan group), and it acts transitively on each of the hyperspheres $x_1 x_2 \cdots x_n = \mathrm{const}$, $x_i > 0$, $i = 1, 2, \dots, n$, that is, the action of SDiag_+ on the cone $\mathbb{R}^n_+$ is not ergodic.

2.2. Computation of some integrals with respect to the measure $\mathscr{L}$. Let us apply the properties of the measure $\mathscr{L}$ to computing the integral

$$I = \int_{l^1_+(X)} \left(\prod_{k=1}^{\infty} \varphi(r_k, x_k) \right) d\mathscr{L}(\xi), \tag{2.4}$$

where $\varphi(r, x)$ is a function on $\mathbb{R}^*_+ \times X$ satisfying the conditions

$$\varphi(0, x) \equiv 1 \quad \text{and} \quad \int_X \int_0^\infty \bigl(\varphi(r, x) - e^{-r}\bigr) r^{-1}\, dr\, dm(x) < \infty. \tag{2.5}$$

Theorem 2.3. *The following equality holds*:

$$\int_{l^1_+(X)} \left(\prod_{k=1}^{\infty} \varphi(r_k, x_k) \right) d\mathscr{L}(\xi) = \exp\left(\int_X \int_0^\infty \bigl(\varphi(r, x) - e^{-r}\bigr) r^{-1}\, dr\, dm(x) \right). \tag{2.6}$$

Proof. Under the projection $D_+(X) \to \Phi_\alpha$ (recall that Φ_α is the finite-dimensional space associated with a partition $\alpha : X = \bigcup_{k=1}^n X_k$) the left-hand side of (2.4) takes the form

$$I_\alpha = \prod_{k=1}^{n} I_\alpha^k, \qquad I_\alpha^k = \frac{1}{\Gamma(\lambda_k)} \int_0^\infty \varphi_{\alpha,k}(r_k) r_k^{\lambda_k - 1}\, dr_k, \tag{2.7}$$

where $\lambda_k = m(X_k)$ and $\varphi_{\alpha,k}(r_k) = \lambda_k^{-1} \displaystyle\int_{X_k} \varphi(r_k, x)\, dm(x)$. The original integral I is the inductive limit of the integrals I_α over the set of partitions α.

Since $\displaystyle\frac{1}{\Gamma(\lambda_k)} \int_0^\infty e^{-t} r_k^{\lambda_k - 1} dr_k = 1$, the integral I_α^k can be written in the form

$$I_\alpha^k = 1 + \frac{1}{\Gamma(\lambda_k)} \int_0^\infty \bigl(\varphi_{\alpha,k}(r_k) - e^{-r_k}\bigr) r_k^{\lambda_k - 1}\, dr_k.$$

It follows that

$$I_\alpha^k = 1 + \lambda_k \int_0^\infty \bigl(\varphi_{\alpha,k}(r_k) - e^{-r_k}\bigr) r_k^{-1}\, dr_k + O(\lambda_k^2),$$

whence

$$I_\alpha^k = \exp\left(\lambda_k \int_0^\infty \bigl(\varphi_{\alpha,k}(r_k) - e^{-r_k}\bigr) r_k^{-1}\, dr_k \right) + O(\lambda_k^2).$$

Thus, up to terms of order greater than 1 with respect to λ_k,

$$I_\alpha \cong \exp\left(\sum_{k=1}^{n} \lambda_k \int_0^\infty \bigl(\varphi_{\alpha,k}(r) - e^{-r}\bigr) r^{-1}\, dr \right).$$

Since $\sum_{k=1}^n \bigl(\lambda_k (\varphi_{\alpha,k}(r) - e^{-r})\bigr) = \int_X \bigl(\varphi(r, x) - e^{-r}\bigr)\, dm(x)$, the expression obtained can be written in the form

$$I_\alpha \cong \exp\left(\int_X \int_0^\infty \bigl(\varphi(r, x) - e^{-r}\bigr) r^{-1}\, dr\, dm(x) \right).$$

The proof is completed by taking the inductive limit over the set of partitions α.[4]

Corollary. *If* $\varphi(r,x)=\sum_{i=1}^n c_i\varphi_i(r,x)$, *where* $c_i>0$, $\sum c_i=1$, *and the functions* $\varphi_i(r,x)$ *satisfy* (2.5), *then*

$$\int_{l^1_+(X)}\biggl(\prod_{k=1}^\infty \varphi(r_k,x_k)\biggr)\,d\mathscr{L}(\xi)=\prod_{i=1}^n \exp\biggl(c_i\int_X\int_0^\infty \bigl(\varphi_i(r,x)-e^{-r}\bigr)r^{-1}\,dr\,dm(x)\biggr). \tag{2.8}$$

Example. Let $\varphi(r,x)=e^{-r^\sigma a(x)}$, where $\sigma\geqslant 1$ and $\operatorname{Re} a(x)>0$. In this case we obtain

$$\int_{l^1_+(X)}\biggl(\prod_{k=1}^\infty e^{-r_k^\sigma a(x_k)}\biggr)\,d\mathscr{L}(\xi)=\exp\biggl(\int_X\int_0^\infty (e^{-r^\sigma a(x)}-e^{-r})r^{-1}\,dr\,dm(x)\biggr).$$

Let us integrate with respect to r. We have

$$\begin{aligned}\int_0^\infty (e^{-r^\sigma a(x)}-e^{-r})r^{-1}\,dr&=\lim_{\lambda\to 0}\biggl(\int_0^\infty (e^{-r^\sigma a(x)}-e^{-r})r^{\lambda-1}\,dr\biggr)\\&=\lim_{\lambda\to 0}\biggl(\sigma^{-1}\Gamma\Bigl(\frac{\lambda}{\sigma}\Bigr)a^{-\lambda/\sigma}(x)-\Gamma(\lambda)\biggr).\end{aligned}$$

Since $\Gamma(\lambda)\sim\lambda^{-1}+\gamma$ as $\lambda\to 0$, where γ is the Euler constant, it follows that

$$\int_0^\infty \bigl(\exp(-r^\sigma a(x))-\exp(-r)\bigr)r^{-1}\,dr=-\sigma^{-1}\log a(x)+(\sigma^{-1}-1)\gamma.$$

Hence,

$$\begin{aligned}&\int_{l^1_+(X)}\biggl(\prod_{k=1}^\infty \exp(-r_k^\sigma a(x_k))\biggr)\,d\mathscr{L}(\xi)\\&\qquad=\exp\bigl((\sigma^{-1}-1)\gamma\bigr)\exp\biggl(-\sigma^{-1}\int_X \log a(x)\,dm(x)\biggr).\end{aligned}\tag{2.9}$$

In particular, for $\sigma=1$ we recover the original formula for the Laplace transform of the measure $\mathscr{L}$:

$$\int_{l^1_+(X)}\prod_{k=1}^\infty \exp\Bigl(-\sum r_k a(x_k)\Bigr)\,d\mathscr{L}(\xi)=\exp\biggl(-\int_X \log a(x)\,dm(x)\biggr).$$

3. The canonical representations of the group P_0 and the associated representations of the current group P_0^X

3.1. The definition of canonical representations. We consider the semidirect products $P=A\leftthreetimes P_0$ of a locally compact group P_0 and the multiplicative group $A\cong\mathbb{R}_+^*$ of automorphisms of P_0. Denote by g^a the image of an element $g\in P_0$ under an automorphism $a\in A$.

Definition 1.

[4] Since the measure $\mathscr{L}$ is absolutely continuous with respect to the measure generated by the Lévy gamma process (see [22]), this result can be obtained by similar computations with Lévy processes.

Unless otherwise stated, a representation is understood in what follows to be an orthogonal or unitary representation of a group.

There is a natural action of the group A of automorphisms on the set of all representations of P_0, which sends a representation $T(g)$ to $T_a(g) = T(g^a)$.

We say that a representation T of the group P_0 on a Hilbert space H is *canonical* if there exists a cyclic vector $h \in H$ of norm $\|h\| = 1$ and an isomorphism $\sigma\colon \mathbb{R}_+^* \to A$ such that

1) the inequality

$$\|T_{\sigma(r)}(g)h - h\| < c(g)r \quad \text{for every } g \in P_0 \tag{3.1}$$

holds for sufficiently small r;

2) the representations $T_a(g) = T(g^a)$ of P_0 are pairwise non-equivalent.

We say that a cyclic vector $h \in H$ satisfying (3.1) is *almost invariant* with respect to T and that the representations $T_a(g) = T(g^a)$ of P_0 are conjugate to T.

In what follows we identify elements $a \in A$ with their pre-images $r \in \mathbb{R}_+^*$ under the isomorphism σ and write g^r and T_r instead of $g^{\sigma(r)}$ and $T_{\sigma(r)}$. Thus, the condition (3.1) takes the form

$$\|T_r(g)h - h\| < c(g)r \quad \text{for every } g \in P_0. \tag{3.2}$$

It follows from Definition 1 that the representations T_r form a deformation of the identity representation of the group P_0, that is, every neighbourhood of the identity representation in the Fell topology on the set of representations contains all the T_r for sufficiently small r.

The definition of a canonical representation also implies the following assertion.

Proposition 3.1. *If a representation T of the group P_0 on a space H is canonical, then for every summable numerical sequence $\{r_k\}$, $r_k > 0$ $\left(\sum r_k < \infty\right)$ and for every $g \in P_0$,*

$$\sum_{k=1}^{\infty} \|T_{r_k}(g)h - h\| < \infty, \tag{3.3}$$

where $h \in H$ is a vector almost invariant with respect to T.

We note that in the space H of a canonical representation T an almost invariant vector h may be not unique; an example (a one-dimensional extension of the Heisenberg group) will be considered below.

The following assertion will be useful for us.

Proposition 3.2. *If in the space H of a representation T of the group P_0 there is a unique, up to a factor, unit cyclic vector h satisfying the condition 1) of Definition 1, then the representation T is canonical, that is, it satisfies also the condition 2) of this definition.*

Proof. Assume that the condition 2) is not satisfied, that is, there exist two equivalent representations T_r, say $T = T_1$ and T_{r_0} with $r_0 < 1$. Hence there exists a unitary operator A such that $A^{-1}T(g)A = T_{r_0}(g)$ for every $g \in P_0$. Then we have

$$A^{-n}T_r(g)A^n = T_{r_0^n r}(g) \quad \text{for any } g \in P_0,\ r > 0, \text{ and } n = 1, 2, \dots .$$

Therefore, if $h \in H$ is a unit cyclic vector satisfying 1), then this condition implies that

$$\|T_r(g)A^n h - A^n h\| < c(g) r_0^n r \quad \text{for every } g \in P_0. \tag{3.4}$$

In view of this estimate, all the vectors $h_n = A^n h$ also satisfy 1). Note also that $Ah \neq ch$. Otherwise we could assume that $Ah = h$, and the estimate obtained would then imply that

$$\|T_r(g)h - h\| < c(g) r_0^n r \quad \text{for every } g \in P_0 \text{ and } n = 1, 2, \ldots,$$

that is, h is an invariant vector, a contradiction to the assumption that it is cyclic. Thus, a vector h satisfying 1) is not unique, a contradiction.

Note. In a series of earlier papers (see, for example, [1], [12], [24]) canonical representations were understood to be a one-parameter family of representations T_r, $r > 0$, with spherical functions of the form $\varphi_r(g) = e^{r\psi(g)}$.

A weaker condition for a family of representations T_r to be canonical is the existence of the derivative of the spherical functions of this family as $r \to 0$,

$$\psi(g) = \left.\frac{d\varphi_r(g)}{dr}\right|_{r=0},$$

which is a conditionally positive-definite function (the generator of the system). This condition is satisfied, for instance, for the family of complementary series representations of the groups $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$. The existence of a generator in this case follows from the estimate $\|T_r(g)h - h\|^2 < c(g)r$, which is weaker than (3.2) and (3.3). Various approaches to the notion of a canonical representation will be discussed elsewhere.[5]

3.2. The representations of the group $l^\infty(P_0)$ and the current group P_0^X associated with canonical representations of the subgroup P_0. We denote by $l^\infty(P_0)$ the group of all infinite bounded sequences $g = \{g_1, g_2, \ldots\}$ of elements in P_0, with coordinatewise multiplication. We will associate with each canonical representation T of P_0 a family of representations of $l^\infty(P_0)$.

To this end we use the following well-known definition.

Definition 2. The countable tensor power of a Hilbert space H with stabilizing unit vector h is the completion of the inductive limit of the finite tensor powers $\bigotimes_{i=1}^n H$ of H under the isometric embeddings $\bigotimes_{i=1}^n H \ni f \mapsto f \otimes h \in \bigotimes_{i=1}^{n+1} H$. We will denote this limit by $\widetilde{H} = \bigotimes_{i=1}^\infty (H, h)$ or, in short, $\bigotimes_{i=1}^\infty H$.

Thus, $\widetilde{H}$ is the Hilbert space gotten by completing the space $\lim\operatorname{ind}_{n\to\infty} \bigotimes_{i=1}^n H$ with respect to the norm.

It is natural to write elements $\bigotimes_{k=1}^n f_k$ in the subspaces $\bigotimes_{k=1}^n H \subset \widetilde{H}$ as infinite products stabilizing at the $(n+1)$th step:

$$y_n = f_1 \otimes f_2 \otimes \cdots \otimes f_n \otimes h \otimes h \otimes \cdots, \quad \text{where } f_k \in H; \tag{3.5}$$

they form a total subset of $\widetilde{H}$.

[5] Canonical representations in a closely related sense have also been considered in other papers (see, for example, [26]).

Lemma. *If $\sum_{n=1}^{\infty} \|f_n - h\| < \infty$, then the sequence $\{y_n\}$ of the form (3.5) converges in the norm of the space $\widetilde{H}$.*

The limits of such sequences $\{y_n\}$ will be written as infinite products $y = \bigotimes_{n=1}^{\infty} f_n$, where $\sum_{n=1}^{\infty} \|f_n - h\| < \infty$. In what follows, when describing the space $\widetilde{H}$, we will confine ourselves only to such elements and their finite linear combinations.

Definition 3. With each canonical representation T of the group P_0 on a space H with an almost invariant vector $h \in H$ and each sequence $\{r_k\}$, $r_k > 0$, such that $\sum r_k < \infty$, we associate the following representation of the group $l^\infty(P_0)$ on the space $\bigotimes_{k=1}^{\infty}(H, h)$:

$$\widetilde{T}_{\{r_n\}}(g)\biggl(\bigotimes_{n=1}^{\infty} f_n\biggr) = \bigotimes_{n=1}^{\infty}\bigl(T_{r_n}(g_n) f_n\bigr) \quad \text{for } g = \{g_1, g_2, \dots\}.$$

Let us check that this representation is well defined, that is, that the condition $\sum_{n=1}^{\infty} \|f_n - h\| < \infty$ implies that $\sum_{n=1}^{\infty} \|T_{r_n}(g_n) f_n - h\| < \infty$ for every $g = \{g_1, g_2, \dots\} \in l^\infty(P_0)$. Indeed,

$$\|T_{r_n}(g_n) f_n - h\| \leqslant \|f_n - h\| + \|T_{r_n}(g_n) h - h\|.$$

Since the representation T is canonical and the sequence $\{g_n\}$ is bounded, it follows that $\|T_{r_n}(g_n) h - h\| \leqslant c r_n$ for every $g \in l^\infty(P_0)$. Hence the condition $\sum r_k < \infty$ implies that $\sum_{n=1}^{\infty} \|T_{r_n}(g_n) h - h\| < \infty$. The assertion follows.

Proposition 3.3. *If a canonical representation T of the group P_0 is irreducible, then the associated representations $T_{\{r_n\}}$ of the group $l^\infty(P_0)$ are irreducible and pairwise non-equivalent.*

Indeed, the irreducibility of the representation $T_{\{r_n\}}$ of $l^\infty(P_0)$ follows at once from the irreducibility of the representation T of P_0. The pairwise non-equivalence of the $T_{\{r_n\}}$ follows from the pairwise non-equivalence of the representations T_r of P_0 conjugate to T.

Starting from the representations $T_{\{r_n\}}$ of $l^\infty(P_0)$, we will now construct representations of the current group P_0^X. For this, we associate with each $\xi = \{r_k, x_k\} \in l^1(X)$ a homomorphism $P_0^X \to l^\infty(P_0)$:

$$\sigma_\xi \colon g(\cdot) \mapsto \bigl(g(x_1), g(x_2), \dots\bigr).$$

Thus, for each element $\xi = \{r_k, x_k\} \in l^1(X)$ we have a representation T_ξ of P_0^X factored through this homomorphism. It acts in the countable tensor product $H_\xi = \bigotimes_{k=1}^{\infty} H_{r_k}$, $H_{r_k} = H$. The operators of T_ξ, $\xi = \{r_k, x_k\}$, are given on the space H_ξ by

$$T_\xi(g(\cdot))\biggl(\bigotimes_{k=1}^{\infty} f_k\biggr) = \bigotimes_{k=1}^{\infty}\bigl(T_{r_k}(g(x_k)) f_k\bigr).$$

Note that the representation T_ξ, $\xi = \{r_n, x_n\}$, of the group P_0^X is the countable tensor product of local representations of P_0^X:

$$T_\xi = \bigotimes_{n=1}^{\infty} T_{r_n, x_n}, \quad \text{where } T_{r_n, x_n}(g(\cdot)) = T_{r_n}(g(x_n)).$$

Proposition 3.4. *If a canonical representation T of the group P_0 is irreducible, then the representations T_ξ of P_0^X are irreducible and pairwise non-equivalent.*

Proof. It suffices to check that the representations $T_{r,x}$ and $T_{r',x'}$ of P_0^X are not equivalent for $(r,x) \neq (r',x')$. For $r \neq r'$ this is a consequence of the pairwise non-equivalence of the representations T_r of P_0. For $r = r'$ it follows from the fact that the points x and x' can be separated by elements of P_0^X, that is, there exists an element $g \in P_0^X$ such that $g(x) \neq g(x')$.

Remark. As noted above, an almost invariant vector $h \in H$ associated with a canonical representation T of P_0 may not be unique; for instance, this is the case if P_0 is the Heisenberg group (see below). Then the constructed representations T_ξ of P_0^X depend also on the choice of an almost invariant vector $h \in H$. It is easy to check that the families of representations associated with almost invariant vectors h and h' are equivalent if and only if $h' = ch$ with $|c| = 1$.

3.3. An example: $P = \mathbb{R}_+^* \leftthreetimes P_0$, where P_0 is the Heisenberg group of dimension $2n-1$. Let us realize P_0 as the group of pairs (t,z), $t \in \mathbb{R}$, $z \in \mathbb{C}^{n-1}$, with the multiplication law $(t_1,z_1)(t_2,z_2) = \bigl(t_1+t_2-\operatorname{Im}(z_1 z_2^*), z_1+z_2\bigr)$. Elements $r \in \mathbb{R}_+^*$ and $(t,z) \in P_0$ are related by $r(t,z)r^{-1} = (r^2 t, rz)$.

Up to conjugacy, there are two infinite-dimensional unitary irreducible canonical representations of P_0 (see [27]), which act in the Hilbert spaces $H^\pm$ of entire analytic and entire anti-analytic functions $f(z) = f(z_1,\dots,z_{n-1})$ on $\mathbb{C}^{n-1}$, respectively, with the norm

$$\|f\|^2 = \int_{\mathbb{C}^{n-1}} |f(z)|^2 \exp(-zz^*)\, d\mu(z), \tag{3.6}$$

where $zz^* = \sum z_i \bar z_i$ and $d\mu(z)$ is the Lebesgue measure on $\mathbb{C}^{n-1}$ normalized by the condition

$$\int_{\mathbb{C}^{n-1}} \exp(-zz^*)\, d\mu(z) = 1.$$

The operators of T^+ on the space H^+ have the form

$$\bigl(T^+(t_0,z_0)f\bigr)(z) = \exp(\zeta_0 - zz_0^*) f(z+z_0), \quad \text{where } \zeta_0 = it_0 - \frac{1}{2} z_0 z_0^*. \tag{3.7}$$

The operators of the second representation T^- are obtained from them by complex conjugation.

In this example the operators of the representations T_r^+ conjugate to T^+ are given by

$$\bigl(T_r^+(t_0,z_0)f\bigr)(z) = \exp(r^2\zeta_0 - rzz_0^*) f(z+rz_0).$$

It is not difficult to check that the representations T^r are pairwise non-equivalent. Further, the definition of the norm in H^+ implies that every monomial $f(z) = z_1^{k_1}\cdots z_{n-1}^{k_{n-1}}$, and hence every finite linear combination of such monomials, is almost invariant with respect to T^+, that is, $\|T_r^+(g)f - f\| < c(g)r$. Therefore, the representation T^+ of the Heisenberg group is canonical, and the set of almost invariant vectors associated with T^+ is dense in the representation space. A similar assertion holds for the second representation T^-.

4. The representations of the group $P = \mathbb{R}_+^* \ltimes P_0$ and its current group P^X that are associated with canonical representations of the subgroup P_0

4.1. The representations of the group $P = \mathbb{R}_+^* \ltimes P_0$ that are associated with representations of the subgroup P_0. With each orthogonal or unitary representation T of P_0 we associate the direct integral with respect to the multiplicative Haar measure $d^*r = r^{-1}\,dr$ on $\mathbb{R}_+^*$ of the representations T_r of P_0 on the spaces $H_r = H$. The representation $\widetilde{T}$ of P_0 thus defined acts in the Hilbert space

$$\mathscr{H} = \int_0^\infty H_r\,d^*r,$$

that is, in the space of sections $f(r)$ of the fibre bundle over $\mathbb{R}_+^*$ with fibre H_r over $r \in \mathbb{R}_+^*$ such that $\int_X \|f(r)\|^2\,d^*r < \infty$. The action of the operators $\widetilde{T}(g_0)$, $g_0 \in P_0$, on $\mathscr{H}$ is fibrewise, that is,

$$\bigl(\widetilde{T}(g_0)f\bigr)(r) = T_r(g_0)f(r) \quad \text{for } g \in P_0. \tag{4.1}$$

This representation of P_0 can be extended to the whole group P. Namely, the operators on $\mathscr{H}$ corresponding to elements of the subgroup $\mathbb{R}_+^*$ are given by

$$\bigl(\widetilde{T}(r_0)f\bigr)(r) = f(r_0 r) \quad \text{for } r_0 \in \mathbb{R}_+^*. \tag{4.2}$$

(In other words, the operators $\widetilde{T}(r_0)$ permute the fibres of the fibre bundle over $\mathbb{R}_+^*$.) Obviously, the operators $\widetilde{T}(r_0)$ preserve the inner product on $\mathscr{H}$, and one can easily check that together with the operators $\widetilde{T}(g_0)$, $g_0 \in P_0$, they generate a representation of the whole group P. We say that this representation of P is *associated* with the original representation T of P_0.

We will write elements $g \in P$ as $g = rg_0$ with $r \in \mathbb{R}_+^*$ and $g_0 \in P_0$.

Proposition 4.1. *If a representation T of the subgroup P_0 is canonical and irreducible, then the associated representation $\widetilde{T}$ of the group P is also irreducible.*

This assertion follows from the irreducibility and pairwise non-equivalence of the representations T_r.

Theorem 4.1. *If a representation T of the subgroup P_0 on a space H is canonical, then the associated representation $\widetilde{T}$ of the group P on the space $\mathscr{H}$ has a non-trivial 1-cocycle $b\colon P \to \mathscr{H}$ of the form*[6]

$$b(g,r) = \bigl(\widetilde{T}(g)f_0\bigr)(r) - f_0(r), \quad \textit{where } f_0(r) = e^{-r/2}h_r \tag{4.3}$$

and $h_r = h$ is a vector in H almost invariant with respect to T.

Indeed, since T is canonical, it follows that $b(g) \in \mathscr{H}$ for every $g \in P$. Further, it is clear that b is a 1-cocycle. Since $f_0 \neq \mathscr{H}$, this 1-cocycle is non-trivial.

With the 1-cocycle $b(g)$ we associate the following function on P:

$$c(g) = \langle b(g), f_0\rangle. \tag{4.4}$$

[6] For information concerning cocycles with values in unitary representations, see [28], [4], [29], and also [7].

Proposition 4.2. *The following relations hold*:

$$\|b(g)\|^2 = -\tau(g) - 2\,\mathrm{Re}\,c(g) \quad \textit{for every } g \in P, \tag{4.5}$$

where

$$\tau(g) = \log r_0 \quad \textit{for } g = r_0 g_0 \tag{4.6}$$

and

$$\langle \widetilde{T}(g) b(g'), b(g) \rangle = -c(gg') + c(g) + c(g') \quad \textit{for any } g, g' \in P. \tag{4.7}$$

Proof. 1) It follows from (4.3) that

$$\|b(g)\|^2 = \int_0^\infty F(r)\, r^{-1}\, dr,$$

where

$$F(r) = \big\| \big(\widetilde{T}(g) f_0\big)(r) - f_0(r) \big\|^2_{H_r}.$$

The expression for $F(r)$ can be transformed into the following form:

$$F(r) = -2\,\mathrm{Re}\big\langle \big(\widetilde{T}(g) f_0\big)(r) - f_0(r), f_0(r) \big\rangle_{H_r} + \big\| \big(\widetilde{T}(g) f_0\big)(r) \big\|^2_{H_r} - \|f_0(r)\|^2_{H_r}.$$

Therefore, since $\|f_0(r)\|^2_{H_r} = e^{-r}$ and $\big\| \big(\widetilde{T}(g) f_0\big)(r) \big\|^2_{H_r} = e^{-r_0 r}$ for $g = r_0 g_0$, we obtain

$$\|b(g)\|^2 = -2\,\mathrm{Re}\langle b(g), f_0 \rangle + \int_0^\infty (e^{-r_0 r} - e^{-r}) r^{-1}\, dr.$$

Since

$$\begin{aligned}\int_0^\infty (e^{-r_0 r} - e^{-r}) r^{-1}\, dr &= \lim_{\lambda \to 0} \int_0^\infty (e^{-r_0 r} - e^{-r}) r^{\lambda - 1}\, dr \\ &= \lim_{\lambda \to 0} (r_0^{-\lambda} - 1)\Gamma(\lambda) = -\log r_0,\end{aligned}$$

this implies (4.5).

2) The equalities $b(g) = \widetilde{T}(g) f_0 - f_0$ and $\widetilde{T}(g) b(g') = b(gg') - b(g)$ imply that

$$\langle \widetilde{T}(g) b(g'), b(g) \rangle = \langle \widetilde{T}(g) b(g'), \widetilde{T}(g) f_0 \rangle - c(gg') + c(g).$$

To prove (4.7), it suffices to check that

$$\langle \widetilde{T}(g) b(g'), \widetilde{T}(g) f_0 \rangle = \langle b(g'), f_0 \rangle = c(g').$$

For $g = r_0 g_0$ and $g' = r_0' g_0'$ we have

$$\begin{aligned}\widetilde{T}(g) b(g')|_{H_r} &= \exp\left(-\frac{1}{2}\, r_0 r_0' r\right) T_{r_0 r}(g_0)\, T_{r_0 r_0' r}(g_0')\, h_r - \exp\left(-\frac{1}{2}\, r_0 r\right) T_{r_0 r}(g_0)\, h_r, \\ \widetilde{T}(g) f_0|_{H_r} &= \exp\left(-\frac{1}{2}\, r_0 r\right) T_{r_0 r}(g_0)\, h_r.\end{aligned}$$

Since the operators $T(g)$ for $g \in P_0$ are unitary and the measure $d^* r$ is multiplicatively invariant, it follows that

$$\langle \widetilde{T}(g) b(g'), \widetilde{T}(g) f_0 \rangle = \int_0^\infty \langle e^{-r_0' r/2} T(r_0' r) h - e^{-r/2} h, e^{-r/2} h \rangle\, d^* r = \langle b(g'), f_0 \rangle.$$

4.2. The integral model of representation of the current group P^X. Let us turn to the main construction of this section. With each canonical representation T of the subgroup P_0 on a Hilbert space H with a vector h almost invariant with respect to T we associate a representation U of the group P^X and call it the *integral model* $\operatorname{INT} T$ associated with T.

The construction of the representation $\operatorname{INT} T$ of P^X is parallel to the construction of the representation $\widetilde{T}$ of P from a representation T of the subgroup P_0 (see above). Namely, in this construction we replace the spaces H_r, $r \in \mathbb{R}_+^*$, of the representations T_r of P_0 by the spaces H_ξ, $\xi = \{r_k, x_k\} \in l_+^1(X)$, of the representations T_ξ of P_0^X, and the direct integral of H_r with respect to the measure $r^{-1}\,dr$ on $\mathbb{R}_+^*$ by the direct integral of H_ξ with respect to the measure $d\mathscr{L}(\xi)$ on the cone $l_+^1(X)$. Thus, according to this construction, the integral model of representation of P^X associated with a canonical representation T of P_0 is realized on the direct integral of the Hilbert spaces H_ξ with respect to the measure $\mathscr{L}$,

$$\operatorname{INT} H = \int_{l_+^1(X)} H_\xi \, d\mathscr{L}(\xi), \qquad H_\xi = \bigotimes_{k=1}^{\infty} H_{r_k},$$

that is, on the space of sections $F(\xi) = F(\{r_k, x_k\})$ of the fibre bundle over the cone $l_+^1(X)$ in which the fibre over a point $\xi = \{r_k, x_k\}$ is the countable tensor product $H_\xi = \bigotimes_{k=1}^{\infty} H_{r_k}$, $H_{r_k} = H$. (This fibre, regarded as a Hilbert space, does not depend on ξ, but the representation itself does depend on ξ.) The action of the group P_0^X in each fibre of this fibre bundle induces a representation of this group on the whole space $\operatorname{INT} H$:

$$U(g)\biggl(\bigotimes_{k=1}^{\infty} f_k(r_k)\biggr) = \bigotimes_{k=1}^{\infty} T(g(x_k)) f_k(r_k) \quad \text{for } g \in P_0^X. \tag{4.8}$$

We define the operators $U(r_0(\cdot))$ on $\operatorname{INT} H$ for elements of the group $\mathbb{R}_+^X$ by the formula

$$\bigl(U(r_0(\cdot))F\bigr)(\{r_k, x_k\}) = \exp\biggl(\frac{1}{2}\int_X \log r_0(x)\, dm(x)\biggr) F\bigl(\{r_0(x_k) r_k, x_k\}\bigr)$$

for any $r_0(\cdot) \in \mathbb{R}_+^X$. So these operators permute the fibres of the fibre bundle $\operatorname{INT} H$.

Proposition 4.3. *The operators $U(r_0(\cdot))$, $r_0(\cdot) \in \mathbb{R}_+^X$, are orthogonal (unitary) and generate, together with the operators $U(g_0)$ for $g_0 \in P_0^X$, an orthogonal (unitary) representation of the current group P^X which is invariant under m-preserving transformations of X:*

$$U(g)\biggl(\bigotimes_{k=1}^{\infty} f_k(r_k)\biggr) = \exp\biggl(\frac{1}{2}\int_X \log r_0(x)\, dm(x)\biggr) \bigotimes_{k=1}^{\infty} \widetilde{T}\bigl(g(x_k) f_k\bigr)(r_k) \tag{4.9}$$

for every $g = r_0 g_0 \in P^X$ $(r_0 \in (\mathbb{R}_+^*)^X$, $g_0 \in P_0^X)$, where $\widetilde{T}$ is the representation of P associated with the representation T of the subgroup P_0.

Proof. The orthogonality (unitarity) of the operators $U(r_0(\cdot))$ follows from the projective invariance of the measure $\mathscr{L}$ (see (2.3)). Indeed,

$$\begin{aligned}\|U(r_0(\cdot))F\|^2 &= \exp\biggl(\int_X \log r_0(x)\,dm(x)\biggr)\int_{l^1_+}\|F(r_0(\cdot)\xi)\|^2\,d\mathscr{L}(\xi)\\ &= \exp\biggl(\int_X \log r_0(x)\,dm(x)\biggr)\int_{l^1_+}\|F(\xi)\|^2\,d\mathscr{L}(r_0^{-1}(\cdot)\xi) = \|F\|^2,\end{aligned}$$

since $d\mathscr{L}(r_0^{-1}(\cdot)\xi) = \exp\bigl(-\int_X \log r_0(x)\,dm(x)\bigr)\,d\mathscr{L}(\xi)$.

Further, from the definition of these operators it follows that

$$U^{-1}(r_0(\cdot))\,U(g_0(\cdot))\,U(r_0(\cdot)) = U\bigl(r_0^{-1}(\cdot)g_0(\cdot)r_0(\cdot)\bigr)$$

for any $g_0(\cdot) \in P_0^X$ and $r_0(\cdot) \in \mathbb{R}_+^X$. Hence these operators generate a representation of the group P^X.

Since the measure $\mathscr{L}$ on $l^1_+(X)$ is preserved by any m-preserving transformations of X, the representation of P^X obtained is also invariant with respect to these transformations. Proposition 4.3 is proved.

Using (4.5), we can write the expression (4.9) for $U(g)$, $g \in P^X$, in the form

$$U(g)\biggl(\bigotimes_{k=1}^{\infty} f_k(r_k)\biggr) = \exp\biggl(-\int_X \lambda(g(x))\,dm(x)\biggr)\bigotimes_{k=1}^{\infty}\widetilde{T}\bigl(g(x_k)f_k\bigr)(r_k), \tag{4.10}$$

where

$$\lambda(g) = \frac{1}{2}\,\|b(g)\|^2 + \operatorname{Re} c(g). \tag{4.11}$$

We call the constructed representation of P^X the integral model associated with the canonical representation T of P_0 and denote it, by analogy with Fock representations, by $\mathrm{INT}\,T$; similarly, we denote by $\mathrm{INT}\,H$ the Hilbert space on which $\mathrm{INT}\,T$ is realized.

The description of the integral model implies the following assertion.

Theorem 4.2. *If T_1 and T_2 are canonical representations of the group P_0 on spaces H_1 and H_2, then*

$$\mathrm{INT}(H_1 \oplus H_2) = \mathrm{INT}\,H_1 \otimes \mathrm{INT}\,H_2,$$

and on this space the integral model $\mathrm{INT}(T_1 \oplus T_2)$ of P^X associated with the representation $T_1 \oplus T_2$ of P_0 is realized.

Theorem 4.3. *If a canonical representation T of the group P_0 is irreducible, then the associated representation $U = \mathrm{INT}\,T$ of the current group P^X is also irreducible.*

Proof. Let us first consider the operators $U(g(\cdot))$, $g(\cdot) \in P_0^X$. They preserve the fibres H_ξ of the fibre bundle $\mathrm{INT}\,H$, and, by Proposition 3.3, the resulting representations of P_0^X are irreducible and pairwise non-equivalent. Hence every operator A on $\mathrm{INT}\,H$ that commutes with these operators is a multiple of the identity operator on each fibre of $\mathrm{INT}\,H$, that is, it is multiplication by a function $a(\xi) = a(\{r_k, x_k\})$. If this operator A also commutes with the operators

$U(r_0(\cdot))$, $r_0(\cdot) \in \mathbb{R}_+^X$, then the function $a(\{r_k, x_k\})$ is constant on the orbits of the group $\mathscr{M}$ of multipliers, which acts in $l_+^1(X)$ by multiplication by positive functions $r_0(\cdot)\colon \{r_k, x_k\} \mapsto \{r_0(x_k)r_k, x_k\}$. Since the measure $\mathscr{L}$ is ergodic, it follows that A is a constant.

4.3. The total subset $M \subset \mathrm{INT}\, H$. We define the *vacuum* vector in the space $\mathrm{INT}\, H$ to be the vector Ω given by

$$\Omega(\xi) = \bigotimes_{k=1}^{\infty} f_0(r_k) \quad \text{for } \xi = \{r_k, x_k\}, \quad \text{where } f_0(r) = e^{-r/2} h_r. \tag{4.12}$$

It follows from the formula for the characteristic functional of the measure $\mathscr{L}$ that $\|\Omega\| = 1$.

Definition 4. With each element $g \in P^X$ we associate the following vector in the space $\mathrm{INT}\, H$:

$$F_g(\xi) = \exp\biggl(\int_X \biggl(\frac{1}{2} \|b(g(x))\|^2 - i \operatorname{Im} c(g(x)) \biggr) \, dm(x) \biggr) U(g)\Omega(\xi), \tag{4.13}$$

where $c(g) = \langle b(g), f_0 \rangle$.

Since Ω is cyclic, the set M consisting of vectors F_g, $g \in P^X$, is total in $\mathrm{INT}\, H$. Further,

$$(U(g)\Omega)(\xi) = \exp\biggl(- \int_X \lambda(g(x)) \, dm(x) \biggr) \bigotimes_{k=1}^{\infty} \bigl(\widetilde{T}(g(x_k)) f_0\bigr)(r_k) \quad \text{for } \xi = \{r_k, x_k\}, \tag{4.14}$$

where $\lambda(g) = \frac{1}{2} \|b(g)\|^2 + \operatorname{Re} c(g)$, so that the expression for F_g can be written in the form

$$F_g(\xi) = \exp\biggl(- \int_X c(g(x)) \, dm(x) \biggr) \bigotimes_{k=1}^{\infty} \bigl(\widetilde{T}(g(x_k)) f_0\bigr)(r_k) \quad \text{for } \xi = \{r_k, x_k\}. \tag{4.15}$$

The vector $\Omega = F_e$ and the set of vectors F_g generated by Ω can be viewed as analogues of the vacuum vector $\mathrm{EXP}\, 0$ and the set of vectors $\mathrm{EXP}\, b^X(g)$ generated by $\mathrm{EXP}\, 0$ in the space of the Fock representation. Let us describe the main properties of the set M.

The definition of F_g implies the following assertion.

Proposition 4.4. *The action of the operators of the representation $U = \mathrm{INT}\, T$ on vectors of the form F_g is given by the formula*

$$U(g) F_{g_1} = \exp\biggl(- \int_X \bigl(c(g_1(x)) - c(g g_1(x)) + \lambda(g(x)) \bigr) \, dm(x) \biggr) F_{g g_1}, \tag{4.16}$$

where $\lambda(g) = \frac{1}{2} \|b(g)\|^2 + \operatorname{Re} c(g)$.

Indeed, we have

$$\begin{aligned}\big(U(g)F_{g_1}\big)(\xi) &= \exp\Big(-\int_X c(g_1(x))\,dm(x)\Big)U(g)\Big(\bigotimes_{k=1}^{\infty}\widetilde{T}\big(g_1(x_k)f_0\big)(r_k)\Big)\\ &= \exp\Big(-\int_X \big(c(g_1(x))+\lambda(g(x))\big)\,dm(x)\Big)\Big(\bigotimes_{k=1}^{\infty}\widetilde{T}\big(gg_1(x_k)f_0\big)(r_k)\Big)\\ &= \exp\Big(-\int_X \big(c(g_1(x))+\lambda(g(x))-c(gg_1(x))\big)\,dm(x)\Big)F_{gg_1}.\end{aligned}$$

Proposition 4.5. *On the set of vectors of the form F_g the inner product is given by the formula*

$$\langle F_{g_1},F_{g_2}\rangle = \exp\Big(\int_X \langle b(g_1(x)),b(g_2(x))\rangle\,dm(x)\Big) \quad \textit{for all } g_1,g_2\in P^X. \tag{4.17}$$

Proof. From the definition of F_g it follows that

$$\langle F_{g_1},F_{g_2}\rangle = \exp\Big(-\int_X \big(c(g_1(x))+\overline{c(g_2(x))}\,\big)\,dm(x)\Big)I, \tag{4.18}$$

where

$$I=\int_{l^1_+(X)}\prod_{k=1}^{\infty}\big\langle\big(\widetilde{T}(g_1(x_k))f_0\big)(r_k),\big(\widetilde{T}(g_2(x_k))f_0\big)(r_k)\big\rangle_H\,d\mathscr{L}(\xi). \tag{4.19}$$

To compute I we use the general formula (2.6). Let

$$\varphi(r,x)=\big\langle\big(\widetilde{T}(g_1(x))f_0\big)(r),\big(\widetilde{T}(g_2(x))f_0\big)(r)\big\rangle_H.$$

In view of (2.6), we obtain

$$I=\exp\Big(\int_X\int_0^{\infty}\big(\varphi(r,x)-e^{-r}\big)\,d^*r\,dm(x)\Big).$$

Let us transform the integrand. Since $\widetilde{T}(g_1(x))f_0 = b(g(x))+f_0$ and $e^{-r} = \langle f_0(r),f_0(r)\rangle_H$, the function $\varphi(r,x)$ can be written in the form

$$\varphi(r,x)=\big\langle b\big(g_1(x),r\big),b\big(g_2(x),r\big)\big\rangle\big|_H+\langle b\big(g_1(x),r\big),f_0(r)\rangle\big|_H+\overline{\big\langle b\big(g_2(x),r\big),f_0(r)\big\rangle}_H.$$

Integrating with respect to r yields

$$I=\exp\Big(\int_X\big(\langle b(g_1(x)),b(g_2(x))\rangle+c(g_1(x))+\overline{c(g_2(x))}\,\big)\,dm(x)\Big).$$

Together with (4.18) this implies (4.17).

Remark. Denote by K the subgroup consisting of the elements $k\in P$ such that $b(k)=0$. Clearly, two vectors F_{g_1} and F_{g_2} coincide if and only if $g_2=g_1k$ with $k\in K^X$. Hence the set of pairwise distinct vectors F_g can be naturally identified with the quotient space G^X/K^X.

4.4. The spherical function of the representation $U = \mathrm{INT}\, T$.

Definition 5. The spherical function of the representation $U = \mathrm{INT}\, T$ of the group P^X is the following function on P^X:

$$\Psi(g) = \langle U(g)\Omega, \Omega\rangle, \quad \text{where } \Omega(\xi) = \bigotimes_{k=1}^{\infty} f_0(r_k) \text{ for } \xi = \{r_k, x_k\}. \tag{4.20}$$

Since the representation U is irreducible, this function uniquely determines it up to equivalence.

Theorem 4.4. *The spherical function $\Psi(g)$ can be written in the form*

$$\Psi(g) = \exp\Big(\int_X \Big(i\,\mathrm{Im}\, c(g(x)) - \frac{1}{2}\,\|b(g(x))\|^2 \Big)\, dm(x) \Big), \tag{4.21}$$

where $b(g)$ is the 1-cocycle

$$b(g) = \big(\widetilde{T}(g) f_0\big)(r) - f_0(r), \qquad f_0(r) = e^{-r/2}, \tag{4.22}$$

of the representation $\widetilde{T}$ of P associated with the representation T of P_0, and $c(g) = \langle b(g(x)), f_0\rangle$.

Proof. The desired formula (4.21) follows from the formula (4.17) for the inner product of vectors of the form F_g and the formula (4.13) which expresses F_g in terms of $U(g)\Omega$. For completeness, let us give also a direct proof. By (4.14) we have

$$\Psi(g) = \exp\Big(-\int_X \lambda(g(x))\, dm(x) \Big) \int_{l^1_+(X)} \Big(\prod_{k=1}^{\infty} \big\langle \widetilde{T}\big(g(x_k) f_0\big)(r_k), f_0(r_k) \big\rangle \Big)\, d\mathscr{L}(\xi), \tag{4.23}$$

where $\lambda(g) = \frac{1}{2}\,\|b(g)\|^2 + \mathrm{Re}\, c(g)$. To compute the integral

$$I = \int_{l^1_+(X)} \Big(\prod_{k=1}^{\infty} \big\langle \widetilde{T}\big(g(x_k) f_0\big)(r_k), f_0(r_k) \big\rangle \Big)\, d\mathscr{L}(\xi),$$

consider the projections of the cone $l^1_+(X)$ on the finite-dimensional cones Φ_α associated with the partitions α: $X = \bigcup_{k=1}^{n} X_k$ of the space X. Under the projection on Φ_α the expression for I takes the form

$$I_\alpha = \prod_{k=1}^{n} I_\alpha^k, \quad \text{where } I_\alpha^k = \int_0^\infty \big\langle \big(\widetilde{T}(g_k) f_0\big)(r_k), f_0(r_k) \big\rangle_H \frac{r_k^{\lambda_k - 1}\, dr_k}{\Gamma(\lambda_k)}\,.$$

Here we have used the notation $g_k = g(x)\big|_{X_k}$. Let us substitute into this formula the expression for $\widetilde{T}(g_k) f_0$ in terms of the non-trivial cocycle $b(g, r)$:

$$\big(\widetilde{T}(g_k) f_0\big)(r_k) = b(g_k, r_k) + f_0(r_k).$$

Taking into account that $\displaystyle\int_0^\infty \|f_0(r)\|^2\big|_H \frac{r^{\lambda_k-1}\,dr_k}{\Gamma(\lambda_k)} = 1$, we obtain

$$I_\alpha^k = 1 + \int_0^\infty \langle b(g_k,r), f_0(r)\rangle_H \frac{r^{\lambda_k-1}\,dr}{\Gamma(\lambda_k)}\,.$$

It follows that

$$I_\alpha^k = 1 + \lambda_k \int_0^\infty \langle b(g_k,r), f_0(r)\rangle r^{-1}\,dr + O(\lambda_k^2) = 1 + \lambda_k\, c(g_k) + O(\lambda_k^2).$$

Therefore, $I_\alpha = \prod_{k=1}^n \bigl(1 + c(g_k) + O(\lambda_k^2)\bigr)$. Since

$$\sum \lambda_k c(g_k) = \int_X c(g(x))\,dm(x),$$

where $g(x)$ is the piecewise constant function that takes the values g_k on the elements of the partition α, the expression obtained can be written in the form $I_\alpha \sim \exp\bigl(\int_X c(g(x))\,dm(x)\bigr)$ up to terms of order greater than 1 with respect to λ_k. Taking the inductive limit over the set of finite partitions α, we obtain the following expression for I:

$$I = \exp\biggl(\int_X c(g(x))\,dm(x)\biggr).$$

Together with (4.23) this implies (4.21).

4.5. The relation between the integral and Fock models of representations of the current group $P^X = (\mathbb{R}_+^* \ltimes P_0)^X$. The Fock construction (see, for example, [1], [3], [30], [28], [31]) associates with each pair $(\widetilde{T}, b)$, where $\widetilde{T}$ is a special orthogonal or unitary representation of an arbitrary locally compact group G on a Hilbert space $\mathscr{H}$ and b is a non-trivial 1-cocycle $b\colon G \to \mathscr{H}$, a unitary representation of the current group G^X on the complex Hilbert space $\operatorname{EXP}\mathscr{H}^X$, where

$$\mathscr{H}^X = \int_X^\oplus \mathscr{H}_x\,dm(x), \qquad \mathscr{H}_x = \mathscr{H}.$$

By definition,

$$\operatorname{EXP}\mathscr{H}^X = \bigoplus_{k=0}^\infty S^k\mathscr{H}^X$$

(here S^k is the kth symmetric tensor power) in the case where $\mathscr{H}$ is a complex Hilbert space; if $\mathscr{H}$ is a real space, then $\operatorname{EXP}\mathscr{H}^X$ is the complexification of the real space $\bigoplus_{k=0}^\infty S^k\mathscr{H}^X$. In the latter case, $\operatorname{EXP}\mathscr{H}^X$ is isomorphic to the Fock space $\operatorname{EXP}(\mathscr{H}_\mathbb{C})^X$, where $\mathscr{H}_\mathbb{C}$ is the complexification of the real space $\mathscr{H}$.

In the space $\operatorname{EXP}\mathscr{H}^X$ we consider the total subset of vectors $\operatorname{EXP} v$, $v \in \mathscr{H}^X$, of the form

$$\operatorname{EXP} v = \mathbb{1} \oplus v \oplus \frac{1}{\sqrt{2!}}\, v\otimes v \oplus \frac{1}{\sqrt{3!}}\, v\otimes v\otimes v \oplus \cdots.$$

On this set the operators of the Fock representation are defined by the following formula:

$$U(g)\,\mathrm{EXP}\,v=\exp\Big(-\frac{1}{2}\,\|b^X(g)\|^2-\big\langle\widetilde{T}^X(g)v,b^X(g)\big\rangle\Big)\,\mathrm{EXP}\big(\widetilde{T}^X(g)v+b^X(g)\big).$$

Here $\widetilde{T}^X$ and b^X denote, respectively, the representation of G^X on $\mathscr{H}^X$ generated by the representation $\widetilde{T}$ of G on $\mathscr{H}$, and the 1-cocycle $G^X\to\mathscr{H}^X$ generated by the 1-cocycle $b\colon G\to\mathscr{H}$.

The operators $U(g)$ are related by

$$U(g_1g_2)=\exp\big(i\,\mathrm{Im}\langle T^X(g_1)b^X(g_2),b^X(g_1)\rangle\big)U(g_1)U(g_2)\quad\text{for any } g_1,g_2\in G^X. \tag{4.24}$$

Thus, the Fock representation of G^X associated with a unitary representation of G is projective if the 2-cocycle

$$\exp\big(i\,\mathrm{Im}\langle T^X(g_1)b^X(g_2),b^X(g_1)\rangle\big)$$

is not identically zero, and it is equivalent to a true representation if and only if this 2-cocycle is trivial.

Theorem 4.5. *The Fock model of representation of the group $P^X=(\mathbb{R}_+^*\leftthreetimes P_0)^X$ on the space $\mathrm{EXP}\,\mathscr{H}^X$ is projectively equivalent to the true representation V of P^X on the same space $\mathrm{EXP}\,\mathscr{H}^X$ whose operators are related to the operators $U(g)$ of the Fock representation by*

$$V(g)=\exp\Big(i\,\mathrm{Im}\int_X\langle b(g(x)),f_0\rangle\,dm(x)\Big)U(g). \tag{4.25}$$

Indeed, it follows from (4.7) that the 2-cocycle

$$\lambda(g_1,g_2)=\exp\big(i\,\mathrm{Im}\langle T^X(g_1)b^X(g_2),b^X(g_1)\rangle\big)$$

in (4.24) is trivial, namely,

$$\lambda(g_1,g_2)=\frac{C(g_1)C(g_2)}{C(g_1g_2)}\,,\quad\text{where } C(g)=\exp\Big(i\,\mathrm{Im}\int_X\langle b(g(x)),f_0\rangle\,dm(x)\Big).$$

The assertion follows.

We define the spherical function of the representation V of P^X on the space $\mathrm{EXP}\,\mathscr{H}^X$ by the formula

$$\Phi(g)=\langle V(g)\,\mathrm{EXP}\,0,\,\mathrm{EXP}\,0\rangle.$$

The definitions of the Fock representation U and the (true) representation V of P^X projectively equivalent to U imply the following assertion.

Proposition 4.6. *The spherical function $\Phi(g)$ of the representation V of P^X is equal to*

$$\Phi(g)=\exp\bigg(\int_X\Big(i\,\mathrm{Im}\langle b(g(x)),f_0\rangle-\frac{1}{2}\,\|b(g(x))\|^2\Big)\,dm(x)\bigg). \tag{4.26}$$

Theorem 4.6. *Let T be a canonical irreducible representation of the group P_0 on a space H, let $\widetilde{T}$ be the associated representation of the group $P = \mathbb{R}_+^* \ltimes P_0$ on the space $\mathscr{H}$, and let $b(g,r) = \big(\widetilde{T}(g)f_0\big)(r) - f_0(r)$, where $f_0(r) = e^{-r/2}h$, be a non-trivial 1-cocycle $P \to \mathscr{H}$. Then the integral model of representation* $\operatorname{INT} T$ *of the group P^X on the space* $\operatorname{INT} H$ *is projectively equivalent to the Fock representation U of P^X on the space* $\operatorname{EXP} \mathscr{H}^X$*. The intertwining operator for these representations is generated by the map* $\Omega \to \operatorname{EXP} 0$ *of the cyclic vectors.*

Proof. Note that the formulae (4.21) and (4.26) for the spherical functions of $\operatorname{INT} T$ and the representation V of P^X on the Fock space coincide. Hence these representations are equivalent, and the intertwining operator for them is generated by the map $\Omega \to \operatorname{EXP} 0$ of the cyclic vectors. The required assertion now follows from Theorem 4.5.

4.6. Extension of the integral model of representation of the group P^X to a representation of the group G^X, where $P \subset G$. We consider an arbitrary locally compact group G that contains $P = \mathbb{R}_+^* \ltimes P_0$ as a subgroup. Let T be an irreducible canonical representation of P_0 on a space H, let $\widetilde{T}$ be the associated special representation of P on the space $\mathscr{H} = \int_0^\infty H_r \, d^*r$, $H_r = H$, and let $b(g)\colon P \to \mathscr{H}$ be the non-trivial 1-cocycle of $\widetilde{T}$ defined by

$$b(g) = \widetilde{T}(g) f_0 - f_0, \quad \text{where } f_0(r) = e^{-r/2} h_r. \tag{4.27}$$

Theorem 4.7. *Assume that there exists an extension of the representation $\widetilde{T}$ of the group P on the space $\mathscr{H}$ to a representation of the group G, and there exists an extension of the 1-cocycle (4.27) of P to a 1-cocycle of the same form of G. Then there exists a corresponding extension of the integral model $U = \operatorname{INT} T$ of representation of the current group P^X to a representation of the current group G^X.*

Let us explicitly describe this extension. In § 4.3 above we defined the total set of vectors of the form

$$F_g(\xi) = \exp\biggl(-\int_X c(g(x))\, dm(x)\biggr) \bigotimes_{k=1}^{\infty} \big(\widetilde{T}(g(x_k)) f_0\big)(r_k) \quad \text{for } \xi = \{r_k, x_k\} \tag{4.28}$$

in the space $\operatorname{INT} H$, where $f_0(r) = \exp(-r/2) h_r$ and $c(g) = \langle b(g), f_0 \rangle$.

We proved that

$$\langle F_{g_1}, F_{g_2} \rangle = \exp\biggl(\int_X c(g_1(x), g_2(x))\, dm(x)\biggr), \tag{4.29}$$

where

$$c(g_1, g_2) = \langle b(g_1), b(g_2) \rangle, \tag{4.30}$$

and the action of the operators of the representation U of P^X on these vectors is given by

$$U(g) F_{g_1} = \exp\biggl(-\int_X \big(c(g_1(x)) - c(g_1(x)) + \lambda(g(x))\big)\, dm(x)\biggr) F_{gg_1}, \tag{4.31}$$

where $\lambda(g) = \frac{1}{2}\, \|b(g)\|^2 + \operatorname{Re} c(g)$.

Definition 6. Let us extend the family of functions F_g, $g \in P^X$, by defining F_g for an arbitrary $g \in G^X$ by the same formula (4.28).

Exactly as in Proposition 4.5, we establish the following fact: *on the set of vectors of the form F_g, $g \in G^X$, the inner product is given by the same formula* (4.29). In particular, $\langle F_g, F_g \rangle = \exp\big(\int_X c(g(x), g(x))\, dm(x) \big) < \infty$, so that the vectors F_g, $g \in G^X$, belong to $\mathrm{INT}\, H$.

Definition 7. On the set of vectors of the form F_g, $g \in G^X$, we define the action of the operators $U(g)$ for $g \in G^X$ by the formula

$$U(g)F_{g_1} = \exp\left(- \int_X \lambda(g(x), g_1(x))\, dm(x) \right) F_{gg_1}, \tag{4.32}$$

where

$$\lambda(g, g_1) = \frac{1}{2}\, \|b(g)\|^2 + \langle \widetilde{T}(g) b(g_1), b(g) \rangle - i \operatorname{Im} c(g). \tag{4.33}$$

Theorem 4.8. *The operators $U(g)$, $g \in G^X$, preserve the inner products of vectors of the form F_g, that is,*

$$\langle U(g)F_{g_1}, U(g)F_{g_2} \rangle = \langle F_{g_1}, F_{g_2} \rangle \quad \textit{for any } g, g_1, g_2 \in G^X, \tag{4.34}$$

and thus they can be extended to orthogonal (unitary) operators on the whole space $\mathrm{INT}\, H$.

Proof. We have

$$\langle U(g)F_{g_1}, U(g)F_{g_2} \rangle = \exp\left(- \int_X u(g(x), g_1(x), g_2(x))\, dm(x) \right),$$

where

$$u(g, g_1, g_2) = \lambda(g, g_1) + \overline{\lambda(g, g_2)} - \langle b(gg_1), b(gg_2) \rangle.$$

Since $b(gg_1) = \widetilde{T}(g) b(g_1) + b(g)$, it follows from the expression (4.33) for $\lambda(g, g_i)$ that $u(g, g_1, g_2) = \langle b(g_1), b(g_2) \rangle$. This implies (4.34).

Theorem 4.9. *The operators $U(g)$ determine a representation (in general, projective) of the group G^X on the space* $\mathrm{INT}\, H$*:*

$$U(g_1 g_2) = \exp\left(i \operatorname{Im} \int_X p(g_1(x), g_2(x))\, dm(x) \right) U(g_1) U(g_2) \ \textit{for any } g_1, g_2 \in G^X, \tag{4.35}$$

where

$$p(g_1, g_2) = \langle \widetilde{T}(g_1) b(g_2), b(g_1) \rangle - c(g_1) - c(g_2) + c(g_1 g_2). \tag{4.36}$$

Proof. For any $g, g_1, g_2 \in G^X$ we have

$$U(g_1)U(g_2)F_g = \exp\left(- \int_X a(g_1(x), g_2(x), g(x))\, dm(x) \right) F_{g_1 g_2 g},$$

$$U(g_1 g_2)F_g = \exp\left(- \int_X a'(g_1(x), g_2(x), g(x))\, dm(x) \right) F_{g_1 g_2 g},$$

where

$$a(g_1, g_2, g) = \lambda(g_2, g) + \lambda(g_1, g_2 g), \qquad a'(g_1, g_2, g) = \lambda(g_1 g_2, g).$$

Let us use the relation

$$l(g_2, g) + l(g_1, g_2 g) - l(g_1 g_2, g) = i\,\mathrm{Im}\langle \widetilde{T}(g_1) b(g_2), b(g_1)\rangle$$

for

$$l(g_1, g_2) = \frac{1}{2}\,\|b(g_1)\|^2 + \langle \widetilde{T}(g_1) b(g_2), b(g_1)\rangle.$$

It implies that

$$a(g_1, g_2, g) - a'(g_1, g_2, g) = i\,\mathrm{Im}\big(\langle \widetilde{T}(g_1) b(g_2), b(g_1)\rangle - c(g_1) - c(g_2) + c(g_1 g_2)\big).$$

Hence,

$$\begin{aligned} &U(g_1)U(g_2)U^{-1}(g_1 g_2)\, F_g \\ &\qquad = \exp\Big(-\mathrm{Im}\int_X p(g_1(x), g_2(x))\, dm(x)\Big) F_g \quad \text{for every } g \in G^X, \end{aligned}$$

where $p(g_1, g_2)$ is given by (4.36). The required assertion follows.

Theorem 4.10. *The restriction of the representation U of the group G^X to the subgroup P^X coincides with the original representation* $\mathrm{INT}\, T$.

Proof. It suffices to check that on the total set of vectors of the form F_{g_1} the operators $U(g)$ for $g \in P^X$ coincide with the original operators. For $g, g_1 \in P$ we have $\langle \widetilde{T}(g) b(g_1), b(g)\rangle = -c(g g_1) + c(g) + c(g_1)$ by (4.7). Hence,

$$\lambda(g, g_1) = c(g_1) - c(g g_1) + \frac{1}{2}\,\|b(g)\|^2 + \mathrm{Re}\, c(g),$$

so that the expression for $U(g)F_{g_1}$ coincides for $g, g_1 \in P^X$ with the original expression (4.16).

5. The integral model of representation of the current group $\mathrm{O}(n,1)^X$, $n > 2$

In this and subsequent sections we describe the integral models of representations of the current groups P^X, where P is the maximal parabolic subgroup of the group $\mathrm{O}(n,1)$, $\mathrm{U}(n,1)$, or $\mathrm{Sp}(n,1)$, and in the first two cases we extend these representations of P^X to representations of the groups $\mathrm{O}(n,1)^X$ and $\mathrm{U}(n,1)^X$, respectively. (In the case of $\mathrm{Sp}(n,1)$ the corresponding current group has no representations.) A separate section is devoted to the case of $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$, in which P is the subgroup of triangular matrices. Each of these groups has a unique (up to conjugacy) maximal parabolic subgroup P, and this subgroup can be written as a semidirect product: $P = \mathbb{R}_+^* \leftthreetimes P_0$. Thus, the description of the integral models essentially reduces to the description of the canonical representations of the subgroup P_0.

We begin with the case $P \subset \mathrm{O}(n,1)$, $n > 2$, because in this case there is a unique, up to conjugacy, canonical representation of P_0 and, accordingly, a unique integral model of representation of P^X.

5.1. Preliminary definitions and notation. By definition, $\mathrm{O}(n,1)$ is the group of linear transformations on $\mathbb{R}^{n+1}$ preserving a non-degenerate quadratic form of signature $(n,1)$. Here we choose $2x_1x_{n+1}+x_2^2+\dots+x_n^2$ as such a form and write elements of the group $\mathrm{O}(n,1)$ as block matrices

$$g=\|g_{ij}\|_{i,j=1,2,3},$$

where the diagonal consists of square matrices of orders 1, $n-1$, and 1, respectively.

This matrix realization of $\mathrm{O}(n,1)$ is convenient for describing its maximal parabolic subgroup $P\subset \mathrm{O}(n,1)$, which is, by definition, the group of linear transformations preserving a subspace E that is isotropic with respect to the quadratic form under consideration. Up to conjugacy, $\mathrm{O}(n,1)$ has a unique maximal parabolic subgroup. In our realization E is the one-dimensional subspace of vectors of the form $(x_1,0,\dots,0)$, and the corresponding maximal parabolic subgroup P of $\mathrm{O}(n,1)$ can be written, as the group of all lower block-triangular matrices, in the form of the semidirect product

$$P=D\leftthreetimes N,$$

where $N\cong\mathbb{R}^{n-1}$ is the maximal nilpotent subgroup consisting of the block matrices of the form

$$h=\begin{pmatrix}1&0&0\\-\gamma^*&e_{n-1}&0\\-\frac{1}{2}\,\gamma\gamma^*&\gamma&1\end{pmatrix},\qquad \gamma\in\mathbb{R}^{n-1},$$

and $D\cong\mathbb{R}^*\times \mathrm{O}(n-1)$ is the group of block-diagonal matrices of the form $d=\operatorname{diag}(s^{-1},u,s)$, $s\in\mathbb{R}^*$, $u\in \mathrm{O}(n-1)$.

Let D be written as the direct product $D=\mathbb{R}_+^*\times D_0$, where D_0 is the subgroup of matrices of the form $d=\operatorname{diag}(\pm1,u,\pm1)$, and let

$$P_0=D_0\leftthreetimes N.$$

Thus,

$$P=\mathbb{R}_+^*\leftthreetimes P_0=(\mathbb{R}_+^*\times D_0)\leftthreetimes N.$$

Elements of $\mathbb{R}_+^*$, D_0, and N will be denoted by r, (ε,u) (with $\varepsilon=\pm1$), and γ (a row vector), respectively. With this notation the group relations take the form

$$(\varepsilon,u)^{-1}g(\varepsilon,u)=\varepsilon\gamma u,\quad rgr^{-1}=r\gamma\quad\text{for } g=\gamma\in N.$$

5.2. Description of the canonical representations of the subgroup P_0. Up to conjugacy with respect to the group $\mathbb{R}_+^*$ of automorphisms, there is a unique canonical irreducible unitary representation T of the subgroup $P_0=D_0\leftthreetimes N$. It is realized on the Hilbert space H of functions on the unit sphere $S^{n-2}\subset\mathbb{R}^{n-1}$ with the norm

$$\|f\|^2=\int_{S^{n-2}}|f(\omega)|^2\,d\omega,$$

where $d\omega$ is the invariant measure on S^{n-2} normalized by the condition $\int_{S^{n-2}}d\omega=1$. The operators of this representation are given by the formulae

$$\big(T(\gamma)f\big)(\omega)=e^{-i\langle\gamma,\omega\rangle}\,f(\omega)\quad\text{for } \gamma\in N,\ N\cong\mathbb{R}^{n-1},\tag{5.1}$$

$$\big(T(\varepsilon\omega u)f\big)(\omega)=f(\varepsilon\omega u)\quad\text{for } (\varepsilon,u)\in D_0,\ D_0=\{\pm1\}\times \mathrm{O}(n-1).\tag{5.2}$$

The operators of the representations T_r, $r \in \mathbb{R}_+^*$, conjugate to T act in the spaces $H_r = H$ and are given by the formulae

$$\bigl(T_r(\gamma)f\bigr)(\omega) = e^{-ir\langle\gamma,\omega\rangle}\, f(\omega), \qquad T_r(g) = T(g) \quad \text{for } g \in D_0. \tag{5.3}$$

Remark. The representation T^- of P_0 defined by the formulae

$$\bigl(T^-(\gamma)f\bigr)(\omega) = e^{i\langle\gamma,\omega\rangle}\, f(\omega), \qquad T^-(g) = T(g) \quad \text{for } g \in D_0,$$

is equivalent to T: $T^- = A^{-1}TA$, where $Af(\omega) = f(-\omega)$.

Proposition 5.1. *The representation T of the group P_0 is canonical.*

Proof. It is clear that the representations T_r are pairwise non-equivalent. Thus, it suffices to check the estimate

$$\|T_r(g)\mathbb{1} - \mathbb{1}\| < c(g)r \quad \text{for every } g \in P_0, \tag{5.4}$$

where $\mathbb{1}$ stands for the vector $f(\omega) \equiv 1$. Since $T_r(g)\mathbb{1} = \mathbb{1}$ for $g \in D_0$, it suffices to prove (5.4) only for the elements $g = \gamma \in N$. For these elements the estimate follows from the obvious equality

$$\|T_r(g)\mathbb{1} - \mathbb{1}\|^2 = 2\int_{S^{n-2}} \bigl(1 - \cos(r\langle\gamma,\omega\rangle)\bigr)\, d\omega.$$

5.3. The special representation of the group *P*. The special irreducible representation $\widetilde{T}$ of P associated with T acts in the direct integral of the Hilbert spaces $H_r = H$ with respect to the measure $d^*r = r^{-1}\,dr$ on $\mathbb{R}_+^*$,

$$\mathscr{H} = \int_0^\infty H_r\, d^*r,$$

that is, in the space of sections $f(r)$ of the fibre bundle over $\mathbb{R}_+^*$ with fibre H_r. The operators corresponding to elements of the subgroup P_0 act in the fibres of this fibre bundle, $\bigl(\widetilde{T}(g)f\bigr)(r) = T_r(gf(r))$ for $g \in P_0$, and the operators corresponding to elements of the subgroup $\mathbb{R}_+^*$ are defined by the formula

$$\bigl(\widetilde{T}(r_0)f\bigr)(r) = f(r_0 r).$$

The non-trivial 1-cocycle $b\colon P \to \mathscr{H}$ associated with this representation will be written in the form

$$b(g) = \widetilde{T}(g)f_0(r,\omega) - f_0(r,\omega), \quad \text{where } f_0(r,\omega) = e^{-r/2}. \tag{5.5}$$

Proposition 5.2. *The functions $\|b(g)\|^2$ and $c(g) = \langle b(g), f_0\rangle$ are given by the following formulae*:

$$\|b(g)\|^2 = \log\frac{(r_0+1)^2}{4r_0}, \quad c(g) = \log\frac{2}{r_0+1} \quad \textit{for } g = r_0 \in \mathbb{R}_+^*; \tag{5.6}$$

$$\|b(g)\| = 2c(g) = \int_0^{\pi/2} (1 + |\gamma|^2\cos^2 t)\, \sin^{n-3} t\, dt \quad \textit{for } g = \gamma \in N. \tag{5.7}$$

Proof. Let us use the equality

$$\int_0^\infty (e^{-ar} - e^{-br}) r^{-1}\, dr = \log\left(\frac{b}{a}\right) \quad \text{for } \operatorname{Re} a, \operatorname{Re} b > 0. \tag{5.8}$$

We have

$$\|b(r_0)\|^2 = \int_0^\infty \int_{S^{n-2}} \bigl(e^{-r_0 r} - 2e^{-(r_0+1)r/2} + e^{-r}\bigr)\, d\omega\, d^*r,$$
$$c(g) = \int_0^\infty \int_{S^{n-2}} \bigl(e^{-(r_0+1)r/2} - e^{-r}\bigr)\, d\omega\, d^*r.$$

In view of (5.8), this implies (5.6).

Further, we have

$$\|b(\gamma)\|^2 = \int_0^\infty \int_{S^{n-2}} \bigl(2e^{-r} - e^{-(1+i\langle\gamma,\omega\rangle)r} - e^{-(1-i\langle\gamma,\omega\rangle)r}\bigr)\, d\omega\, d^*r.$$

Integrating first with respect to r, we see in view of (5.8) that

$$\|b(\gamma)\|^2 = \int_{S^{n-2}} \log(1 + \langle\gamma,\omega\rangle^2)\, d\omega.$$

Converting to spherical coordinates and integrating over S^{n-3}, we obtain

$$\|b(\gamma)\|^2 = \int_0^\pi \log(1 + |\gamma|^2 \cos^2 t) \sin^{n-3} t\, dt.$$

A similar calculation gives the expression for $c(g)$.

5.4. Extension of the special representation of the group P to a representation of the group $\mathrm{O}(n,1)$. In order to construct this extension, we first describe the realization of the special representation of $\mathrm{O}(n,1)$ on the space of functions on $N \cong \mathbb{R}^{n-1}$; in what follows, we identify elements of N with points $x \in \mathbb{R}^{n-1}$. Using the decomposition $\mathrm{O}(n,1) = P^+N$, where $P^+ \cong P$ is the subgroup of *upper* block-triangular matrices, we can interpret N as a section of the fibre bundle $\mathrm{O}(n,1) \to P^+ \setminus \mathrm{O}(n,1)$. Thus, on N there is an action $x \mapsto xg$ of the group $\mathrm{O}(n,1)$:

$$xg = \left(-\frac{|x|^2}{2} g_{13} + xg_{23} + g_{33}\right)^{-1} \left(-\frac{|x|^2}{2} g_{12} + xg_{22} + g_{32}\right), \tag{5.9}$$

where the g_{ij} are elements of a block matrix $g \in \mathrm{O}(n,1)$. In particular,

$$xg = x + x_0 \quad \text{for } g = x_0 \in N; \qquad xg = \varepsilon^{-1}\gamma u \quad \text{for } g = \operatorname{diag}(\varepsilon^{-1}, u, \varepsilon);$$
$$xg = -\frac{2x}{|x|^2} \quad \text{for } g = s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & e_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Further, we define a function $\beta(x,g)$ by the formula

$$\beta(x,g) = \left|-\frac{|x|^2}{2} g_{13} + xg_{23} + g_{33}\right|, \qquad x \in \mathbb{R}^{n-1}, \quad g \in \mathrm{O}(n,1). \tag{5.10}$$

In particular, $\beta(x,g)=1$ for $g\in N$; $\beta(x,g)=|\varepsilon|$ for $g=\operatorname{diag}(\varepsilon^{-1},u,\varepsilon)$; $\beta(x,s)=|x|^2/2$.

Definition 8 (see [24]). The special representation of the group $\mathrm{O}(n,1)$ is realized on the Hilbert space $\widetilde{\mathscr{H}}$ of functions $\varphi(x)$ on $\mathbb{R}^{n-1}$ satisfying the condition

$$\int_{\mathbb{R}^{n-1}}\varphi(x)\,dx=0$$

(where dx is the Lebesgue measure on $\mathbb{R}^{n-1}$), with the inner product

$$\langle\varphi_1,\varphi_2\rangle=-\int_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}}\log|x'-x''|\,\varphi_1(x')\overline{\varphi_2(x'')}\,dx'\,dx''. \tag{5.11}$$

The operators of this representation have the form

$$\bigl(T(g)\varphi\bigr)(x)=\varphi(xg)\beta^{1-n}(x,g). \tag{5.12}$$

In particular,

$$\bigl(T(g)\varphi\bigr)(x)=\varphi(x+x_0)\quad\text{for } g=x_0\in N; \tag{5.13}$$

$$\bigl(T(g)\varphi\bigr)(x)=|\varepsilon|^{1-n}\varphi(\varepsilon^{-1}\gamma u)\quad\text{for } g=\operatorname{diag}(\varepsilon^{-1},u,\varepsilon); \tag{5.14}$$

$$\bigl(T(g)\varphi\bigr)(x)=\varphi\biggl(-\frac{2x}{|x|^2}\biggr)\biggl(\frac{|x|^2}{2}\biggr)^{1-n}\quad\text{for } g=s=\begin{pmatrix}0&0&1\\0&e_{n-1}&0\\1&0&0\end{pmatrix}. \tag{5.15}$$

The fact that these operators are unitary and satisfy the group property follows from the relations

$$\beta(x,g_1g_2)=\beta(x,g_1)\,\beta(xg_1,g_2) \tag{5.16}$$

for any $x\in\mathbb{R}^{n-1}$ and $g_1,g_2\in\mathrm{O}(n,1)$,

$$d(xg)=\beta^{1-n}(x,g)\,dx \tag{5.17}$$

for any $g\in\mathrm{O}(n,1)$, and

$$|x'-x''|^2=|x'g-x''g|^2\beta(x',g)\beta(x'',g) \tag{5.18}$$

for any $x',x''\in\mathbb{R}^{n-1}$ and $g\in\mathrm{O}(n,1)$. It is convenient to define a non-trivial 1-cocycle of this representation by the formula

$$b(g,x)=T(g)\varphi_0-\varphi_0, \tag{5.19}$$

where $\varphi_0(x)$ is the Fourier transform of the function $e^{-|\gamma|/2}$ on $\mathbb{R}^{n-1}$; the motivation for such a choice of $\varphi_0(x)$ will be explained below.

The required realization of the special representation of the group $\mathrm{O}(n,1)$ is obtained by passing from functions $\varphi(x)$ to their Fourier transforms

$$f(\gamma)=\int_{\mathbb{R}^{n-1}}\varphi(x)e^{i\langle\gamma,x\rangle}\varphi(x)\,dx.$$

One can easily check that under this transformation the space $\widetilde{\mathscr{H}}$ turns into the Hilbert space of functions on $\mathbb{R}^{n-1}$ with the norm given in spherical coordinates on $\mathbb{R}^{n-1}$ by the formula

$$\|f\|^2 = \int_0^\infty \int_{S^{n-2}} |f(r,\omega)|^2 \, d\omega \, d^*r,$$

that is, into the space $\mathscr{H}$ of the special representation of the group P. Further, it is clear that in this new realization the operators corresponding to elements of the subgroup P have the form

$$\big(\widetilde{T}(g)\varphi\big)(\gamma) = e^{-i\langle\gamma,\gamma_0\rangle} f(\gamma) \quad \text{for } g = \gamma_0 \in N, \tag{5.20}$$

$$\big(\widetilde{T}(g)\varphi\big)(\gamma) = f(\varepsilon\gamma u) \quad \text{for } g = \operatorname{diag}(\varepsilon^{-1}, u, \varepsilon), \tag{5.21}$$

that is, they coincide with the operators of the original special representation of P. Thus, the resulting representation of the group $\mathrm{O}(n,1)$ is the required extension to $\mathrm{O}(n,1)$ of the original special representation of P.

Further, it is clear that the 1-cocycle in the space of functions $\varphi(x)$ defined by (5.19) turns into the 1-cocycle $b(g) = \widetilde{T}(g)f_0 - f_0$ of the original representation, where $f_0(r) = e^{-r/2}$.

The operators of the extension obtained can be written in an integral form:

$$\big(\widetilde{T}(g)\varphi\big)(\gamma) = \int_{\mathbb{R}^{n-1}} A(\gamma,\gamma',g) f(\gamma') \, d\gamma', \tag{5.22}$$

where

$$A(\gamma,\gamma',g) = \int_{\mathbb{R}^{n-1}} \exp\big(i(\langle\gamma,x\rangle - \langle\gamma',xg\rangle)\big)\beta^{1-n}(x,g) \, dx. \tag{5.23}$$

These expressions simplify only for elements of the subgroup P.

5.5. Description of the integral model of representation INT T of the current group P^X associated with the representation T of the group P_0. According to the general construction, the representation $\mathrm{INT}\,T$ of P^X is realized on the direct integral of the Hilbert spaces $\mathscr{H}_\xi$ with respect to the measure $\mathscr{L}$,

$$\mathrm{INT}\,H = \int_{l^1_+(X)}^{\oplus} \mathscr{H}_\xi \, d\mathscr{L}(\xi),$$

where the $\mathscr{H}_\xi$, $\xi = \{r_k, x_k\}$, are countable tensor powers of the Hilbert space $H_r = H$ of functions $f(\omega)$ on S^{n-2} with stabilizing vector $f(\omega) \equiv 1$:

$$\mathscr{H}_\xi = \bigotimes_{k=1}^\infty H_{r_k}, \qquad H_{r_k} = H.$$

Thus, elements of the space $\mathrm{INT}\,H$ are sections $F(\xi)$ of the fibre bundle over $l^1_+(X)$ with fibre $\mathscr{H}_\xi$.

The operators $U(g)$, $g \in P_0^X$, act in the fibres $\mathscr{H}_\xi$ as

$$U(g(\cdot)) = \bigotimes_{k=1}^\infty T_{r_k}(g(x_k)). \tag{5.24}$$

The operators $U(r_0(\cdot))$, $r_0 \in \mathbb{R}^X$, are given by the formula

$$\bigl(U(r_0(\cdot))F\bigr)(\xi) = \exp\biggl(\frac{1}{2}\int_X \log r_0(x)\,dm(x)\biggr) F\bigl(r_0(\cdot)\xi\bigr). \tag{5.25}$$

5.6. A formula for the spherical function and the relation between the representation INT T of the group P^X and its Fock representation. According to § 4, the spherical function of the representation INT T of P^X is defined by the formula

$$\Psi(g) = \langle U(g)\Omega, \Omega\rangle, \quad \text{where } \Omega(\xi) = \bigotimes_{k=1}^{\infty}(e^{-r_k/2}h_{r_k}) \quad \text{for } \xi = \{r_k, x_k\}.$$

Theorem 5.1. *The spherical function of the representation* INT T *is*

$$\Psi(g) = \exp\biggl(-\frac{1}{2}\int_X \|b(g(x))\|^2\,dm(x)\biggr), \tag{5.26}$$

where $b(g)$ is the 1-cocycle of the special representation of the group $\mathrm{O}(n,1)$.

Indeed, in the case of $\mathrm{O}(n,1)$ we have $\operatorname{Im} c(g) = 0$, so that (5.26) follows immediately from the general formula (4.21) for the spherical function of an integral model.

According to the general construction of Fock models, the spherical function $\Phi(g) = \langle U(g)\,\mathrm{EXP}\,0, \mathrm{EXP}\,0\rangle$ of the Fock representation of P^X associated with the representation $\widetilde{T}$ of P and the 1-cocycle b is given by the same formula (5.26). Thus, Theorem 5.1 implies the following result.

Corollary. *The integral model of representation* INT T *of the group P^X is equivalent to the Fock representation of P^X associated with the representation $\widetilde{T}$ of the group P and the 1-cocycle b. The intertwining operator for these representations is generated by the map $\Omega \mapsto \mathrm{EXP}\,0$ of the cyclic vectors.*

5.7. Extension of the integral model of representation of the group P^X to a representation of the group $\mathrm{O}(n,1)^X$. Let $\widetilde{T}$ be the extension (described in § 5.4) to $\mathrm{O}(n,1)$ of the special representation $\widetilde{T}$ of the group P, and let $b(g) = \widetilde{T}(g)f_0 - f_0$, where $f_0(r) = e^{-r/2}$, be a non-trivial cocycle.

According to § 4.6, the extension of the representation INT T of P^X on INT H to a representation of the group $\mathrm{O}(n,1)^X$ is constructed as follows. In the space INT H we consider the total set of vectors F_g, $g \in \mathrm{O}(n,1)^X$, of the form

$$F_g(\xi) = \exp\biggl(-\int_X c(g(x))\,dm(x)\biggr)\bigotimes_{k=1}^{\infty}\bigl(\widetilde{T}(g(x_k))f_0\bigr)(r_k) \quad \text{for } \xi = \{r_k, x_k\}, \tag{5.27}$$

where $c(g) = \langle b(g), f_0\rangle$. Note that $\operatorname{Im} c(g) = 0$.

The vectors F_g lie in the space INT H, and

$$\langle F_{g_1}, F_{g_2}\rangle = \exp\biggl(\int_X c(g_1(x), g_2(x))\,dm(x)\biggr), \quad \text{where } c(g_1, g_2) = \langle b(g_1), b(g_2)\rangle.$$

We define the action of the operators $U(g)$, $g \in \mathrm{O}(n,1)^X$, on the set of vectors of the form F_g by the formula

$$U(g)F_{g_1} = \exp\biggl(-\int_X \lambda(g(x), g_1(x))\, dm(x)\biggr) F_{gg_1},$$

where

$$\lambda(g, g_1) = \frac{1}{2}\, \|b(g)\|^2 + \langle \widetilde{T}(g) b(g_1), b(g)\rangle.$$

According to § 4.6, these operators preserve the inner products $\langle F_{g_1}, F_{g_2}\rangle$ and generate a representation of $\mathrm{O}(n,1)^X$ on $\mathrm{INT}\, H$ which is an extension of the original representation of P^X.

Since $\operatorname{Im} c(g) = 0$, this representation is a true (non-projective) representation of $\mathrm{O}(n,1)^X$.

6. Integral models of representations of the current group $\mathrm{SL}(2,\mathbb{R})^X$

We consider the subgroup $P \subset \mathrm{SL}(2,\mathbb{R})$ of real matrices of the form $g = \begin{pmatrix} \alpha^{-1} & 0 \\ \gamma & \alpha \end{pmatrix}$. Let us write its elements as $g = \varepsilon(r,\gamma)$, where $\varepsilon = \pm 1$ and $(r,\gamma) = \begin{pmatrix} r^{-1/2} & 0 \\ r^{1/2}\gamma & r^{1/2} \end{pmatrix}$, $r > 0$. With this notation the group operation on P takes the form

$$(r_1, \gamma_1)(r_2, \gamma_2) = \bigl(r_1 r_2, r_2^{-1}\gamma_1 + \gamma_2\bigr).$$

The group P can be written as the semidirect product $P = \mathbb{R}_+^* \ltimes P_0$ of commutative groups, where $P_0 = \{\pm 1\} \times \mathbb{R}$ is the subgroup of elements $\varepsilon(1,\gamma)$ and $\mathbb{R}_+^*$ is the subgroup of pairs $(r, 0)$, $r > 0$. The group $\mathbb{R}_+^*$ acts on P_0 by the transformations $\varepsilon(1,\gamma) \to \varepsilon(r,0)(1,\gamma)(r,0)^{-1} = \varepsilon(1, r\gamma)$.

6.1. The canonical representations of the subgroup P_0 and the associated representations of the group P. The group P_0 has a unique, up to passage to conjugate representations, canonical irreducible orthogonal representation T^0 on a two-dimensional space H^0. Upon complexification, it splits into the direct sum of two canonical unitary representations $T^\pm$ on spaces $H^\pm \cong \mathbb{C}$, and H^0 is the subspace of $H^+ \oplus H^-$ consisting of the vectors $(x, \bar{x}) \in \mathbb{C}^2$.

The operators $T^\pm(\varepsilon(1,\gamma))$ act by multiplication by $e^{\pm i\gamma}$. Accordingly, the operators $T_r^\pm(\varepsilon(1,\gamma))$, $r \in \mathbb{R}_+^*$, of the conjugate representations act by multiplication by $e^{\pm ir\gamma}$. The fact that the representations $T^\pm$ are canonical follows from the relation $|e^{\pm ir\gamma} - 1| \sim |\gamma| r$ as $r \to 0$.

The representations $T^\pm$ of P_0 give rise to special irreducible unitary representations $\widetilde{T}^\pm$ of P. They act in the complex Hilbert space $\mathscr{H} = \mathscr{H}^\pm$ of functions $f(r)$ on the half-line $r > 0$, with the norm

$$\|f\|^2 = \int_0^\infty |f(r)|^2\, d^*r, \qquad d^*r = r^{-1}\, dr,$$

and they are given by the formulae $\widetilde{T}^\pm(\varepsilon) = \mathrm{id}$ (the triviality of the operators of $\widetilde{T}^\pm$ on the centre of the group) and

$$\bigl(\widetilde{T}^\pm(r_0, r_0\gamma) f\bigr)(r) = e^{\pm i r_0 r\gamma} f(r_0 r). \tag{6.1}$$

In particular,

$$\big(\widetilde{T}^{\pm}(1,\gamma)f\big)(r) = e^{\pm ir\gamma} f(r), \tag{6.2}$$

$$\big(\widetilde{T}^{\pm}(r_0,0)f\big)(r) = f(r_0 r), \qquad r_0 \in \mathbb{R}_+^*. \tag{6.3}$$

These representations have non-trivial 1-cocycles $b^{\pm}\colon P \to \mathscr{H}$ whose structure will be discussed in the next subsection.

The orthogonal canonical representation T^0 of P_0 gives rise to a special orthogonal representation $\widetilde{T}^0$ of P on the space $\mathscr{H}^0 \subset \mathscr{H}^+ \oplus \mathscr{H}^-$ of functions $f(r)\colon \mathbb{R}_+^* \to H^0$ with the norm

$$\|f\|^2 = \int_0^\infty \|f(r)\|^2 \, d^* r.$$

The operators $\widetilde{T}^0(g)$ are obtained by restricting to $\mathscr{H}^0$ the operators $\widetilde{T}^+(g) \oplus \widetilde{T}^-(g)$ on the space $\mathscr{H}^+ \oplus \mathscr{H}^-$.

6.2. Extension of the representations $\widetilde{T}^{\pm}$ and $\widetilde{T}^0$ of the group P to representations of the group $\mathrm{SL}(2,\mathbb{R})$. We will introduce dense invariant subspaces of the spaces of the special representations of P. Thus, it will suffice to construct the required extensions only on these subspaces.

Let us begin with the case of the representation $\widetilde{T}^+$ of P on the space $\mathscr{H}^+$.

Denote by L the upper complex half-plane ($\operatorname{Im} z > 0$) with the action of the group $\mathrm{SL}(2,\mathbb{R})$:

$$z \mapsto gz = \frac{\delta z + \gamma}{\beta z + \alpha} \quad \text{for } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

(the Lobachevskii plane); in particular, $gz = z$ for $g = \pm e$ (where e is the identity element of the group) and

$$gz = r(z+\gamma) \quad \text{for } g = (r,\gamma) \in P. \tag{6.4}$$

We associate with each point $z = u + iv \in L$ a function $f_z(r)$ on the right half-line:

$$f_z(r) = e^{irz} = e^{-r(v-iu)}, \quad \text{where } z = u+iv,\ v > 0.$$

In particular,

$$f_{z_0}(r) = e^{-r} \quad \text{for } z_0 = i.$$

Proposition 6.1. *The functions $f_{z_1} - f_{z_2}$ lie in the space $\mathscr{H}^+$, and the inner product of a pair of these functions is given by*

$$\langle f_{z_1} - f_{z_2},\, f_{z_1'} - f_{z_2'} \rangle = \sum_{j,k=1,2} (-1)^{j+k-1} c(z_j, z_k'), \tag{6.5}$$

where

$$c(z_1,z_2) = \log\big(-i(z_1 - \bar z_2)\big) = \log\big((v_1+v_2) + i(u_1-u_2)\big) \quad \textit{for } z_k = u_k + iv_k. \tag{6.6}$$

In particular,

$$\|f_{z_1} - f_{z_2}\|^2 = \log \frac{|z_1 - \bar z_2|^2}{4 \operatorname{Im} z_1 \operatorname{Im} z_2}\,. \tag{6.7}$$

Hereafter, log stands for the principal branch of the logarithm with $\log 1 = 0$ on the plane cut along the negative real axis.

Proof. It follows from the definition of the inner product in $\mathscr{H}^+$ that if $z_j = u_j + iv_j$ and $z'_j = u'_j + iv'_j$, $j = 1, 2$, then

$$\langle f_{z_1} - f_{z_2}, f_{z'_1} - f_{z'_2} \rangle = \int_0^\infty \Big(\sum_{j,k=1,2} (-1)^{j+k} \exp\big(-r((v_1 + v_2) - i(u_1 - u_2))\big) \Big) r^{-1}\, dr.$$

The convergence of this integral and the formula (6.5) follow from the relation

$$\int_0^\infty (e^{-ar} - e^{-br})\, r^{-1}\, dr = \log b - \log a \quad \text{for } \operatorname{Re} a, \operatorname{Re} b > 0.$$

Corollary. *The following equality holds*:

$$\langle f_{z_1} - f_{z_0}, f_{z_2} - f_{z_0} \rangle = c(z_1) + \overline{c(z_2)} - c(z_1, z_2) - c(z_0), \tag{6.8}$$

where

$$c(z) = c(z, z_0), \qquad z_0 = i. \tag{6.9}$$

In particular,

$$\|f_z - f_{z_0}\|^2 = 2 \operatorname{Re} c(g) - c(z, z) - c(z_0). \tag{6.10}$$

Remark. The expression for $\|f_{z_1} - f_{z_2}\|^2$ can be written in the form

$$\|f_{z_1} - f_{z_2}\|^2 = 2 \log \left[\cosh \frac{d(z_1, z_2)}{2} \right], \tag{6.11}$$

where $d(z_1, z_2)$ is the Lobachevskii distance between z_1 and z_2.

Indeed, let us transform the expression $I = \dfrac{4 \operatorname{Im} z_1 \operatorname{Im} z_2}{|z_1 - \bar z_2|^2}$ using the formula $\tanh \dfrac{d(z_1, z_2)}{2} = \dfrac{|z_1 - z_2|}{|z_1 - \bar z_2|}$. We have

$$I = \frac{|z_1 - \bar z_2|^2 - |z_1 - z_2|^2}{|z_1 - \bar z_2|^2} = 1 - \tanh^2 \frac{d(z_1, z_2)}{2} = \cosh^{-2} \frac{d(z_1, z_2)}{2}.$$

This implies (6.11).

Definition 9. Denote by M^+ the pre-Hilbert subspace of $\mathscr{H}^+$ linearly spanned by the functions $f_{z_1} - f_{z_2}$, or, equivalently, by the functions $f_z - f_{z_0}$.

The following assertion is a consequence of (6.4).

Proposition 6.2. *The subspace M^+ is invariant under the action of the operators corresponding to elements of the group P; namely, for any $z_1, z_2 \in L$*

$$\widetilde{T}^+(g)(f_{z_1} - f_{z_2}) = f_{gz_1} - f_{gz_2}. \tag{6.12}$$

It is also clear that the subspace M^+ is dense in $\mathscr{H}^+$. Thus, in order to extend the representation $\widetilde{T}^+$ of P to a representation of the group $\mathrm{SL}(2, \mathbb{R})^X$, it suffices to define the action of the operators of the representation only on this subspace.

Definition 10. We define the operators $\widetilde{T}^{+}(g)$ for $g \in \mathrm{SL}(2,\mathbb{R})$ on elements of the space M^{+} by the same formula (6.12).

Theorem 6.1. *The operators $\widetilde{T}^{+}(g)$, $g \in \mathrm{SL}(2,\mathbb{R})$, preserve the inner product on M^{+} and satisfy the group property. Thus, they generate an extension of the representation $\widetilde{T}^{+}$ of the group P to a unitary representation of the group* $\mathrm{SL}(2,\mathbb{R})$.

Proof. The group relation for these operators is obvious. Further, the relation

$$gz_1 - g\bar{z}_2 = (z_1 - \bar{z}_2)(\beta z_1 + \alpha)^{-1}\,(\beta \bar{z}_2 + \alpha)^{-1} \quad \text{for every } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

implies that

$$\sum_{j,k=1,2} (-1)^{j+k-1} c(gz_j, gz'_k) = \sum_{j,k=1,2} (-1)^{j+k-1} c(z_j, z'_k) \quad \text{for every } g \in \mathrm{SL}(2,\mathbb{R}), \tag{6.13}$$

that is, the inner product on M^{+} is invariant under $\widetilde{T}^{+}(g)$ for $g \in \mathrm{SL}(2,\mathbb{R})$.

Proposition 6.3. *The representation $\widetilde{T}^{+}$ of the group* $\mathrm{SL}(2,\mathbb{R})$ *has a non-trivial* 1-*cocycle $b^{+}\colon \mathrm{SL}(2,\mathbb{R}) \to \mathscr{H}^{+}$ of the form*

$$b^{+}(g) = f_{gz_0} - f_{z_0}, \quad \textit{where } z_0 = i. \tag{6.14}$$

Indeed, $b^{+}(g) \in \mathscr{H}^{+}$ for every $g \in \mathrm{SL}(2,\mathbb{R})$. Further, since $f_{gz_0} = \widetilde{T}^{+}(g) f_{z_0}$, it follows that $b^{+}(g) = \widetilde{T}^{+}(g) f_{z_0} - f_{z_0}$, so that $b^{+}(g)$ is a 1-cocycle. Since $f_{z_0} \notin \mathscr{H}^{0}$, this 1-cocycle is non-trivial.

The formula for the inner product in H implies the following assertion.

Proposition 6.4. *For any $g_1, g_2 \in \mathrm{SL}(2,\mathbb{R})$*

$$\langle b(g_1), b(g_2)\rangle = c(g_1 z_0) + \overline{c(g_2 z_0)} - c(g_1 z_0, g_2 z_0) - c(z_0, z_0); \tag{6.15}$$

in particular,

$$\|b(g)\|^2 = 2\,\mathrm{Re}\, c(gz_0) - c(gz_0, gz_0) - c(z_0, z_0). \tag{6.16}$$

Corollary. *The following relation holds*:

$$\begin{aligned} -\frac{1}{2}\,\|b(g)\|^2 - \langle \widetilde{T}^{+}(g) b(g_1), b(g)\rangle = i\,\mathrm{Im}\, c(gz_0) + \bigl(c(gg_1 z_0, gz_0) - c(gg_1 z_0, z_0)\bigr) \\ - \frac{1}{2}\bigl(c(gz_0, gz_0) - c(z_0, z_0)\bigr). \end{aligned} \tag{6.17}$$

In a similar way we can construct an extension of the representation $\widetilde{T}^{-}$ of P to a unitary representation of $\mathrm{SL}(2,\mathbb{R})$. Namely, we replace the space M^{+} of functions of the form $f_{z_1} - f_{z_2}$ with the space M^{-} of functions of the form $\overline{f_{z_1}} - \overline{f_{z_2}}$. Obviously, M^{-} is total in $\mathscr{H}^{-}$ and invariant under the action of the operators corresponding to elements of the group P:

$$\widetilde{T}^{-}(g)(\overline{f_{z_1}} - \overline{f_{z_2}}\,) = \overline{f_{gz_1}} - \overline{f_{gz_2}}. \tag{6.18}$$

The same formula (6.18) defines an extension of the representation $\widetilde{T}^-$ of P to the whole group $\mathrm{SL}(2,\mathbb{R})$. The non-trivial 1-cocycle b^- of the representation obtained is related to the 1-cocycle b^+ by the equality $b^-(g)=\overline{b^+(g)}$.

The extensions thus defined of the representations $\widetilde{T}^\pm$ of P to unitary representations of $\mathrm{SL}(2,\mathbb{R})$ induce an extension of the orthogonal representation of P on the space $\mathscr{H}^0\subset\mathscr{H}^+\oplus\mathscr{H}^-$ to an orthogonal representation of $\mathrm{SL}(2,\mathbb{R})$. The non-trivial 1-cocycle associated with this representation is

$$b^0(g)=\bigl(b^+(g),b^-(g)\bigr).$$

6.3. The integral models of representations of $\boldsymbol{P^X}$ associated with canonical representations of $\boldsymbol{P_0}$. The spaces $H_r^\pm$ of the representations $T_r^\pm$ of P_0 are one-dimensional, so the countable tensor products $H_\xi^\pm$, $\xi=\{r_k,x_k\}$, of these spaces are also one-dimensional. Thus, the representations $\mathrm{INT}\,T^\pm$ of P^X act in the direct integrals with respect to $\mathscr{L}$ of the one-dimensional spaces $H_\xi^\pm$, that is, in the Hilbert spaces $\mathrm{INT}\,H^\pm$ of complex-valued functionals $F^\pm(\xi)=F^\pm(\{r_k,x_k\})$ on $l_+^1(X)$ with the norm

$$\|F\|^2=\int_{l_+^1(X)}|F(\xi)|^2\,d\mathscr{L}(\xi).$$

The operators of these representations are equal to the identity on the centre of P^X and are uniquely determined by the formulae

$$\bigl(U^\pm(1,\gamma(\cdot))F^\pm\bigr)(\xi)=\exp\Bigl(\pm i\sum r_k\gamma(x_k)\Bigr)F^\pm(\xi), \tag{6.19}$$

$$\bigl(U^\pm(r_0(\cdot),0)F^\pm\bigr)(\xi)=\exp\biggl(\frac12\int_X\log r_0(x)\,dm(x)\biggr)F^\pm(r_0(\cdot)\,\xi) \tag{6.20}$$

for $\xi=\{r_k,x_k\}$.

Let us proceed to the description of the orthogonal representation $\mathrm{INT}\,T^0$ of P^X associated with the canonical representation T^0 of P_0. Here elements of the spaces $H_r^0=H^0$ of the representations T_r^0 are vectors $(s,\bar s)$, $s\in\mathbb{C}$, so that the spaces H_ξ^0 of the representations of the group P_0^X are countable tensor products of two-dimensional real spaces with stabilizing vector $2^{-1/2}(1,1)$. Obviously, $H_\xi^0\subset H_\xi^0\otimes H_\xi^0$.

Thus, the representation $\mathrm{INT}\,T^0$ associated with the canonical representation T^0 of P_0 is realized on the real orthogonal space $\mathrm{INT}\,H^0$ of functionals $F(\xi)$ on $l_+^1(X)$ with values in the spaces H_ξ^0 equipped with the norm

$$\|F\|^2=\int_{l_+^1(X)}\|F(\xi)\|^2\,d\mathscr{L}(\xi).$$

The operators $U^0(1,\gamma(\cdot))$ are given by the formula

$$U^0(1,\gamma(\cdot))\biggl(\bigotimes_{k=1}^\infty(s_k,\overline{s_k})\biggr)=\bigotimes_{k=1}^\infty\bigl(e^{ir_k^2\gamma(x_k)}s_k,e^{-ir_k^2\gamma(x_k)}\overline{s_k}\bigr), \tag{6.21}$$

and the operators $U^0(r_0(\cdot),0)$, as in the case of the representations $U^{\pm}$, are given by (6.20).

Note that under the natural embedding

$$\mathrm{INT}\, H^0 \subset \mathrm{INT}\, H^+ \otimes \mathrm{INT}\, H^-,$$

the operators $U^0(g)$ are the restrictions to $\mathrm{INT}\, H^0$ of the operators $U^+(g)\otimes U^-(g)$ on the space $\mathrm{INT}\, H^+ \otimes \mathrm{INT}\, H^-$.

6.4. Extension of the unitary representations $U^{\pm} = \mathrm{INT}\, T^{\pm}$ of the current group P^X to projective unitary representations of the group $\mathrm{SL}(2,\mathbb{R})^X$. Let us construct an extension of the unitary representation $U^+ = \mathrm{INT}\, T^+$ of P^X on the space $U^+ = \mathrm{INT}\, H^+$ to a unitary projective representation of $\mathrm{SL}(2,\mathbb{R})^X$. As in the case of the group of coefficients $\mathrm{SL}(2,\mathbb{R})$, the action of the current group $\mathrm{SL}(2,\mathbb{R})^X$ will be defined on some total subset $\widetilde{M}^+ \subset \mathrm{INT}\, H^+$.

Denote by L^X, where L is the upper complex half-plane, the space of bounded functions $z\colon X \to L$, $z(x) = u(x) + iv(x)$, $v(x) > 0$. The action of the group $\mathrm{SL}(2,\mathbb{R})$ on L induces a pointwise action on L^X of the current group $\mathrm{SL}(2,\mathbb{R})^X$.

We associate with each function $z \in L^X$ the functional $F_z(\xi) = F_z(\{r_k, x_k\})$ on $l^1_+(X)$ given by

$$F_z^+(\xi) = \exp\Bigl(i\sum r_k z(x_k)\Bigr) = \exp\Bigl(-\sum r_k\bigl(v(x_k) - iu(x_k)\bigr)\Bigr) \quad \text{for } z = u+iv. \tag{6.22}$$

It follows from the definition of the characteristic functional of the measure $\mathscr{L}$ that

$$\langle F_{z_1}^+, F_{z_2}^+\rangle = \exp\Bigl(-\int_X c\bigl(z_1(x), z_2(x)\bigr)\, dm(x)\Bigr), \tag{6.23}$$

where $c(z_1,z_2)$ is given by (6.6). In particular,

$$\|F_z^+\|^2 = \exp\Bigl(-\int_X \log(2v(x))\, dm(x)\Bigr) \quad \text{for } z = u+iv.$$

Since the functions $z \in L^X$ are bounded, the functionals F_z^+ lie in the space $\mathrm{INT}\, H^+$, and one can easily check that the set of them is total in $\mathrm{INT}\, H^+$.

Definition 11. We define the action of the operators $U^+(g)$ for $g \in \mathrm{SL}(2,\mathbb{R})^X$ on the set of functionals F_z^+ by the formula

$$U^+(g)F_z^+ = \exp\Bigl(\int_X \varphi(g(x), z(x))\, dm(x)\Bigr) F_{gz}^+, \tag{6.24}$$

where

$$\varphi(g,z) = c(gz, gz_0) - c(z,z_0) - \frac{1}{2}\bigl(c(gz_0, gz_0) - c(z_0,z_0)\bigr), \qquad z_0 = i. \tag{6.25}$$

Proposition 6.5. *On the elements of the subgroup P^X the operators $U^+(g)$ coincide with the operators of the original representation of P^X.*

Proof. For $g = (r_0, \gamma) \in P$, we have $c(gz_1, gz_2) = \log r_0 + c(z_1, z_2)$. It follows that the factor in the formula (6.24) for $U^+(g)$ is equal to one for $g \in P_0$ and to $\exp\bigl(\frac{1}{2}\int_X \log r_0(x)\, dm(x)\bigr)$ for $g = \bigl(r_0(\cdot), \gamma(\cdot)\bigr) \in P^X$. Further, we have $F^+_{gz}(\xi) = \exp\bigl(i\sum r_k\gamma(x_k)\bigr)F^+_z(\xi)$ for $g = \bigl(1, \gamma(\cdot)\bigr)$ and $F^+_{gz}(\xi) = F^+_z\bigl(r_0(\cdot)\xi\bigr)$ for $g = \bigl(r_0(\cdot), 0\bigr)$. Therefore,

$$U^+(g)F^+_z(\xi) = \exp\biggl(i\sum r_k\gamma(x_k)\biggr)F^+_z(\xi) \quad \text{for } g = \bigl(1, \gamma(\cdot)\bigr),$$
$$U^+(g)F^+_z(\xi) = \exp\biggl(\frac{1}{2}\int_X \log r_0(x)\, dm(x)\biggr)F^+_z\bigl(r_0(\cdot)\xi\bigr) \quad \text{for } g = \bigl(r_0(\cdot), 0\bigr)$$

for every $z \in L^X$. The proposition follows.

Let us check that the operators U^+ determine an extension of the representation of the group P^X to the whole group $\mathrm{SL}(2,\mathbb{R})^X$. For this, replace the set of functionals of the form F^+_z with the set $\widetilde{M}^+$ of functionals of the form

$$\Psi^+_g = 2^{-1/2}\exp\biggl(\int_X c(z(x))\, dm(x)\biggr)F_z, \qquad g \in \mathrm{SL}(2,\mathbb{R})^X, \quad \text{where } z = gz_0.$$

Proposition 6.6. *On the set $\widetilde{M}^+$ the inner product and the operators of the representation are given by the following formulae:*

$$\langle \Psi^+_{g_1}, \Psi^+_{g_2}\rangle = \exp\biggl(\int_X \langle b(g_1(x)), b(g_2(x))\rangle\, dm(x)\biggr), \tag{6.26}$$

where $b(g)$ is the 1-cocycle $P \to \mathscr{H}^+$ defined by (6.14);

$$U^+(g_1)\Psi^+_g = \exp\biggl(-\int_X u\bigl(g_1(x), g(x)\bigr)\, dm(x)\biggr)\Psi^+_{g_1 g}, \tag{6.27}$$

where

$$u(g_1, g) = i\,\mathrm{Im}\, c(g_1 z_0) + \frac{1}{2}\,\|b(g_1)\|^2 + \langle \widetilde{T}(g_1)b(g), b(g_1)\rangle. \tag{6.28}$$

Proof. Equation (6.26) follows from (6.15). Equation (6.27) follows from (6.17) with g and g_1 interchanged.

Theorem 6.2. *The operators $U^+(g)$ preserve the inner products $\langle \Psi^+_{g_1}, \Psi^+_{g_2}\rangle$ and thus can be extended to unitary operators on the whole space* $\mathrm{INT}\, H^+$.

Proof. We have

$$\langle U^+(g)\Psi^+_{g_1}, U^+(g)\Psi^+_{g_2}\rangle = \exp\biggl(\int_X v\bigl(g(x), g_1(x), g_2(x)\bigr)\, dm(x)\biggr),$$

where

$$v(g, g_1, g_2) = -\bigl(u(g, g_1) + \overline{u(g, g_2)}\,\bigr) + \langle b(gg_1), b(gg_2)\rangle.$$

This, along with the equality

$$\langle b(gg_1), b(gg_2)\rangle = \|b(g)\|^2 + \langle \widetilde{T}^+(g)b(g_1), b(g)\rangle + \langle b(g), \widetilde{T}^+(g)b(g_2)\rangle + \langle b(g_1), b(g_2)\rangle,$$

implies that $\langle U^+(g)\Psi^+_{g_1}, U^+(g)\Psi^+_{g_2}\rangle = \exp\bigl(\int_X \langle b(g_1(x)), b(g_2(x))\rangle\, dm(x)\bigr)$.

Theorem 6.3. *The operators U^+ form a projective representation of the group* $\mathrm{SL}(2,\mathbb{R})^X$; *namely, for any $g_1, g_2 \in \mathrm{SL}(2,\mathbb{R})^X$*

$$U^+(g_1g_2) = \exp\biggl(i \operatorname{Im} \int_X a\bigl(g_1(x), g_2(x)\bigr)\, dm(x)\biggr)\, U^+(g_1)U^+(g_2), \tag{6.29}$$

where

$$a(g_1, g_2) = c(g_1z_0) + c(g_2z_0) - c(g_1g_2z_0) + \langle T(g_1)b(g_2), b(g_1)\rangle. \tag{6.30}$$

Proof. By (6.27) we have

$$U^+(g_1)U^+(g_2)\Psi_g^+ = \exp\biggl(-\int_X a_1\bigl(g_1(x), g_2(x)\bigr)\, dm(x)\biggr)\Psi^+_{g_1g_2g},$$
$$U^+(g_1g_2)\Psi_g^+ = \exp\biggl(-\int_X a_2(g_1g_2(x))\, dm(x)\biggr)F^+_{g_1g_2g},$$

where

$$a_1(g_1, g_2) = u(g_2, g) + u(g_1, g_2g), \qquad a_2(g_1g_2) = u(g_1g_2, g).$$

Hence,

$$U^+(g_1)U^+(g_2)\Psi_g = \exp\biggl(-\int_X a'\bigl(g_1(x), g_2(x)\bigr)\, dm(x)\biggr)U^+(g_1g_2)\Psi_g,$$

where

$$a'(g_1, g_2) = u(g_2, g) + u(g_1, g_2g) - u(g_1g_2, g),$$

that is, by (6.28),

$$a'(g_1, g_2) = i \operatorname{Im}\bigl(c(g_1z_0) + c(g_2z_0) - c(g_1g_2z_0)\bigr) + v(g_2, g) + v(g_1, g_2g) - v(g_1g_2, g),$$

where $v(g_1, g) = \frac{1}{2}\,\|b(g_1)\|^2 + \langle \widetilde{T}^+(g_1)b(g), b(g_1)\rangle$. This, along with the relation

$$v(g_2, g) + v(g_1, g_2g) - v(g_1g_2, g) = i \operatorname{Im}\langle \widetilde{T}^+(g_1)b(g_2), b(g_1)\rangle,$$

implies that

$$a'(g_1, g_2) = i \operatorname{Im}\bigl(c(g_1z_0) + c(g_2z_0) - c(g_1g_2z_0) + \langle \widetilde{T}^+(g_1)b(g_2), b(g_1)\rangle\bigr).$$

Thus,

$$\begin{aligned} &U^+(g_1g_2)\Psi_g \\ &\quad= \exp\biggl(i \operatorname{Im}\int_X a\bigl(g_1(x), g_2(x)\bigr)\, dm(x)\biggr)U^+(g_1)U^+(g_2)\Psi_g \text{ for every } g \in P^X, \end{aligned}$$

where $a(g_1, g_2)$ is given by (6.30) and does not depend on g. This implies (6.29). Theorem 6.3 follows.

An extension of the second representation U^- of P^X to a representation of $\mathrm{SL}(2,\mathbb{R})^X$ is obtained by replacing the total set $\widetilde{M}^+ \subset \widetilde{H}^+$ by the total set $\widetilde{M}^- \subset \widetilde{H}^-$ of functionals $\Psi_g^- = \overline{\Psi_g^+}$. The formulae for the inner products $\langle \Psi_{g_1}^-, \Psi_{g_2}^- \rangle$ and for the operators $U^-(g)$, as well as the relation between $U^-(g_1 g_2)$ and $U^-(g_1)U^-(g_2)$, are obtained from the corresponding formulae (6.26), (6.27), and (6.29) for the case of U^+ by complex conjugation:

$$\langle \Psi_{g_1}^-, \Psi_{g_2}^- \rangle = \exp\biggl(\int_X \langle b(g_2(x)), b(g_1(x)) \rangle \, dm(x) \biggr), \tag{6.31}$$

$$U^-(g_1)\Psi_g^- = \exp\biggl(-\int_X \overline{u\bigl(g_1(x), g(x)\bigr)} \, dm(x) \biggr) \Psi_{g_1 g}^-, \tag{6.32}$$

$$U^-(g_1 g_2) = \exp\biggl(-i \operatorname{Im} \int_X a\bigl(g_1(x), g_2(x)\bigr) \, dm(x) \biggr) U^-(g_1)U^-(g_2). \tag{6.33}$$

6.5. Extension of the orthogonal representation $U^0 = \mathrm{INT}\, T^0$ of the current group P^X to an orthogonal representation of the group $\mathrm{SL}(2,\mathbb{R})^X$. As mentioned above, under the natural embedding $\mathrm{INT}\, H^0 \subset \mathrm{INT}\, H^+ \otimes \mathrm{INT}\, H^-$ the operators $U^0(g)$ for $g \in P^X$ are obtained by restricting to $\mathrm{INT}\, H^0$ the operators $U^+(g) \otimes U^-(g)$ on the space $\mathrm{INT}\, H^+ \otimes \mathrm{INT}\, H^-$. Thus, the resulting extensions of the representations $U^\pm$ of P^X to representations of $\mathrm{SL}(2,\mathbb{R})^X$ induce an extension of the representation U^0 of P^X on the space $\mathrm{INT}\, H^+$ to an orthogonal representation of $\mathrm{SL}(2,\mathbb{R})^X$. Its complexification is a unitary non-projective representation $\mathrm{INT}(T^+ \oplus T^-) = \mathrm{INT}\, T^+ \otimes \mathrm{INT}\, T^-$ equivalent to the representations of $\mathrm{SL}(2,\mathbb{R})$ constructed earlier in [1].

Let us give an independent description of an extension of the representation U^0 of P^X to a representation of $\mathrm{SL}(2,\mathbb{R})^X$.

With each pair $z \in L^X$ and $(r,x) \in \mathbb{R}_+^* \times X$ we associate a vector $f_{z,r,x}^0 \in H_r^0$,

$$f_{z,r,x}^0 = 2^{-1/2}\bigl(e^{ir_k z(x_k)}, e^{-ir_k \overline{z(x_k)}}\bigr),$$

and we define functionals $F_z^0(\xi)$, $z \in L^X$, on $l_+^1(X)$ by the formula

$$F_z^0(\xi) = \bigotimes_{k=1}^{\infty} f_{z,r_k,x_k}^0. \tag{6.34}$$

Theorem 6.4. *For any $z_1, z_2 \in L^X$*

$$\begin{aligned} \langle F_{z_1}^0, F_{z_2}^0 \rangle &= \exp\biggl(-\int_X \log |z_1(x) - \overline{z_2(x)}| \, dm(x) \biggr) \\ &= \exp\biggl(-\int_X \operatorname{Re} c\bigl(z_1(x), z_2(x)\bigr) \, dm(x) \biggr). \end{aligned} \tag{6.35}$$

In particular,

$$\|F_z^0\|^2 = \exp\biggl(-\int_X \log(2 \operatorname{Im} z(x)) \, dm(x) \biggr). \tag{6.36}$$

Proof. Note that

$$\langle f^0_{z_1,r,x}, f^0_{z_2,r,x}\rangle = \frac{1}{2}\big(e^{irz(x)}, e^{-i\overline{rz(x)}}\big), \quad \text{where } z = z_1 - \overline{z_2}, \quad \text{for any } z_1, z_2 \in L^X.$$

Hence,

$$\langle F^0_{z_1}, F^0_{z_2}\rangle = \int_{l^1_+(X)} \prod_{k=1}^{\infty}\Big(\frac{1}{2}\, e^{ir_k z(x_k)} + \frac{1}{2}\, e^{-ir_k \bar z(x_k)}\Big)\, d\mathscr{L}(\xi), \quad \text{where } z = z_1 - z_2.$$

To compute the integral, we use Theorem 2.3. By this theorem, we have

$$\langle F^0_{z_1}, F^0_{z_2}\rangle = \exp\bigg(\int_X \int_0^{\infty}\Big(\frac{1}{2}\, e^{irz(x)} + \frac{1}{2}\, e^{-ir\bar z(x)} - e^{-r}\Big) r^{-1}\, dr\, dm(x)\bigg),$$

that is,

$$\langle F^0_{z_1}, F^0_{z_2}\rangle = \exp\bigg(\frac{1}{2}\int_X \big(a(x) + \overline{a(x)}\,\big)\, dm(x)\bigg),$$

where

$$a(x) = \int_0^{\infty} (e^{irz(x)} - e^{-r})\, r^{-1}\, dr.$$

Since $e^{irz(x)} = e^{-r(v(x) - iu(x))}$ for $z = u + iv$, where $v > 0$, it follows that

$$a(x) = -\log\big(v(x) - iu(x)\big).$$

This implies (6.35).

Corollary. *The functionals* F^0_z *lie in the space* $\mathrm{INT}\, H^0$.

One can easily check that the set of these functionals is total in $\mathrm{INT}\, H^0$.

Definition 12. We define the operators $U^0(g)$, $g \in \mathrm{SL}(2,\mathbb{R})^X$, on $\widetilde{M}^0$ by the formula

$$U^0(g) F^0_z = \exp\bigg(\mathrm{Re}\int_X \varphi(g(x), z(x))\, dm(x)\bigg)\, F^0_{gz}, \tag{6.37}$$

where $\varphi(g,z)$ is given by (6.25). In other words, the formula for $U^0(g)$ is obtained from the formula for $U^{\pm}(g)$ by replacing the function $\varphi(g,z)$ with its real part.

As in the case of $U^{\pm}$, on the elements of P^X these operators coincide with the operators of the original representation of P^X.

By analogy with the case of $U^{\pm}$, we consider the total set $\widetilde{M}^0 \subset \mathrm{INT}\, H^0$ of functionals of the form

$$\Psi^0_g = 2^{-1/2} \exp\bigg(\mathrm{Re}\int_X c(z(x))\, dm(x)\bigg) F^0_z, \qquad g \in \mathrm{SL}(2,\mathbb{R})^X, \quad \text{where } z = gz_0.$$

Proposition 6.7. *On the set* $\widetilde{M}^0$ *the inner product and the operators of the representation are given by the following formulae*:

$$\langle \Psi^0_{g_1}, \Psi^0_{g_2}\rangle = \exp\bigg(\int_X \langle b^0(g_1(x)), b^0(g_2(x))\rangle\, dm(x)\bigg), \tag{6.38}$$

where $b^0(g)$ *is the* 1*-cocycle* $P \to \mathscr{H}^0$ *defined by*

$$b^0(g) = 2^{-1/2}\big(b(g), \overline{b(g)}\,\big); \tag{6.39}$$

$$U^0(g_1)\Psi^0_g = \exp\biggl(-\int_X \widetilde{u}\big(g_1(x), g(x)\big)\,dm(x)\biggr)\Psi^0_{g_1 g}, \tag{6.40}$$

where

$$\widetilde{u}(g_1, g) = \operatorname{Re} u(g_1, z) = \frac{1}{2}\,\|b(g_1)\|^2 + \langle \widetilde{T}^0(g_1)b^0(g), b^0(g_1)\rangle. \tag{6.41}$$

Proof. It follows from the definition of the functionals Ψ^0 that

$$\langle \Psi^0_{g_1}, \Psi^0_{g_2}\rangle = \exp\biggl(\operatorname{Re}\int_X \langle b(g_1(x)), b(g_2(x))\rangle\,dm(x)\biggr),$$

$$U^0(g_1)\Psi^0_g = \exp\biggl(-\operatorname{Re}\int_X u\big(g_1(x), g(x)\big)\,dm(x)\biggr)\Psi^0_{g_1 g},$$

where $u(g_1, g)$ is given by (6.28). It remains to observe that $\|b(g)\| = \|b^0(g)\|$ and $\operatorname{Re}\langle \widetilde{T}^+(g_1)b(g), b(g_1)\rangle = \langle \widetilde{T}^0(g_1)b^0(g), b^0(g_1)\rangle$.

By analogy with Theorems 6.2 and 6.3, we obtain the following assertion.

Theorem 6.5. *The operators* $U^0(g)$ *preserve the inner products* $\langle \Psi^0_{g_1}, \Psi^0_{g_2}\rangle$ *and can be extended to orthogonal (non-projective) operators on the whole space* $\mathrm{INT}\,H^0$.

6.6. The relation to the Fock representations of the group $\mathrm{SL}(2,\mathbb{R})^X$. We establish a connection between the integral model of representation U^+ of $\mathrm{SL}(2,\mathbb{R})^X$ and the Fock representation of this group.

By definition, the Fock representation V^+ of $\mathrm{SL}(2,\mathbb{R})^X$ associated with the unitary representation $\widetilde{T}^+$ of $\mathrm{SL}(2,\mathbb{R})$ on $\mathscr{H}^+$ and the 1-cocycle $b\colon \mathrm{SL}(2,\mathbb{R}) \to \mathscr{H}^+$ acts in the Hilbert space $\widetilde{H} = \mathrm{EXP}\,\mathscr{H}^X$, where

$$\mathrm{EXP}\,\mathscr{H}^X = \bigoplus_{k=0}^{\infty} S^k \mathscr{H}^X$$

and

$$\mathscr{H}^X = \int_X^{\oplus} \mathscr{H}^+_x\,dm(x), \qquad \mathscr{H}^+_x = \mathscr{H}^+.$$

Let $\mathscr{M}^+ \subset \mathrm{EXP}\,\mathscr{H}^X$ be the total subset of vectors of the form $\Phi^+_{g_1} = \mathrm{EXP}\,b^X(g_1)$, $g_1 \in \mathrm{SL}(2,\mathbb{R})^X$, where $b^X(g)$ is the 1-cocycle $\mathrm{SL}(2,\mathbb{R})^X \to \mathscr{H}^X$ generated by the 1-cocycle $b^+\colon \mathrm{SL}(2,\mathbb{R}) \to \mathscr{H}^+$.

On this subset the inner products and the operators of the Fock representation are given by the formulae

$$\langle \Phi^+_{g_1}, \Phi^+_{g_2}\rangle = \exp\biggl(\int_X \langle b(g_1(x)), b(g_2(x))\rangle\,dm(x)\biggr), \tag{6.42}$$

$$V^+(g)\Phi^+_{g_1} = \exp\biggl(\int_X v\big(g(x), g_1(x)\big)\,dm(x)\biggr)\Phi^+_{g g_1}, \tag{6.43}$$

where

$$v(g,g_1)=-\frac{1}{2}\,\|b(g)\|^2-\langle T(g)b(g_1),b(g)\rangle \quad \text{for any } g,g_1\in \mathrm{SL}(2,\mathbb{R}). \tag{6.44}$$

Theorem 6.6. *The extension to* $\mathrm{SL}(2,\mathbb{R})^X$ *of the integral model of representation* U^+ *of the group* P^X *on the space* $\mathrm{INT}\,H^+$ *is projectively equivalent to the Fock model of representation* V^+ *of* $\mathrm{SL}(2,\mathbb{R})^X$.

Indeed, the bijection $\Psi_g^+\mapsto\Phi_g^+$ of the total subsets $\widetilde{M}^+$ and $\mathscr{M}^+$ in the spaces of these representations preserves the inner products, and the formulae for the corresponding operators $U^+(g)$ and $V^+(g)$ differ only by a factor:

$$U^+(g)=\exp\biggl(-i\int_X c(g(x)z_0)\,dm(x)\biggr)V^+(g).$$

A similar argument holds for the integral model U^-.

Now let us compare the integral model of the representation U^0 of $\mathrm{SL}(2,\mathbb{R})^X$ and the Fock representation V^0 of this group associated with the orthogonal representation $\widetilde{T}^0$ of $\mathrm{SL}(2,\mathbb{R})$ on the space $\mathscr{H}^0$ and the 1-cocycle $b^0\colon \mathrm{SL}(2,\mathbb{R})\to\mathscr{H}^0$.

In this case the Fock representation V^0 is a true (non-projective) representation, and on the corresponding total subsets M^0 and $\mathscr{M}^0$ in the spaces of U^0 and V^0 the inner products and the formulae for the operators coincide. This implies the following theorem.

Theorem 6.7. *The extension to* $\mathrm{SL}(2,\mathbb{R})^X$ *of the integral model of orthogonal representation* U^0 *of* P^X *on the space* $\widetilde{H}^0$ *is equivalent to the Fock model of representation* V^0 *of* $\mathrm{SL}(2,\mathbb{R})^X$. *The intertwining operator for these representations is generated by the map* $\Psi_e^0\mapsto\Phi_e^0=\mathrm{EXP}\,0$ *of the cyclic vectors.*

6.7. Addendum: Unitary representations of the group $\widetilde{G}^X$, where $\widetilde{G}$ is the universal cover of the group $G=\mathrm{SL}(2,\mathbb{R})$. In this subsection G stands for the group $\mathrm{SL}(2,\mathbb{R})$ and $\widetilde{G}$ for the universal cover of G, that is, the covering space over G in which the fibre over an element $g\in G$ is the set $\mathbb{Z}$ of homotopy classes of paths in G from the identity element e to g. Elements of $\widetilde{G}$ will be denoted by $\tilde{g}$, and their images in G by g.

Since G is a quotient of $\widetilde{G}$, the integral models of projective representations $U^\pm$ of G^X induce projective representations $\widetilde{U}^\pm$ of the current group $\widetilde{G}^X$ on the same Hilbert spaces $\mathrm{INT}\,H^\pm$. We will show that the projective representations of $\widetilde{G}^X$ thus defined are projectively equivalent to unitary non-projective representations $V^\pm$ of $\widetilde{G}^X$ on the same spaces $\widetilde{H}^\pm$. Let us describe these representations $V^\pm$; for definiteness, we restrict ourselves to the case of $V^+=V$. To describe the representation V, it suffices to determine the action of the operators of this representation on the elements of the total subset of functionals of the form F_z.

We introduce a function $\psi(\tilde{g},z)$ on $\widetilde{G}\times L$, where L is the upper half-plane. Let

$$\varphi(g,z)=-\log(\beta z+\alpha) \quad \text{for } g=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in G \text{ and } z\in L,$$

where, as above, log is the branch of the logarithm with $\log 1=0$ on the plane cut along the negative real axis. The function φ is everywhere finite, and for every fixed $z \in L$ it is a single-valued analytic function of $g \in G$ in a sufficiently small neighbourhood of the identity element e. Hence for each $g \in G$ and each path $\tilde{g}$ in G from e to g, this function can be analytically continued along the path. Denote this analytic continuation by $\psi(\tilde{g}, z)$. The function $\psi(\tilde{g}, z)$ defined in this way depends only on the homotopy class of the path $\tilde{g}$, and thus is a function on $\widetilde{G} \times L$.

It follows from the definition that

$$\psi(\tilde{g}, z) = -\log(\beta z + \alpha) \quad \text{for } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{6.45}$$

provided that $g \in G$ and the path $\tilde{g}$ from e to g lies in a sufficiently small neighbourhood of the identity element e.

Proposition 6.8. *For any $\tilde{g}_1, \tilde{g}_2 \in \widetilde{G}$ and $z \in L^+$*

$$\psi(\tilde{g}_1 \tilde{g}_2, z) = \psi(\tilde{g}_1, g_2 z) + \psi(\tilde{g}_2, z). \tag{6.46}$$

Proof. Let us use the equality

$$\beta z + \alpha = \big(\beta_1(g_2 z) + \alpha_1\big)(\beta_2 z + \alpha_2),$$

where (α_1, β_1), (α_2, β_2), and (α, β) are elements of the matrices g_1, g_2, and $g_1 g_2$, respectively. For g_1, g_2 sufficiently close to the identity element, this equality implies that

$$\log(\beta z + \alpha) = \log\big(\beta_i(g_2 z) + \alpha_1\big) + \log(\beta_2 z + \alpha_2).$$

Thus, in view of (6.45), the desired relation (6.46) holds for elements g_1, g_2 and paths $\tilde{g}_1$, $\tilde{g}_2$ sufficiently close to the identity element. Hence it is preserved under analytic continuation with respect to g, that is, it remains valid for any $\tilde{g}_1$, $\tilde{g}_2$. Proposition 6.8 follows.

With each pair $\tilde{g} \in \widetilde{G}^X$ and $z \in L^X$, we associate the following function on X:

$$\Psi_{\tilde{g},z}(x) = \psi\big(\tilde{g}(x), z(x)\big). \tag{6.47}$$

Proposition 6.8 implies the next assertion.

Proposition 6.9. *The functions $\Psi_{\tilde{g},z}$ are related by*

$$\Psi_{\tilde{g}_1 \tilde{g}_2, z} = \Psi_{\tilde{g}_1, g_2 z} + \Psi_{\tilde{g}_2, z}. \tag{6.48}$$

Definition 13. We define the action of the operators $V(\tilde{g})$, $\tilde{g} \in \widetilde{G}^X$, on the functions F_z of the total set M by the formula

$$V(\tilde{g}) F_z = \exp\bigg(\int_X \Psi_{\tilde{g},z}(x)\, dm(x) \bigg) F_{gz}. \tag{6.49}$$

Theorem 6.8. *The operators $V(\tilde{g})$ are unitary on M, that is,*

$$\langle V(\tilde{g}) F_{z_1}, V(\tilde{g}) F_{z_2} \rangle = \langle F_{z_1}, F_{z_2} \rangle \quad \textit{for any } z_1, z_2 \in L^X \textit{ and } \tilde{g} \in \widetilde{G}^X, \tag{6.50}$$

and they satisfy the relation

$$V(\tilde{g}_1 \tilde{g}_2) F_z = V(\tilde{g}_1) V(\tilde{g}_2) F_z \quad \textit{for any } \tilde{g}_1, \tilde{g}_2 \in \widetilde{G}^X \textit{ and } z \in L^X. \tag{6.51}$$

Thus, they generate a unitary (non-projective) representation of the group $\widetilde{G}^X$ on the space $\widetilde{H}$.

Proof. The group property (6.51) follows at once from Proposition 6.9. Namely,

$$V(\tilde g_1)\,V(\tilde g_2)\,F_z = \exp\biggl(\int_X \bigl(\Psi_{\tilde g_1, g_2 z}(x) + \Psi_{\tilde g_2, z}(x)\bigr)\,dm(x)\biggr)F_{g_1 g_2 z}$$
$$= \exp\biggl(\int_X \Psi_{\tilde g_1 \tilde g_2, z}(x)\,dm(x)\biggr)F_{g_1 g_2 z} = U(\tilde g_1 \tilde g_2)F_z.$$

Further, since the group $\widetilde G$ is simply connected, it suffices to establish the unitarity (6.50) only for elements $\tilde g \in \widetilde G^X$ sufficiently close to the identity element. From the definition of the operators $V(\tilde g)$ and (6.23) it follows that

$$\langle V(\tilde g)F_{z_1}, V(\tilde g)F_{z_2}\rangle = \exp\biggl(\int_X u(\tilde g, z_1, z_2)\,dm(x)\biggr),$$

where

$$u(\tilde g, z_1, z_2) = \Psi_{\tilde g, z_1} + \overline{\Psi_{\tilde g, z_2}} - c(gz_1, gz_2).$$

Under our assumption we have by (6.45) that

$$\Psi_{\tilde g, z_1} + \overline{\Psi_{\tilde g, z_2}} = -\log(\beta z_1 + \alpha) - \log(\beta \bar z_2 + \alpha).$$

On the other hand, the equation

$$gz_1 - \overline{gz_2} = \frac{z_1 - \bar z_2}{(\beta z_1 + \alpha)(\beta \bar z_2 + \alpha)}$$

implies that

$$c(gz_1, gz_2) = c(z_1, z_2) - \log(\beta z_1 + \alpha) - \log(\beta \bar z_2 + \alpha).$$

Thus, $u(\tilde g, z_1, z_2) = -c(z_1, z_2)$. This implies (6.50).

Theorem 6.9. *The constructed representation V of the group $\widetilde G^X$ is projectively equivalent to the representation $\widetilde U$ of this group.*

This assertion follows immediately from the formulae for the operators of these representations on the total set M^+.

7. Integral models of representations of the current group $\mathrm{U}(n,1)^X$

7.1. Initial definitions. Let us realize $\mathrm{U}(n,1)$ as the group of linear transformations on $\mathbb{C}^{n+1}$ that preserve the Hermitian form $x_1\bar x_{n+1} + x_{n+1}\bar x_1 + |x_2|^2 + \dots + |x_n|^2$, and write its elements as block matrices

$$g = \|g_{ij}\|_{i,j=1,2,3},$$

where the diagonal contains matrices of orders 1, $n-1$, and 1, respectively. In this realization, the maximal parabolic subgroup P is the group of lower block-triangular matrices and can be written as the semidirect product

$$P = D \leftthreetimes N,$$

where N is the maximal nilpotent subgroup consisting of the block matrices of the form

$$h = \begin{pmatrix} 1 & 0 & 0 \\ -z^* & e_{n-1} & 0 \\ it - \dfrac{zz^*}{2} & z & 1 \end{pmatrix}, \qquad t \in \mathbb{R}, \quad z \in \mathbb{C}^{n-1}$$

(the Heisenberg group of dimension $2n-1$) and $D \cong \mathbb{C}^* \times U(n-1)$ is the group of block-diagonal matrices of the form $d = \operatorname{diag}(\bar{s}^{-1}, u, s)$, $s \in \mathbb{C}^*$, $u \in U(n-1)$.

We write D as the direct product $D = \mathbb{R}_+^* \times D_0$, where D_0 is the subgroup of matrices of the form $d = \operatorname{diag}(\varepsilon, u, \varepsilon)$, $|\varepsilon| = 1$, and we set

$$P_0 = D_0 \leftthreetimes N.$$

Thus,

$$P = \mathbb{R}_+^* \leftthreetimes P_0 = (\mathbb{R}_+^* \times D_0) \leftthreetimes N.$$

Elements of D_0 and N will be identified with pairs (ε, u) with $\varepsilon \in U(1)$ and $u \in U(n-1)$, and pairs (t, z) with $t \in \mathbb{R}$ and $z \in \mathbb{C}^{n-1}$ (a row vector), respectively. Sometimes instead of $(t, z) \in N$ we will also write (ζ, z), where $\zeta = it - |z|^2/2$. With this notation the group relations take the form

$$\begin{aligned} (\zeta_1, z_1)(\zeta_2, z_2) &= (\zeta_1 + \zeta_2 - z_1 z_2^*, z_1, + z_2), \\ (\varepsilon, u)^{-1}(\zeta, z)(\varepsilon, u) &= (\zeta, \bar{\varepsilon} z u), \\ r\,(\zeta, z) r^{-1} &= (r^2 \zeta, rz) \quad \text{for } r \in \mathbb{R}_+^*. \end{aligned}$$

7.2. Description of the canonical irreducible representations of P_0. Up to conjugacy with respect to the group $\mathbb{R}_+^*$ of automorphisms, there are two countable series $T_m^{\pm}$, $m = 0, 1, \ldots$, of canonical irreducible representations of P_0. Let us first describe the representations T_m^+.

We consider a unitary representation of P_0 on the Hilbert space $H = H^+$ of functions $f(z)$ on $\mathbb{C}^{n-1}$ with the norm

$$\|f\|^2 = \int_{\mathbb{C}^{n-1}} |f(z)|^2 \exp(-zz^*)\, d\mu(z), \qquad zz^* = \sum z_i \bar{z}_i, \tag{7.1}$$

where $d\mu(z)$ is the Lebesgue measure on $\mathbb{C}^{n-1}$ normalized by the condition

$$\int_{\mathbb{C}^{n-1}} \exp(-zz^*)\, d\mu(z) = 1.$$

The operators of the representation T^+ of the group P_0 are given by the formulae

$$\big(T^+(g) f\big)(z) = \exp(\zeta_0 - z z_0^*) f(z + z_0) \quad \text{for } g = (\zeta_0, z_0) \in N, \tag{7.2}$$

$$\big(T^+(g) f\big)(z) = f(\bar{\varepsilon} z u) \quad \text{for } g = (\varepsilon, u) \in D_0. \tag{7.3}$$

Correspondingly, the operators of the representations T_r^+ conjugate to T^+ with respect to the group $\mathbb{R}_+^*$ of automorphisms are given by the formulae

$$\big(T_r^+(g) f\big)(z) = \exp\big(r^2 \zeta_0 - r z z_0^*\big) f(z + r z_0) \quad \text{for } g = (\zeta_0, z_0) \in N, \tag{7.4}$$

$$\big(T_r^+(g) f\big)(z) = f(\bar{\varepsilon} z u) \quad \text{for } g = (\varepsilon, u) \in D_0. \tag{7.5}$$

We note that the multiplier $e^{\zeta_0 - zz_0^*}$ in (7.4) is an entire analytic function of z. It follows that *the space H^+ is the direct sum*

$$H^+ = \bigoplus_{m=1}^{\infty} H_m^+$$

of irreducible pairwise non-equivalent invariant subspaces, where H_m^+ is the invariant subspace cyclically generated by the homogeneous polynomials in $\bar{z}_1, \dots, \bar{z}_{n-1}$ of homogeneity degree m. In particular, elements of the space H_0^+ are entire analytic functions on $\mathbb{C}^{n-1}$.

Denote by $T_m^+(g)$ the restrictions of the operators $T^+(g)$, $g \in P_0$, to the subspaces H_m^+.

Proposition 7.1. *The representations T_m^+ of the group P_0 on the spaces H_m^+ are canonical, and each of them has a unique almost invariant vector $\varphi_m(z) = l_m^{n-2}(zz^*)$, where l_m^{n-2} is a Laguerre polynomial.*

Proof. Let us find all almost invariant vectors in H_m^+. Obviously, every such vector $f_m(z)$ is invariant under the subgroup $U(n-1)$, and thus is a function of zz^*, $f(z) = \varphi(zz^*)$. Further, observe that for every m the direct sum $\bigoplus_{k=1}^m H_k^+$ contains all vectors $(zz^*)^k$ with $k \leqslant m$, but does not contain vectors $(zz^*)^k$ with $k > m$. It follows that H_m contains a unique, up to a factor, vector $f_m(z)$ of the form $\varphi(zz^*)$, and this vector is obtained at the mth step of orthogonalization of the sequence of vectors $1, zz^*, (zz^*)^2, \dots$ with respect to the norm in H^+. One can easily see that the vector obtained by such an orthogonalization is, up to a factor, a Laguerre polynomial in zz^*: $f_m(z) = l_m^{n-2}(zz^*)$. Further, the obvious relation $\bigl(T_r^+(g) f_m\bigr)(z) = f_m(z) + O(r)$ for $g \in N$ implies that the vector $f_m(z)$ is almost invariant. Therefore, by Proposition 3.2, the representation T_m^+ is canonical. Proposition 7.1 follows.

The second family T_m^- of canonical irreducible unitary representations of the group P_0 is obtained from the family T_m^+ by complex conjugation; namely, the representation T_m^- acts in the space of functions $\overline{f(z)}$, where $f(z) \in H_m^+$, as

$$T^-(g)\bar{f} = \overline{T^+(g)f}.$$

In particular, T_0^- acts in the space of entire anti-analytic functions on $\mathbb{C}^{n-1}$.

Every pair T_m^+, T_m^- of irreducible unitary representations of P_0 gives rise to an irreducible orthogonal representation T_m^0 of P_0 on the space $H_m^0 \subset H_m^+ \oplus H_m^-$ of vectors of the form $(f, \bar{f})$, $f \in H_m^+$. The operators of this representation are defined as the restrictions to H_m^0 of the operators $T_m^+(g) \oplus T_m^-(g)$ on $H_m^+ \oplus H_m^-$, that is,

$$T_m^0(g)(f, \bar{f}) = \bigl(T_m^+(g) f, T_m^-(g)\bar{f}\bigr) = \bigl(T_m^+(g) f, \overline{T_m^+(g) f}\bigr).$$

The representations $T_m^\pm$ of P_0 give rise to pairwise non-equivalent irreducible unitary representations $\widetilde{T}_m^\pm$ of the maximal parabolic subgroup P on the spaces $\mathscr{H}_m^\pm = \int_0^\infty H_{m,r}^\pm r^{-1}\, dr$, $H_{m,r}^\pm = H_m^\pm$, and pairwise non-equivalent integral models $U_m^\pm = \mathrm{INT}\, T_m^\pm$ of irreducible unitary representations of the current group P^X. The representations $\widetilde{T}_m^{\pm}$ of P have non-trivial 1-cocycles $b_m^{\pm}\colon P \to \mathscr{H}_m^{\pm}$, which are given by

$$b_m^{\pm}(g)(g) = T^{\pm}(g) f_m(r,z) - f_m(r,z), \quad \text{where } f_m(r,z) = e^{-r}\, l_m^{n-2}(zz^*),$$

where l_m^{n-2} is a Laguerre polynomial.

Thus, the representations T_m^0 of P_0 give rise to pairwise non-equivalent irreducible orthogonal representations $\widetilde{T}_m^0$ of P on the spaces $\mathscr{H}_m^0 \subset \mathscr{H}_m^+ \oplus \mathscr{H}_m^-$ and pairwise non-equivalent integral models $U_m^0 = \mathrm{INT}\, T_m^0$ of irreducible orthogonal representations of P^X. The representations $\widetilde{T}_m^0$ of P have non-trivial 1-cocycles, which are given by $b^0(g) = \bigl(b^+(g), b^-(g)\bigr)$.

7.3. The representations of P associated with the representations $T^{\pm}$ and T^0 of P_0. Further in this section we restrict ourselves to canonical representations of the subgroup P_0 such that the associated representation of the group P can be extended to a representation of the whole group $\mathrm{U}(n,1)$. This property is satisfied for the unitary representations $T_0^{\pm}$, for the orthogonal representation T_0^0, and only for them. We describe these representations in detail. In what follows, the subscript 0 will be omitted.

The representation T^+ of P_0 is realized on the Hilbert space H^+ of *entire analytic functions* $f(z)$ on $\mathbb{C}^{n-1}$ with the norm (7.1); its operators are defined on H by (7.2), (7.3). Correspondingly, the operators of the representations of P_0 conjugate to T^+ act in the spaces $H_r^+ = H^+$ according to (7.4), (7.5).

The function $f(z) \equiv 1$ is a vector in H^+ almost invariant with respect to the family of conjugate representations T_r^+ of P_0, and it will be denoted by $\mathbb{1}$ in what follows. For this function we have

$$\|T_r^+(g)\mathbb{1} - \mathbb{1}\| < c(g) r \quad \text{for every } g \in P_0.$$

The corresponding representation $\widetilde{T}^+$ of the maximal parabolic subgroup P is realized on the direct integral of the Hilbert spaces H_r^+ with respect to the measure $d^* r = r^{-1}\, dr$ on $\mathbb{R}^+$,

$$\mathscr{H}^+ = \int_0^\infty H_r^+ \, d^* r,$$

that is, elements of $\mathscr{H}^+$ are sections $f(r)$ of the fibre bundle over $\mathbb{R}_+^*$ with fibre H_r^+ over $r \in \mathbb{R}_+^*$ that satisfy the condition

$$\int_0^\infty \|f(r)\|^2 r^{-1}\, dr < \infty.$$

In this realization the operators corresponding to elements of the subgroup P_0 act in each space H_r^+ according to (7.4), (7.5), and the homothety operators $T(r_0)$, $r_0 \in \mathbb{R}_+^*$, map H_r isometrically to $H_{r_0 r}$.

Thus, the representation $\widetilde{T}^+$ of the group P acts in the space $\mathscr{H}^+$ according to the formulae

$$\big(\widetilde{T}^+(g)f\big)(r,z)=\exp\bigg(r^2\bigg(it_0-\frac{|z_0|^2}{2}\bigg)-rzz_0^*\bigg)\,f(r,z+rz_0)\ \text{for}\ g=(t_0,z_0)\in N, \tag{7.6}$$

$$\big(\widetilde{T}^+(g)f\big)(r,z)=f(r,\bar{\varepsilon}zu)\quad \text{for}\ g=(\varepsilon,u)\in D_0, \tag{7.7}$$

$$\big(\widetilde{T}^+(g)f\big)(r,z)=f(r_0r,z)\quad \text{for}\ g=r_0\in\mathbb{R}_+^*. \tag{7.8}$$

This representation has a non-trivial 1-cocycle $b^+\colon P\to\mathscr{H}^+$:

$$b^+(g)=\widetilde{T}^+(g)\,f_0(r,z)-f_0(r,z),\quad \text{where}\ f_0(r,z)=e^{-r^2}.$$

The second canonical unitary representation T^- of P_0 and the associated unitary representation of P act, respectively, in the spaces H^- and $\mathscr{H}^-=\int_0^\infty H_r^-\,d^*r$, where H^- is the space of *entire anti-analytic functions* $f(z)$ on $\mathbb{C}^{n-1}$ with the norm (7.1).

Finally, the canonical orthogonal representation T^0 of P_0 and the associated orthogonal representation of P act, respectively, in the spaces H^0 and $\mathscr{H}^0=\int_0^\infty H_r^- r^{-1}dr$, where $H^0\subset H^+\oplus H^-$ is the subspace of vectors of the form $(f,\bar{f})$, $f\in H^+$. The operators $T^0(g)$ on H^0 are obtained by restricting to H^0 the operators $T^+(g)\oplus T^-(g)$ on $H^+\oplus H^-$, where H^- is the space of *entire anti-analytic functions* $f(z)$ on $\mathbb{C}^{n-1}$ with the norm (7.1).

7.4. The representations of P^X associated with the representations $T^\pm$ and T^0 of P_0. According to the general definitions, the unitary representations $U^\pm$ of P^X associated with the representations $T^\pm$ of P_0 are realized on the direct integrals

$$\operatorname{INT}H^\pm=\int_{l_+^1(X)}^{\oplus}H_\xi^\pm\,d\mathscr{L}(\xi),$$

where $H_\xi^\pm$, $\xi=\{r_k,x_k\}$, are countable tensor powers of the Hilbert space $H^\pm$ with stabilizing vector $\mathbb{1}$:

$$H_\xi^\pm=\bigotimes_{k=1}^\infty H_{r_k}^\pm,\qquad H_{r_k}^\pm=H^\pm.$$

Thus, elements of $\operatorname{INT}H^\pm$ are sections $F(\tilde{r})$ of the fibre bundle over D_+ with fibre $H_{\tilde{r}}^\pm$.

The operators $U^\pm(g)$, $g\in P_0^X$, act in the fibres $\mathscr{H}_{\tilde{r}}^\pm$ of this fibre bundle according to the formula

$$U^\pm(g(\cdot))=\bigotimes_{k=1}^\infty T_{r_k}^\pm(g(x_k)). \tag{7.9}$$

The operators $U^\pm(r_0(\cdot))$, $r_0\in\mathbb{R}^X$, are given by

$$\big(U^\pm(r_0(\cdot))f\big)(\xi)=\exp\bigg(\frac12\int_X\log r_0(x)\,dm(x)\bigg)f\big(r_0(\cdot)\xi\big). \tag{7.10}$$

The fact that the operators $U^{\pm}(g)$ for $g \in P_0^X$ are unitary and satisfy the group property is obvious, and the unitarity of the operators $U^{\pm}(r_0(\cdot))$, $r_0 \in \mathbb{R}^X$, follows from the projective invariance of the measure $\mathscr{L}$.

Analogously, the orthogonal representation of P^X associated with the representation T^0 of P_0 is realized on the direct integral

$$\mathrm{INT}\, H^0 = \int_{l_+^1(X)}^{\oplus} H_\xi^0 \, d\mathscr{L}(\xi),$$

where the H_ξ^0, $\xi = \{r_k, x_k\}$, are countable tensor powers of the real Hilbert space $H^0 \subset H^+ \oplus H^-$ with stabilizing vector $2^{-1/2}(\mathbb{1}, \mathbb{1})$, and the operators $U^0(r_0(\cdot))$, $r_0 \in \mathbb{R}^X$, are given by a formula similar to (7.10).

7.5. Extension of the representations $\widetilde{T}^{\pm}$ and $\widetilde{T}^0$ of the subgroup P to representations of the group $\mathrm{U}(n,1)$. The construction of these extensions is similar to that for the case of $\mathrm{SL}(2,\mathbb{R})$ considered in the previous section. First we describe the extension to $\mathrm{U}(n,1)$ of the representation $\widetilde{T}^+$ of P.

By analogy with the Lobachevskii plane in the case of $\mathrm{SL}(2,\mathbb{R})$, let us consider the homogeneous space $L = \mathrm{U}(n,1)/K$, where K is the maximal compact subgroup of $\mathrm{U}(n,1)$ (the n-dimensional complex Lobachevskii space). In the matrix realization we adopt, K is the subgroup of $\mathrm{U}(n,1)$ consisting of the block matrices of the form

$$\begin{pmatrix} \lambda & a & \mu \\ b & c & b \\ \mu & a & \lambda \end{pmatrix},$$

$$L = \{v = (a,b) \in \mathbb{C} \oplus \mathbb{C}^{n-1} \mid a + \bar{a} + b^* b < 0\}, \quad \text{where } b^* b = \sum \bar{b}_i b_i$$

(b is a column vector), and the action $v \mapsto gv$ on L of elements $g = \|g_{ij}\|_{i,j=1,2,3}$ in $\mathrm{U}(n,1)$ is defined as follows: $g(a,b) = (a', b')$, where

$$\begin{aligned} a' &= (g_{11} + g_{12} b + g_{13} a)^{-1} (g_{31} + g_{32} b + g_{33} a), \\ b' &= (g_{11} + g_{12} b + g_{13} a)^{-1} (g_{21} + g_{22} b + g_{23} a). \end{aligned}$$

In particular,

$$\begin{aligned} &g(a,b) = (a + \zeta_0 + (z_0, b),\ b - z_0^*) \quad \text{for } g = (\zeta_0, z_0) \in N, \\ &g(a,b) = (a, ub\bar{\varepsilon}) \quad \text{for } g = (\varepsilon, u) \in D_0, \\ &g(a,b) = (r^2 a, rb) \quad \text{for } g = r \in \mathbb{R}_+^*, \\ &s(a,b) = (a^{-1}, a^{-1} b) \quad \text{for the involution } s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & e_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.11}$$

The point $v_0 = (-1, 0)$ is a fixed point on L with respect to the action of K.

Consider the following function on $L \times L$:

$$p(v_1, v_2) = -(a_1 + \bar{a}_2 + b_2^* b_1) \quad \text{for } v_i = (a_i, b_i) \in L. \tag{7.12}$$

Obviously, $p(v_0, v_0) = 2$ for $v_0 = (-1, 0)$, and one can easily check that

$$\operatorname{Re} p(v_1, v_2) > 0 \quad \text{for any } v_1, v_2 \in L. \tag{7.13}$$

Proposition 7.2. *The function $p(v_1, v_2)$, where $v_1 = (a_1, b_1)$, $v_2 = (a_2, b_2) \in L$, and $g \in \mathrm{U}(n, 1)$, satisfies the relation*

$$p(gv_1, gv_2) = p(v_1, v_2)(g_{11} + g_{12}b_1 + g_{13}a_1)^{-1}\big[\,\overline{(g_{11} + g_{12}b_2 + g_{13}a_2)}\,\big]^{-1}. \tag{7.14}$$

In particular, $p(v_1, v_2)$ is invariant under the action of the subgroup P_0, and the equality $p(gv_1, gv_2) = r^2\, p(v_1, v_2)$ holds for $g = r \in \mathbb{R}_+^*$.

Proof. For elements $g \in P_0$, $r \in \mathbb{R}_+^*$, and the involution s, the above relations follow immediately from (7.11). Since these elements generate $\mathrm{U}(n, 1)$ algebraically, the relations hold for any $g \in \mathrm{U}(n, 1)$.

We associate with each $v = (a, b) \in L$ the following function on $\mathbb{R}_+^* \oplus \mathbb{C}^{n-1}$:

$$f_v(r, z) = \exp\bigl(r^2 a + r(z, b)\bigr), \quad \text{where } (z, b) = \sum z_i b_i. \tag{7.15}$$

Note that in this notation the expression for the 1-cocycle on $\mathscr{H}^+$ takes the form

$$b^+(g) = \widetilde{T}^+(g) f_{v_0} - f_{v_0}, \quad \text{where } v_0 = (-1, 0). \tag{7.16}$$

Proposition 7.3. *For any fixed $v \in L$ and r, the function $f_v(r, z)$ lies in the space H^+, and the inner product in H^+ of two such functions has the form*

$$\langle f_{v_1}, f_{v_2}\rangle_{H^+} = \exp\bigl(-r^2 p(v_1, v_2)\bigr) \quad \textit{for } v_i = (a_i, b_i), \tag{7.17}$$

where $p(v_1, v_2)$ is given by (7.12). In particular,

$$\|f_v\|^2_{H^+} = \exp\bigl(-r^2 p(v, v)\bigr) < \infty.$$

Proof. We have

$$\langle f_{v_1}, f_{v_2}\rangle_{H^+} = \int_{\mathbb{C}^{n-1}} \exp\bigl(r^2(a_1 + \bar{a}_2) + r(z, b_1) + r\overline{(z, b_2)} - zz^*\bigr)\, d\mu(z).$$

To prove (7.17), it suffices to use the equality

$$\int_{\mathbb{C}^{n-1}} \exp\bigl(r(z, b_1) + r\overline{(z, b_2)} - zz^*\bigr)\, d\mu(z) = \exp(r^2 b_2^* b_1).$$

Proposition 7.4. *For any $v_1, v_2 \in L^X$ the function $f_{v_1} - f_{v_2}$ lies in $\mathscr{H}^+$, and the inner product of functions $f_{v_1} - f_{v_2}$ and $f_{v_1'} - f_{v_2'}$ in the space $\mathscr{H}$ is equal to*

$$\langle f_{v_1} - f_{v_2},\, f_{v_1'} - f_{v_2'}\rangle = \frac{1}{2}\sum_{i,j=1,2} (-1)^{i+j-1} \log p(v_i, v_j'). \tag{7.18}$$

In particular,

$$\|f_{v_1} - f_{v_2}\|^2 = \frac{1}{2}\log\frac{|p(v_1, v_2)|^2}{p(v_1, v_1)p(v_2, v_2)} < \infty. \tag{7.19}$$

As above, log stands for the branch of the logarithm with $\log 1 = 0$ on the plane $\mathbb{C}$ cut along the negative real axis.

Proof. By definition,

$$\langle f_{v_1} - f_{v_2},\, f_{v_1'} - f_{v_2'}\rangle = \int_0^\infty \langle f_{v_1} - f_{v_2},\, f_{v_1'} - f_{v_2'}\rangle_{H_r^*} r^{-1}\, dr.$$

Hence (7.19) follows immediately from (7.15) and the relation

$$\int_0^\infty (e^{-ar^2} - e^{-br^2}) r^{-1}\, dr = \frac{1}{2}(\log b - \log a) \quad \text{for } \operatorname{Re} a, \operatorname{Re} b > 0.$$

Definition 14. Denote by M^+ the set of functions of the form $f_{v_1} - f_{v_2}$, $v_1, v_2 \in \mathscr{H}^+$, in the space $\mathscr{H}^+$.

One can easily check that the set M^+ is total in $\mathscr{H}^+$.

Proposition 7.5. *The set of functions of the form f_v, $v \in L$, and hence the set M^+, are invariant under the action of the operators corresponding to elements of the group P; namely,*

$$\widetilde{T}^+(g) f_v = f_{gv} \quad \textit{for any } v \in L \textit{ and } g \in P,$$

where $gv = g(a,b)$ is defined for $g \in P$ by (7.11).

One can easily check this assertion by comparing the expressions for $\widetilde{T}^+(g) f_v$ and f_{gv}.

Thus, the operators corresponding to elements of P act on the set M^+ according to the formula

$$\widetilde{T}^+(g)(f_{v_1} - f_{v_2}) = f_{gv_1} - f_{gv_2} \quad \text{for any } v_1, v_2 \in L. \tag{7.20}$$

Definition 15. We define the action of the operators $\widetilde{T}^+(g)$, $g \in \mathrm{U}(n,1)$, on the set M^+ by the same formula (7.20).

Theorem 7.1. *The operators $\widetilde{T}^+(g)$ preserve the inner product on M^+, satisfy the group relation, and thus define an extension of the original representation $\widetilde{T}^+$ of P to a unitary representation of* $\mathrm{U}(n,1)$.

Proof. The group property is obvious. The invariance of the inner product follows from the explicit formula (7.18) for the inner product and the relation (7.14) for the function $p(v_1, v_2)$.

The representation $\widetilde{T}^+$ of $\mathrm{U}(n,1)$ thus defined is special, and the non-trivial 1-cocycle $b^+ \colon \mathrm{U}(n,1) \to \mathscr{H}^+$ is an extension to $\mathrm{U}(n,1)$ of the 1-cocycle of P; that is, in our notation it is given, as in the case of P, by the formula $b^+(g) = T(g) f_{v_0} - f_{v_0}$.

In a similar way we define an extension of the representation $\widetilde{T}^-$ of P to a unitary representation of $\mathrm{U}(n,1)$. Namely, the set M^+ of functions of the form $f_{v_1} - f_{v_2}$ should be replaced by the set M^- of functions of the form $\overline{f_{v_1}} - \overline{f_{v_2}}$. Obviously, M^- is total in $\mathscr{H}^-$ and invariant under the action of the operators $\widetilde{T}^-(g)$ for $g \in P$:

$$\widetilde{T}^-(g)(\overline{f_{z_1}} - \overline{f_{z_2}}) = \overline{f_{gz_1}} - \overline{f_{gz_2}}. \tag{7.21}$$

The same formula (7.21) defines also an extension of the representation $\widetilde{T}^-$ of P to the whole group $\mathrm{U}(n,1)$.

Finally, these extensions of the representations $\widetilde{T}^\pm$ of P to unitary representations of $\mathrm{U}(n,1)$ induce an extension of the orthogonal representation of P on the space $\mathscr{H}^0 \subset \mathscr{H}^+ \oplus \mathscr{H}^-$ to an orthogonal representation of $\mathrm{U}(n,1)$.

In conclusion, we present some relations that will be used in what follows.

Let

$$c(v_1,v_2) = \log\bigl(p(v_1,v_2)\bigr), \qquad v_1,v_2 \in L. \tag{7.22}$$

With this notation, the invariance condition for the inner product on M^+ can be written in the form

$$\begin{aligned} &c(gv_1,gv_0) + \overline{c(gv_2,gv_0)} - c(gv_1,gv_2) - c(gv_0,gv_0) \\ &\qquad = c(v_1,v_0) + \overline{c(v_2,v_0)} - c(v_1,v_2) - c(v_0,v_0) \end{aligned} \tag{7.23}$$

for every $g \in \mathrm{U}(n,1)$, where $v_0 = (-1,0)$. Further, by Proposition 7.4 we have

$$\|b(g)\|^2 = \frac{1}{2}\bigl(2\operatorname{Re} c(gv_0,v_0) - c(gv_0,gv_0) - \log 2\bigr), \quad \text{where } v_0 = (-1,0), \tag{7.24}$$

$$\langle T(g)b(g_1), b(g)\rangle = \frac{1}{2}\bigl(c(gg_1v_0,v_0) + c(gv_0,gv_0) - c(gg_1v_0,gv_0) - c(gv_0,v_0)\bigr). \tag{7.25}$$

It also follows from (7.24) and (7.25) that

$$\begin{aligned} -\frac{1}{2}\,\|b(g)\|^2 - \langle T(g)b(g_1), b(g)\rangle &= \frac{i}{2}\operatorname{Im} c(gz_0,v_0) \\ &+ \frac{1}{2}\bigl(c(gg_1v_0,gv_0) - c(gg_1v_0,v_0)\bigr) - \frac{1}{4}\bigl(c(gv_0,gv_0) - c(v_0,v_0)\bigr). \end{aligned} \tag{7.26}$$

7.6. Extension of the unitary representations $U^\pm$ of the current group P^X to projective unitary representations of the group $\mathrm{U}(n,1)^X$. The construction of these extensions is analogous to the case of $\mathrm{SL}(2,\mathbb{R})$. We describe the extension to $\mathrm{U}(n,1)^X$ of the representation U^+ of the group P^X.

Denote by L^X the space of bounded functions $v\colon X \to L$, $v(x) = \bigl(a(x), b(x)\bigr)$. The action of the group $\mathrm{U}(n,1)$ on L induces a pointwise action on L^X of the current group $\mathrm{U}(n,1)^X$ and, in particular, of the subgroup P^X.

We associate with each function $v(x) = \bigl(a(x), b(x)\bigr) \in L^X$ a functional F^+ on $l^1_+(X)$. Namely, we associate with each function $v(x) = \bigl(a(x), b(x)\bigr) \in L^X$ and each pair $(r,x) \in \mathbb{R}^*_+ \times X$ an entire function on $\mathbb{C}^{n-1}$,

$$f_{v,r,x}(z) = \exp\bigl(r^2 a(x) + r(z, b(x))\bigr), \tag{7.27}$$

and we define functionals $F^+_v(\xi) = F^+_v(\{r_k, x_k\})$ on $l^1_+(X)$ by the formula

$$F^+_v(\xi) = \bigotimes_{k=1}^{\infty} f_{v,r_k,x_k} \quad \text{for } \xi = \{r_k, x_k\}. \tag{7.28}$$

Proposition 7.6. *For each $v \in L^X$ and each $\xi \in l^1_+(X)$, the infinite tensor product* (7.28) *converges and lies in the space H^+_ξ. For any $v_1, v_2 \in L^X$*

$$\langle F^+_{v_1}(\xi), F^+_{v_2}(\xi)\rangle_{H^+_\xi} = \exp\Bigl(-\sum r_k^2 p\bigl(v_1(x_k), v_2(x_k)\bigr)\Bigr), \tag{7.29}$$

where the function $p(v_1, v_2)$ is defined by (7.12).

Proof. First of all, note that the functions $f_{v,r_k,x_k}(z)$ lie in the spaces $H^+_{r_k} = H^+$, and for any $v_1, v_2 \in L^X$ we have by (7.15) that

$$\langle f_{v_1,r_k,x_k}, f_{v_2,r_k,x_k}\rangle_{H^+} = \exp\bigl(-r^2 p(v_1(x_k), v_2(x_k))\bigr). \tag{7.30}$$

In particular, $\|f_v\|^2_{H^+} = \exp\bigl(-r^2 p(v,v)\bigr) < \infty$.

Further, Proposition 7.1 implies that the functions f_{v,r_k,x_k} satisfy the estimate

$$\|f_{v,r_k,x_k} - \mathbb{I}\| < c\, r_k.$$

This estimate implies that the infinite tensor product $F^+_v(\xi)$ converges and lies in H^+_ξ. Now (7.29) follows immediately from (7.30).

Note that (7.29) can be written in the form

$$\langle F^+_{v_1}(\xi), F^+_{v_2}(\xi)\rangle_{H^+_{\tilde r}} = \exp\bigl(-\langle \tilde r^2, p(v_1, v_2)\rangle\bigr). \tag{7.31}$$

Theorem 7.2. *For every $v \in L^X$, the functional $F^+_v(\xi)$ on $l^1_+(X)$ with values in H^+_ξ lies in the space* INT H^+, *and for any $v_1, v_2 \in L^X$*

$$\langle F^+_{v_1}, F^+_{v_2}\rangle = c\exp\biggl(-\frac{1}{2}\int_X \log p\bigl(v_1(x), v_2(x)\bigr)\, dm(x)\biggr), \qquad c = e^{\gamma/2} \tag{7.32}$$

(*where γ is Euler's constant*).

Indeed, (7.32) follows immediately from (7.31) and the relation (2.9) for the measure $\mathscr{L}$, according to which

$$\int_{l^1_+(X)} \exp\Bigl(-\sum r_k^2 a(x_k)\Bigr)\, d\mathscr{L}(\xi) = e^{\gamma/2}\exp\biggl(-\frac{1}{2}\int_X \log a(x)\, dm(x)\biggr).$$

The convergence of the integral on the right-hand side of (7.32) follows from the boundedness of the function $v \in L^X$.

Denote by M^+ the subset of functionals $F_v \in \operatorname{INT} H^+$, $v \in L^X$, defined in this way. One can easily check that this subset is total in $\operatorname{INT} H^+$.

Theorem 7.3. *The set M^+ is invariant under the operators corresponding to elements of the subgroup P^X_0, namely, $U^+(g)F^+_v = F^+_{gv}$ for every $v = (a, b) \in L^X$, where $gv = \bigl(a + \zeta_0 + (z_0, b),\, b - z_0^*\bigr)$ for $g = (\zeta_0, z_0) \in N^X$ and $gv = (a, ub\bar\varepsilon)$ for $g = (\varepsilon, u) \in D^X_0$.*

For $g = r_0 \in (\mathbb{R}^*_+)^X$

$$U^+(r_0)F^+_v = \exp\biggl(\int_X \log r_0(x)\, dm(x)\biggr) F^+_{gv}, \quad \text{where } gv = (r_0^2 a, r_0 b).$$

The assertions follow immediately from the formulae for the operators of U^+ and the definition of F_v^+.

Definition 16. We define the action of the operators $U^+(g)$, $g \in \mathrm{U}(n,1)^X$, on the set M^+ by the formula

$$U^+(g)F_v^+ = \exp\biggl(\frac{1}{2}\int_X \varphi\bigl(g(x), v(x)\bigr)\,dm(x)\biggr)F_{gv}^+, \tag{7.33}$$

where

$$\varphi(g,v) = \bigl(c(gv, gv_0) - c(v, v_0)\bigr) - \frac{1}{2}\bigl(c(gv_0, gv_0) - c(v_0, v_0)\bigr). \tag{7.34}$$

Observe that this definition is similar to the corresponding definition of the functionals F_z^+ for the case of the group $\mathrm{SL}(2,\mathbb{R})^X$ (see (6.24) and (6.25)). The function $c(v_1, v_2)/2$ satisfies the same relations as the function $c(z_1, z_2)$ in the case of $\mathrm{SL}(2,\mathbb{R})$. Hence the assertions and constructions based on this definition in the case of $\mathrm{SL}(2,\mathbb{R})$ can be carried over to the case of $\mathrm{U}(n,1)$. We describe them more briefly.

First of all, the restrictions of the operators $U^+(g)$ to the subgroup P^X coincide with the operators of the original representation of P^X.

Further, as in the case of $\mathrm{SL}(2,\mathbb{R})$, we replace the set of functionals F_v^+ by the set $\widetilde{M}^+$ of functionals of the form

$$\Psi_g^+ = (2c)^{-1/2}\exp\biggl(\frac{1}{2}\int_X c(v(x), v_0)\,dm(x)\biggr)F_z, \quad g \in \mathrm{U}(n,1)^X,$$

where $v = gv_0$ and $c = e^{\gamma/2}$.

On this set the inner product and the operators of the representation are given by the following formulae:

$$\langle \Psi_{g_1}^+, \Psi_{g_2}^+\rangle = \exp\biggl(\int_X \langle b^+(g_1(x)), b^+(g_2(x))\rangle\,dm(x)\biggr), \tag{7.35}$$

where $b^+(g)$ is the 1-cocycle $G \to \mathscr{H}^+$ defined by (7.16);

$$U^+(g_1)\Psi_g^+ = \exp\biggl(-\int_X u\bigl(g_1(x), g(x)\bigr)\,dm(x)\biggr)\Psi_{g_1 g}^+, \tag{7.36}$$

where

$$u(g_1, g) = \frac{i}{2}\operatorname{Im} c(g_1 v_0, v_0) + \frac{1}{2}\,\|b(g_1)\|^2 + \langle \widetilde{T}(g_1)b(g), b(g_1)\rangle. \tag{7.37}$$

As in the case of $\mathrm{SL}(2,\mathbb{R})$, this implies the following theorem.

Theorem 7.4. *The operators $U^+(g)$ preserve the inner products $\langle \Psi_{g_1}^+, \Psi_{g_2}^+\rangle$, and their extensions to the whole space $\mathrm{INT}\, H^+$ form a projective unitary representation of the group $\mathrm{U}(n,1)^X$.*

The extension of the second representation U^- of P^X to a representation of $\mathrm{U}(n,1)^X$ is obtained by replacing the total set $\widetilde{M}^+ \subset \mathrm{INT}\, H^+$ by the total set $\widetilde{M}^- \subset \mathrm{INT}\, H^-$ of functionals $\Psi_g^- = \overline{\Psi_g^+}$. Obviously, the inner products $\langle \Psi_{g_1}^+, \Psi_{g_2}^+ \rangle$ and $\langle \Psi_{g_1}^-, \Psi_{g_2}^- \rangle$ are complex conjugates. The action of the operators $U^-(g)$, $g \in \mathrm{U}(n,1)^X$, on $\mathrm{INT}\, M^-$ is given by

$$U^-(g)\Psi_g^- = \overline{U^+(g)\Psi_g^+}.$$

As in the case of U^+, the restrictions of the operators $U^-(g)$ to P^X coincide with the operators of the original representation of P^X on the space $\mathrm{INT}\, H^+$, and Theorem 7.4 also holds for them.

7.7. Extension of the orthogonal representation U^0 of the current group P^X to an orthogonal representation of the group $\mathrm{U}(n,1)^X$. We associate with each pair $v(x) = \big(a(x), b(x)\big) \in L^X$ and $(r, x) \in \mathbb{R}_+^* \times X$ the vector $f_{v,r,x}^0 \in H_r^0$ given by

$$f_{v,r,x}^0 = 2^{-1/2}\big(e^{ir^2 z(x)}, e^{-ir^2 \overline{z(x)}}\,\big),$$

and we define functionals $F_v^0(\xi)$ on $l_+^1(X)$ by

$$F_v^0(\xi) = \bigotimes_{k=1}^{\infty} f_{v,r_k,x_k}^0 \quad \text{for } \xi = \{r_k, x_k\}. \tag{7.38}$$

As in the case of $\mathrm{SL}(2,\mathbb{R})$, the infinite tensor product $\bigotimes_{k=1}^\infty f_{v,r_k,x_k}^0$ converges and $F_v^0(\xi) \in H_\xi^0$ for any $v \in L^X$ and $\xi \in l_+^1(X)$. The functionals F_v^0 thus defined lie in the space $\mathrm{INT}\, H^0$ and form a total subset M^0 in $\mathrm{INT}\, H^0$. Moreover,

$$\langle F_{v_1}^0, F_{v_2}^0 \rangle = c \exp\left(-\frac{1}{2} \operatorname{Re} \int_X \log(p(v_1(x), v_2(x))\, dm(x)\right), \qquad c = \exp\left(\frac{\gamma}{2}\right). \tag{7.39}$$

We define the operators $U^0(g)$, $g \in \mathrm{U}(n,1)^X$, on $\widetilde{M}^0$ by

$$U^0(g)F_v^0 = \exp\left(\frac{1}{2} \operatorname{Re} \int_X \varphi\big(g(x), v(x)\big)\, dm(x)\right) F_{gv}^0, \tag{7.40}$$

where $\varphi(g, v)$ is given by (7.34).

As in the case of $U^\pm$, for the elements of P^X these operators coincide with the operators of the original representation U^0 of P^X.

Further, by analogy with the case of $U^\pm$, we consider the total set $\widetilde{M}^0 \subset \mathrm{INT}\, H^0$ of functionals of the form

$$\Psi_g^0 = (2c)^{-1/2} \exp\left(\frac{1}{2} \operatorname{Re} \int_X c(v(x), v_0)\, dm(x)\right) F_v^0, \quad g \in \mathrm{U}(n,1)^X, \text{ where } v = gv_0.$$

Proposition 7.7. *On the set $\widetilde{M}^0$ the inner product and the operators of the representation are given by the following formulae*:

$$\langle \Psi_{g_1}^0, \Psi_{g_2}^0 \rangle = \exp\left(\int_X \langle b^0(g_1(x)), b^0(g_2(x)) \rangle\, dm(x)\right), \tag{7.41}$$

where $b^0(g)$ *is the* 1*-cocycle* $G \to \mathscr{H}^0$ *defined by*

$$b^0(g) = 2^{-1/2}\big(b(g), \overline{b(g)}\,\big); \tag{7.42}$$

$$U^0(g_1)\Psi^0_g = \exp\biggl(-\int_X \widetilde{u}\big(g_1(x), g(x)\big)\,dm(x)\biggr)\Psi^0_{g_1 g}, \tag{7.43}$$

where

$$\widetilde{u}(g_1, g) = \frac{1}{2}\,\|b(g_1)\|^2 + \langle \widetilde{T}^0(g_1) b^0(g), b^0(g_1)\rangle. \tag{7.44}$$

The proof is the same as in the case of $\mathrm{SL}(2,\mathbb{R})$.

From this we deduce the next theorem by analogy with Theorems 6.2 and 6.3.

Theorem 7.5. *The operators* $U^0(g)$ *preserve the inner products* $\langle \Psi^0_{g_1}, \Psi^0_{g_2}\rangle$ *and can be extended to orthogonal* (*non-projective*) *operators on the whole space* $\mathrm{INT}\,H^0$.

7.8. The relation between the integral and Fock models of representation of the group $\mathrm{U}(n,1)^X$. This relation is similar to the relation established above for the case of the group $\mathrm{SL}(2,\mathbb{R})$.

Denote by $V^{\pm}$ the Fock projective unitary representations of $\mathrm{U}(n,1)^X$ corresponding to the pairs $(\widetilde{T}^{\pm}, b^{\pm})$, where $\widetilde{T}^{\pm}$ are the special unitary representations of $\mathrm{U}(n,1)$ and $b^{\pm}\colon \mathrm{U}(n,1) \to \mathscr{H}^{\pm}$ are the 1-cocycles defined by (7.16). Similarly, denote by V^0 the Fock orthogonal representation corresponding to the pair $(\widetilde{T}^0, b^0)$.

Theorem 7.6. *The extensions to* $\mathrm{U}(n,1)^X$ *of the integral models of unitary representations* $\mathrm{INT}\,T^{\pm}$ *of the group* P^X *are projectively equivalent to the Fock projective unitary representations* $V^{\pm}$ *of the group* $\mathrm{U}(n,1)^X$.

The extension to $\mathrm{U}(n,1)^X$ *of the integral model of orthogonal representation* $\mathrm{INT}\,T^0$ *of* P^X *is equivalent to the Fock orthogonal representation* V^0 *of* $\mathrm{U}(n,1)^X$. *The intertwining operator for these representations is generated by the map* $\Psi^0_e \mapsto \mathrm{EXP}\,0$ *of the cyclic vectors.*

Proof. By definition, the Fock representation V^+ of $\mathrm{U}(n,1)$ is realized on the complex Hilbert space $\mathrm{EXP}\,\mathscr{H}^X$, where

$$\mathrm{EXP}\,\mathscr{H}^X = \bigoplus_{k=0}^{\infty} S^k \mathscr{H}^X$$

and

$$\mathscr{H}^X = \int_X^{\oplus} \mathscr{H}^+_x\,dm(x), \qquad \mathscr{H}^+_x = \mathscr{H}^+,$$

with $\mathscr{H}^+$ the space of the representation $\widetilde{T}^+$ of P.

Let us introduce in $\mathrm{EXP}\,\mathscr{H}^X$ the total subset $\mathscr{M}^+ \subset \mathrm{EXP}\,\mathscr{H}^X$ of vectors of the form

$$\Phi^+_g = \mathrm{EXP}\,b^X(g), \qquad g \in \mathrm{U}(n,1)^X,$$

where $b^X : \mathrm{U}(n,1)^X \to \mathscr{H}^X$ is the 1-cocycle generated by the 1-cocycle $b^+ : \mathrm{U}(n,1) \to \mathscr{H}$. On this set the inner product and the operators of the representation of

$\mathrm{U}(n,1)^X$ are given by the formulae

$$\langle \Phi^+_{g_1}, \Phi^+_{g_2}\rangle = \exp\biggl(\int_X \langle b^+(g_1(x)), b^+(g_2(x))\rangle\, dm(x)\biggr),$$

$$V^+(g_1)\Phi^+_g = \exp\biggl(-\int_X u'\bigl(g_1(x), g(x)\bigr)\, dm(x)\biggr)\Psi^+_{g_1 g},$$

where

$$u'(g_1, g) = \frac{i}{2}\operatorname{Im} c(g_1 v_0, v_0) + \frac{1}{2}\,\|b(g_1)\|^2 + \langle \widetilde{T}(g_1) b(g), b(g_1)\rangle.$$

We consider the natural bijection $\widetilde{M}^+ \to \mathscr{M}^+$ of the total subsets in the spaces $\mathrm{INT}\,H^+$ and $\mathrm{EXP}\,\mathscr{H}^X$. It follows from the explicit formulae for the inner products and the operators $U^+(g)$ and $V^+(g)$ on $\widetilde{M}^+$ and $\mathscr{M}^+$ that under this bijection the inner products are preserved and the corresponding operators differ only by a factor:

$$U^+(g) = \exp\biggl(-\frac{i}{2}\operatorname{Im}\int_X c\bigl(g(x)v_0, v_0\bigr)\, dm(x)\biggr) V^+(g).$$

Hence the representations U^+ and V^+ are projectively equivalent. The same is true for the representations U^- and V^-.

In a similar way, comparing the formulae for the inner products and the operators on the total subsets $\widetilde{M}^0 \subset \mathrm{INT}\,H^0$ and $\mathscr{M}^0 \subset \mathrm{EXP}\,\mathscr{H}^X$, we see that these formulae are preserved under the natural bijection $\widetilde{M}^0 \to \mathscr{M}^0$. This implies the assertion of the theorem for the case of the representations U^0 and V^0.

7.9. Addendum: a unitary representation of the group $\widetilde{G}^X$, where $\widetilde{G}$ is the universal cover of the group $G = \mathrm{U}(n,1)$. By definition, $\widetilde{G}$ is the covering space over G in which the fibre over an element $g \in G$ is the set of homotopy classes of paths in G from the identity element e to g. Elements of $\widetilde{G}$ will be denoted by $\tilde{g}$, and their images in G by g. As in the case of $\mathrm{SL}(2,\mathbb{R})$, the integral models of representations $U^\pm$ of the current group $G^X = \mathrm{U}(n,1)^X$ induce representations $\widetilde{U}^\pm$ of the current group $\widetilde{G}^X$ on the same Hilbert spaces $\widetilde{H}^\pm$. These representations of $\widetilde{G}^X$ are projectively equivalent to non-projective unitary representations $V^\pm$ of $\widetilde{G}^X$ on the same spaces $\widetilde{H}^\pm$ which will be described explicitly. For definiteness, we restrict ourselves to the representation V^+.

The construction of V^+ is similar to the case of $\mathrm{SL}(2,\mathbb{R})$. It suffices to define the operators of V^+ on the elements of the total subset of functionals F^+_v. To this end, we first set

$$\varphi(g, v) = -\log(g_{11} + g_{12} b + g_{13} a)$$

for any $v = (a, b) \in L$ and $g = \|g_{ij}\|_{i,j=1,2,3} \in \mathrm{U}(n,1)$, where log as usual stands for the branch of the logarithm with $\log 1 = 0$ on the plane cut along the negative real axis. This function φ is everywhere finite, and for any fixed $v \in L$ it is a single-valued analytic function of $g \in G$ in a sufficiently small neighbourhood of the identity element. Hence for every $g \in G$ and every path $\tilde{g}$ in G from e to g, this function can be analytically continued along the path. Denote this analytic continuation by $\psi(\tilde{g}, v)$. The function $\psi(\tilde{g}, v)$ thus defined depends only on the homotopy class of $\tilde{g}$, and hence is a function on $\widetilde{G} \times L$.

It follows from the definition that

$$\psi(\tilde{g}, v) = -\log(g_{11} + g_{12}b + g_{13}a) \tag{7.45}$$

provided that $g \in G$ and the path $\tilde{g}$ from e to g lies in a sufficiently small neighbourhood of the identity element e.

As in the case of $\mathrm{SL}(2,\mathbb{R})$, the following assertion holds.

Proposition 7.8. *For any $\tilde{g}_1, \tilde{g}_2 \in \widetilde{G}$ and $v \in L^+$,*

$$\psi(\tilde{g}_1\tilde{g}_2, v) = \psi(\tilde{g}_1, g_2 v) + \psi(\tilde{g}_2, v). \tag{7.46}$$

We now associate with each pair $\tilde{g} \in \widetilde{G}^X$, $v \in L^X$ the following function on X:

$$\Psi_{\tilde{g},v}(x) = \psi\big(\tilde{g}(x), v(x)\big). \tag{7.47}$$

It follows from Proposition 7.8 that the functions $\Psi_{\tilde{g},v}$ are connected by the relation

$$\Psi_{\tilde{g}_1\tilde{g}_2,v} = \Psi_{\tilde{g}_1,g_2 v} + \Psi_{\tilde{g}_2,v}. \tag{7.48}$$

Definition 17. We define the action of the operators $V(\tilde{g})$, $\tilde{g} \in \widetilde{G}^X$, on the functions F_v of the total set M by the formula

$$V(\tilde{g})F_z = \exp\biggl(\int_X \Psi_{\tilde{g},v}(x)\, dm(x)\biggr)F_{gv}. \tag{7.49}$$

Then, as in the case of $\mathrm{SL}(2,\mathbb{R})$, we have the following theorem.

Theorem 7.7. *The operators $V(\tilde{g})$ are unitary on M, that is,*

$$\langle V(\tilde{g})F_{v_1}, V(\tilde{g})F_{v_2}\rangle = \langle F_{v_1}, F_{v_2}\rangle \quad \text{for any } v_1, v_2 \in L^X \text{ and } \tilde{g} \in \widetilde{G}^X, \tag{7.50}$$

and they satisfy the relation

$$V(\tilde{g}_1\tilde{g}_2)\, F_v = V(\tilde{g}_1)\, V(\tilde{g}_2)\, F_v \quad \text{for any } \tilde{g}_1, \tilde{g}_2 \in \widetilde{G}^X \text{ and } v \in L^X. \tag{7.51}$$

Thus, they generate a unitary linear representation of the group $\widetilde{G}$ on the space $\widetilde{H}$.

Obviously, the constructed representation V of $\widetilde{G}^X$ is projectively equivalent to the representation $\widetilde{U}$ of this group.

Remark. Another model of unitary representation of the group $\widetilde{G}^X$ was constructed in [32].

8. Integral models of representations of the group P^X, where P is the maximal parabolic subgroup of $\mathrm{Sp}(n,1)$

For the case of the group $\mathrm{Sp}(n,1)$ we describe the canonical representations of the subgroup $P_0 \subset P$. According to the general construction, each of them gives rise to an irreducible unitary representation of P^X. Note that, in contrast to the cases of $\mathrm{O}(n,1)$ and $\mathrm{U}(n,1)$, these representations cannot be extended to representations of the group $\mathrm{Sp}(n,1)^X$.

8.1. Initial definitions and notation. Let us realize $\mathrm{Sp}(n,1)$ as the group of linear transformations on $\mathbb{H}^{n+1}$, where $\mathbb{H}$ is the space of quaternions, that preserve the form $x_1\bar{y}_{n+1}+x_{n+1}\bar{y}_1+x_2\bar{y}_2+\cdots+x_n\bar{y}_n$ over $\mathbb{H}$, and write its elements in the form of block matrices

$$g=\|g_{ij}\|_{i,j=1,2,3},$$

where the diagonal contains square matrices of orders 1, $n-1$, and 1, respectively.

In this realization

$$P=D\leftthreetimes N,$$

where $N\cong\mathbb{R}^{n-1}$ is the subgroup consisting of the block matrices of the form

$$h=\begin{pmatrix}1&0&0\\-w^*&e_{n-1}&0\\t-\dfrac{ww^*}{2}&w&1\end{pmatrix},\qquad t\in\mathbb{H}_0,\quad w\in\mathbb{H}^{n-1}$$

(here $\mathbb{H}_0$ is the space of imaginary quaternions), and $D\cong\mathbb{H}^*\times\mathrm{Sp}(n-1)$ is the subgroup of block-diagonal matrices of the form $d=\mathrm{diag}(\bar{s}^{\,-1},u,s)$, $s\in\mathbb{H}^*$, $u\in\mathrm{Sp}(n-1)$.

We write D as the direct product $D=\mathbb{R}_+^*\times D_0$, where D_0 is the subgroup of matrices of the form $d=\mathrm{diag}(\varepsilon,u,\varepsilon)$, $|\varepsilon|=1$, and we set

$$P_0=D_0\leftthreetimes N.$$

Thus,

$$P=\mathbb{R}_+^*\rightthreetimes P_0=(\mathbb{R}_+^*\times D_0)\leftthreetimes N.$$

By $\mathrm{Sc}\,x$ and $\mathrm{Vec}\,x$ we will denote the real and imaginary parts of a quaternion x, respectively, that is, $\mathrm{Sc}\,x=(x+\bar{x})/2$ and $\mathrm{Vec}\,x=(x-\bar{x})/2$.

Let us identify elements of D_0 and N, respectively, with pairs (ε,u), where $\varepsilon\in\mathrm{Sp}(1)$ and $u\in\mathrm{Sp}(n-1)$, and pairs (t,w), where $t\in\mathbb{H}_0$ and $w\in\mathbb{H}^{n-1}$ (a row vector). Sometimes instead of $(t,w)\in N$ we will also write (ζ,w), where $\zeta=t-ww^*/2$.

With this notation the group relations take the form

$$\begin{aligned}(\zeta_1,w_1)(\zeta_2,w_2)&=(\zeta_1+\zeta_2-w_1w_2^*,w_1+w_2),\\(\varepsilon,u)^{-1}(\zeta,w)(\varepsilon,u)&=(\bar{\varepsilon}\zeta\varepsilon,\bar{\varepsilon}zu),\\r(\zeta,w)r^{-1}&=(r^2\zeta,rw)\quad\text{for } r\in\mathbb{R}_+^*.\end{aligned}$$

8.2. Description of the canonical representations of the group P_0. To describe the canonical representations of P_0, we first introduce, as in the case of $\mathrm{U}(n,1)$, a reducible unitary representation of this group and then show that the irreducible components of this representation are canonical.

Denote by S^2 the space of imaginary quaternions s of norm 1, which is isomorphic to the two-dimensional sphere ($s\in\mathbb{H}_0$, $s^2=-1$). On S^2 there is a natural action of the group of quaternions ε of norm 1: $s\mapsto\bar{\varepsilon}s\varepsilon$.

Let us introduce a unitary representation T of P_0 on the Hilbert space of functions $f(s,w)$ on $S^2\times\mathbb{H}^{n-1}$ with the norm

$$\|f\|^2=\int_{S^2}\int_{\mathbb{H}^{n-1}}|f(s,w)|^2\,d\mu(w)\,d\mu(s),\tag{8.1}$$

where $d\mu(w)$ is the Lebesgue measure on $\mathbb{H}^{n-1} \cong \mathbb{R}^{4(n-1)}$ and $d\mu(s)$ is the invariant measure on S^2.

The operators of this representation are defined by

$$\bigl(\widetilde{T}(g)f\bigr)(s,w) = \exp\bigl(-i\,\mathrm{Sc}[s(\zeta_0 - ww_0^*)]\bigr)\,f(s, w+w_0) \quad \text{for } g=(\zeta_0,w_0)\in N; \tag{8.2}$$

$$\bigl(\widetilde{T}(g)f\bigr)(s,w) = f(\bar\varepsilon s\varepsilon, \bar\varepsilon wu) \quad \text{for } g=(\varepsilon,u)\in D_0. \tag{8.3}$$

The group property and unitarity of the operators $\widetilde{T}(g)$ follow immediately from these formulae. Obviously, the operators $\widetilde{T}_r(g) = \widetilde{T}(rgr^{-1})$ of the representations conjugate to T with respect to the group $\mathbb{R}_+^*$ of automorphisms are given by the formulae

$$\bigl(\widetilde{T}_r(g)f\bigr)(s,w) = \exp\bigl(-i\,\mathrm{Sc}[s(r^2\zeta_0 - rww_0^*)]\bigr)\,f(s, w+rw_0) \text{ for } g=(\zeta_0,w_0)\in N;$$
$$\bigl(\widetilde{T}_r(g)f\bigr)(s,w) = f(\bar\varepsilon s\varepsilon, \bar\varepsilon wu) \quad \text{for } g=(\varepsilon,u)\in D_0.$$

The representation of P associated with $\widetilde{T}$ is realized on the direct integral with respect to the measure $d^*r = r^{-1}\,dr$ on $\mathbb{R}_+^*$ of the Hilbert spaces $H_r = H$ with the representations $\widetilde{T}_r$ of P_0 defined on them,

$$\mathscr{H} = \int_0^\infty H_r\, d^*r,$$

that is, on the fibre bundle over $\mathbb{R}_+^*$ with fibre H_r. On this fibre bundle the action of the operators corresponding to elements of P_0 is fibrewise, and the operators $\widetilde{T}(r_0)$, $r_0\in\mathbb{R}_+^*$, act according to the formula $\bigl(\widetilde{T}(r_0)f\bigr)(r) = f(r_0r)$.

Theorem 8.1. *The space H is the direct sum of invariant pairwise non-equivalent irreducible subspaces H_m:*

$$H = \bigoplus_{m=0}^{\infty} H_m.$$

For every $m \geqslant 0$ the representation of the group P_0 on the space H_m is canonical and has a unique, up to a factor, almost invariant vector

$$f_m(\omega,w) = l_m^{2n-3}(ww^*)e^{-\frac{1}{2}ww^*},$$

where $l_m^{2n-3}(x)$ is a Laguerre polynomial.

To prove the theorem, we write H as the direct integral

$$H = \int_{S^2} H(\omega)\,d\omega$$

of the Hilbert spaces of functions $f(w)$ on $\mathbb{H}^{n-1}$ with the norm

$$\|f\|^2 = \int_{\mathbb{H}^{n-1}} |f(w)|^2\,d\mu(w).$$

It is clear that these spaces $H(\omega)$ are invariant under the subgroup $P_1 = D_1 \leftthreetimes N \subset P_0$, where $D_1 \subset D_0$ is the subgroup of elements of the form $(1, u)$, and that the representations of P_1 are transformed one to another by the action of the subgroup of automorphisms $g \mapsto (\varepsilon, e_{n-1})^{-1} g (\varepsilon, e_{n-1})$, $(\varepsilon, e_{n-1}) \in D_0$.

The assertion of Theorem 8.1 follows immediately from the analogous assertion for the spaces $H(\omega)$:

Proposition 8.1. *Each space $H(\omega)$ is the direct sum of pairwise non-equivalent invariant subspaces $H_m(\omega)$ irreducible with respect to P_1:*

$$H(\omega) = \bigoplus_{m=0}^{\infty} H_m(\omega).$$

For every $m \geqslant 0$ the representation of P_1 on $H_m(\omega)$ is canonical and has a unique, up to a factor, almost invariant vector

$$f_m(w) = l_m^{2n-3}(ww^*) e^{-\frac{1}{2} ww^*}. \tag{8.4}$$

8.3. Proof of Proposition 8.1. It suffices to prove the assertion for one fixed $\omega \in S^2$, for example, for $\omega = e_1$, where e_1 is a basis vector in $\mathbb{H}$. For brevity let $H(e_1) = \mathscr{H}$.

We write quaternions $\sum_{k=0}^{3} a_k e_k \in \mathbb{H}$ as elements of an algebra over $\mathbb{C}$: $a = (a_0 + i a_1) + (a_2 + i a_3) j$, where $j^2 = -1$ and $ij = -ji$, and we interpret functions $f(w)$ on $\mathbb{H}^{n-1}$ as functions $f(z) = f(z^1, z^2)$ on $\mathbb{C}^{2n-2} = \mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$, where $w \in \mathbb{H}^{n-1}$ and $z = (z^1, z^2)$ are connected by the relation $w = z^1 + z^2 j$. Thus, in the new realization the representation of P_1 acts in the space of functions $f(z)$ on $\mathbb{C}^{2n-2}$ with the norm

$$\|f\|^2 = \int_{\mathbb{C}^{2n-2}} |f(z)|^2 \, d\mu(z).$$

Lemma. *In the new realization the operators of the representation of P_1 have the following form:*

$$\bigl(T(g) f\bigr)(t_0, w_0) = \exp\bigl(i(t_1 - \operatorname{Im}(z z_0^*))\bigr) f(z + z_0) \quad \textit{for } g = (t_0, w_0) \in N, \tag{8.5}$$

where $t_1 \in \mathbb{R}$ and $z_0 = (z_0^1, z_0^2) \in \mathbb{C}^{2n-2}$ are determined from the relations $t_0 = t_1 e_1 + t_2 e_2 + t_3 e_3$ and $w_0 = z_0^1 + z_0^2 j$;

$$\bigl(T(g) f\bigr)(z) = f(zv) \quad \textit{for } g = (1, u) \in D_1, \tag{8.6}$$

where

$$v = \begin{pmatrix} u_1 & u_2 \\ -u_2' & u_1' \end{pmatrix} \quad \textit{for } u = u_1 + u_2 j \in \operatorname{Sp}(n-1) \tag{8.7}$$

(*the prime indicates the transpose*).

Proof. For $g = (t_0, w_0) \in N$ the operator $T(g)$ in the original realization has the form

$$\bigl(\widetilde{T}(g) f\bigr)(w) = \exp\bigl(-i \operatorname{Sc}[e_1(t_0 - w w_0^*)]\bigr) f(w + w_0).$$

It is clear that $\operatorname{Sc}(e_1 t_0) = -t_1$ in the new realization. Further, we have $w w_0^* = z^1 (z_0^1)^* + z^2 (z_0^2)^* + z^1 (z_0^2 j)^* + (z_0^2 j)(z^1)^*$. One can easily check that $\operatorname{Sc}\bigl[e_1\bigl(z^1 (z_0^2 j)^* +$

$(z_0^2 j)(z^1)^*)\big] = 0$. Hence, $\mathrm{Sc}[e_1(ww_0^*)] = \mathrm{Re}\big(iz^1(z_0^1)^* + z^2(z_0^2)^*\big) = -\operatorname{Im}(zz_0^*)$. This implies (8.5).

Further, for $g = (1, u) \in D_1$ the operator $T(g)$ in the original realization has the form

$$\big(\widetilde{T}(g)f\big)(w) = f(wu).$$

We have $wu = (z^1 + z^2 j)(u_1 + u_2 j) = z^1 u_1 + z^2(ju_2 j) + \big(z^1 u_2 - z^2(ju_1 j)\big)j$. Thus, since $ju_i j = -u_i'$, the vector $(z^1, z^2)v$ with v a block matrix of form (8.7) corresponds to the vector $wu \in \mathbb{H}^{n-1}$. The lemma follows.

Denote by V_n the group of all transformations v of the form (8.7) on $\mathbb{C}^{2n-2}$. Obviously, $V_n \equiv \mathrm{Sp}(n-1)$.

Let us check that $V_n \subset U(n-1)$.

Indeed, the condition $u \in \mathrm{Sp}(n-1)$ is equivalent to

$$(u_1 + u_2 j)(u_1 + u_2 j)^* = e_{m-1},$$

which in turn is equivalent to the relations $u_1 u_1^* + u_2 u_2^* = e_{n-1}$ and $u_1 \bar{u}_2 = u_2 \bar{u}_1$, where the bar stands for complex conjugation. These relations immediately imply that matrices of the form (8.7) belong to the group $U(2n-2)$.

The formulae obtained for the operators $T(g)$ with $g \in P_1 \subset \mathrm{Sp}(n, 1)$ coincide with the formulae (8.4) and (8.5) for the operators T^+ of the representation of the subgroup P_0 in the case of $\mathrm{U}(n, 1)$ with n replaced by $2n-1$. Hence the decomposition of the representation into irreducible canonical components can be obtained according to the same scheme.

First we pass to a new realization of this representation by setting

$$f(z) = \varphi(z) \exp\left(-\frac{zz^*}{2}\right).$$

In the new realization the representation acts in the Hilbert space of functions $f(z)$ with the norm

$$\|f\|^2 = \int_{\mathbb{C}^{2n-2}} |f(z)|^2 \exp(-zz^*)\, d\mu(z).$$

The formulae (8.6) for the operators $T(g)$, $g \in D_1$, remain valid, and the formulae for the operators $T(g)$, $g = (t_0, z_0) \in N$, take the form

$$\big(T(g)f\big)(z) = \exp\left(i\,\frac{t_1 - z_0 z_0^*}{2} - zz_0^*\right) f(z + z_0). \tag{8.8}$$

In this realization the multiplier in the formula for $T(g)$, $g \in N$, is an entire analytic function of z. As in the case of $\mathrm{U}(n, 1)$, this implies that the representation space $\mathscr{H}$ decomposes into the direct sum

$$\mathscr{H} = \bigoplus_{m=0}^{\infty} H_m$$

of irreducible pairwise non-equivalent invariant subspaces, where H_m is the subspace cyclically generated by the homogeneous polynomials in $\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{2n-2}$ of degree m.

We note that in the case of $\mathrm{U}(n,1)$ the irreducibility of the subspaces H_m followed from the irreducibility of the space of homogeneous polynomials in $\bar z_1, \bar z_2, \dots, \bar z_{2n-2}$ with respect to the action of the whole unitary group $U(2n-2)$. However, this property remains true also when we replace the group $U(2n-2)$ by its subgroup $V_n \equiv \mathrm{Sp}(n-1)$.

Further, as in the case of $\mathrm{U}(n,1)$, it can be proved that in each space H_m there is a unique almost invariant vector, which is equal to $l_m^{2n-3}(zz^*)$.

Since $zz^* = ww^*$, this vector is given by (8.4) in the original realization of the representation and in the original coordinates w.

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