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Traces on Infinite-Dimensional Brauer Algebras*

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Abstract. We prove a theorem describing central measures for random walks on graded graphs. Using this theorem, we obtain the list of all finite traces on three infinite-dimensional algebras, namely, on the Brauer algebra, the walled Brauer algebra, and the partition algebra. The main result is that these lists coincide with the list of traces of the symmetric group or (for the walled Brauer algebra) of the square of the symmetric group.

Key words: Brauer algebra, walled Brauer algebra, partition algebra, central measure, finite trace.

1. Brauer Algebras and Pascalized Graphs

Consider the diagonal action of the complex orthogonal group $O_k(\mathbb{C})$ on the tensor power $V^{\otimes n}$ of the space $V=\mathbb{C}^k$:

$$M\cdot(v_1\otimes\dots\otimes v_n)=Mv_1\otimes\dots\otimes Mv_n,\qquad M\in O_k(\mathbb{C}).$$

Brauer (see [3], [20]) defined a family of finite-dimensional algebras $Br_n(k)$ (*Brauer algebras*) depending on a complex parameter k and a positive integer n. For integer $k\geqslant n$, the algebra $Br_n(k)$ is isomorphic to the centralizer of the above-described action of $O_k(\mathbb{C})$. For $k\in(\mathbb{C}\setminus\mathbb{Z})\cup(\mathbb{N}\setminus\{1,\dots,n-1\})$ and n fixed, the algebras $Br_n(k)$ are semisimple and pairwise isomorphic [19]. From now on, we consider only these values of the parameter k and denote the corresponding algebra by Br_n, omitting k in the notation. Generally speaking, the algebra $Br_n(k)$ is not semisimple for $k\in\mathbb{Z}$. Q1

We shall also study the *walled Brauer algebra* $Br_{n,m}(k)$, $n,m\in\mathbb{Z}_+$. The history of the definition of this algebra is as follows. Turaev [15] was the first to define it by a corepresentation; he also pointed out to the second author that it is $(n+m)!$-dimensional and resembles the group algebra of the symmetric group. The walled Brauer algebra was independently defined in [9] and was later studied in [1] as the centralizer of the diagonal action of the group $GL_k(\mathbb{C})$ on the product $V^{\otimes n}\otimes V^{*\otimes m}$. It is clear from its diagrammatic definition that this algebra is a subalgebra of the Brauer algebra. The walled Brauer algebras are also semisimple and pairwise isomorphic for generic k, $k\in(\mathbb{C}\setminus\mathbb{Z})\cup\{x\in\mathbb{Z}\mid |x|\geqslant m+n\}$. (See [14] for details.) Here we again consider only these parameter values and omit k in the notation, $Br_{n,m}=Br_{n,m}(k)$.

Martin [12] introduced the *partition algebras* $Part_n(k)$, $n\in\mathbb{Z}_+$, $k\in\mathbb{C}$. For sufficiently large $k\in\mathbb{N}$, the algebras $Part_{2n}(k)$ and $Part_{2n+1}(k)$ are isomorphic to the centralizers of the diagonal action of the subgroups $S_k\subset GL_k(\mathbb{C})$ and $S_{k-1}\subset GL_k(\mathbb{C})$, respectively, on $V^{\otimes n}$. For $k\in(\mathbb{C}\setminus\mathbb{Z})\cup\{x\in\mathbb{N}\mid x\geqslant 2n-1\}$ and fixed n, the algebras $Part_n(k)$ are semisimple and pairwise isomorphic; we denote them by $Part_n$.

Each of the finite-dimensional algebras in question contains the ideal J (with the appropriate subscript) spanned by all noninvertible standard generators of the corresponding algebra (see [19], [14], [12]), and the following relations hold:

$$Br_n/J_n\cong\mathbb{C}[S_n],\quad Br_{n,m}/J_{n,m}\cong\mathbb{C}[S_n\times S_m],\quad Part_{2n}/J_{2n}\cong Part_{2n+1}/J_{2n+1}\cong\mathbb{C}[S_n].\qquad(1)$$

The algebras Br_n, $Br_{n,m}$, and $Part_n$ form inductive families with natural embeddings. This permits one to consider their inductive limits $Br_\infty=\varinjlim Br_n$, $Br_{\infty,\infty}=\varinjlim Br_{n,m}$, and $Part_\infty=\varinjlim Part_n$, which are *locally semisimple algebras*. From the combinatorial point of view, every locally

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semisimple algebra (i.e., an inductive limit of finite-dimensional semisimple algebras) $A = \varinjlim A_n$ is completely determined by its *Bratteli diagram* (*branching graph*) $\Gamma(A)$, i.e., the $\mathbb{Z}_+$-graded graph whose nth-level vertices are simple A_n-modules and whose edges joining the $(n-1)$st- and nth-level vertices are determined by the decomposition of simple A_n-modules (treated as A_{n-1}-modules) into simple A_{n-1}-modules. (See the surveys [16] and [18] for definitions concerning locally semisimple algebras.) Finding the Bratteli diagram of a locally semisimple algebra defined by a corepresentation or by some other explicit method is analogous to finding the spectrum of a commutative algebra.

To describe the Bratteli diagrams of the Brauer algebras and the partition algebra, it is convenient to use a new operation on graphs, *pascalization*, whose definition generalizes the Jones *basic construction*. Concerning the latter, see the papers [7] and [19] by Jones and Wenzl, the paper [4], the survey [6] by Ram and Halverson on partition algebras, and the literature therein. Let us describe this operation. Suppose that Γ is a $\mathbb{Z}_+$-graded locally finite graph with a single vertex at the zero level and without pendant vertices; by Γ_k denote the set of kth-level vertices of Γ, $k \in \mathbb{Z}_+$. We also write $|\lambda| = i$ for $\lambda \in \Gamma_i$ and $\lambda \nearrow \nu$ ($\lambda \searrow \nu$) if the vertex ν follows (precedes) the vertex λ. Now we define a $\mathbb{Z}_+$-graded graph $\Pi(\Gamma)$ whose kth level $\Pi(\Gamma)_k$ is the union of Γ_k and the sets Γ_i for all previous levels of the same parity. We denote the vertices of $\Pi(\Gamma)_k$ by (k, λ), where $\lambda \in \Gamma_i$, $i \leqslant k$, and $k - i = 0 \pmod 2$. The edges of $\Pi(\Gamma)$ are defined as follows:

$$(k, \lambda) \nearrow (k+1, \nu) \iff \lambda \nearrow \nu \text{ or } \lambda \searrow \nu. \tag{2}$$

Definition 1.1. The transition from Γ to $\Pi(\Gamma)$ is called the *pascalization of* Γ.

It is easily seen that this definition is equivalent to the following. To obtain the kth level of $\Pi(\Gamma)$ we reflect the $(k-2)$nd level of the $\Pi(\Gamma)$ with respect to the $(k-1)$st level (together with the corresponding edges) and supplement the reflection by the kth level Γ_k of the original graph together with the edges that join the levels Γ_{k-1} and Γ_k in Γ.

Obviously, Γ is a subgraph of $\Pi(\Gamma)$, which however constitutes a very small part of the whole pascalized graph.

Example 1.2. Consider the graph Γ^0 whose vertex set is $\mathbb{Z}_+$ and whose edges join the vertices n and $n+1$ for each $n \in \mathbb{Z}_+$. Then $\Pi(\Gamma^0)$ is a "half" of the Pascal graph; that is, $\Pi(\Gamma^0)_{2k} = \{0, 2, \dots, 2k\}$ and $\Pi(\Gamma^0)_{2k+1} = \{1, 3, \dots, 2k+1\}$, and the edges are $(0,1)$ and also $(i, i-1)$ and $(i, i+1)$ for $i > 0$; this explains the name. Note that the graph $\Pi(\Gamma^0)$ corresponds to the locally semisimple Temperley–Lieb algebra (see [4]).

By Y_Γ we denote the set of all paths on a graph Γ; then $Y_\Gamma \subset Y_{\Pi(\Gamma)}$. The space Y_Γ naturally possesses the structure of a totally disconnected compact set, a basis of open-closed sets being formed by cylinder sets. The cylinder consisting of all paths through a vertex d will be denoted by $C_d = C_d^\Gamma$.

It is worth explaining what the path in $\Pi(\Gamma)$ look like. It follows from (2) that every path in $\Pi(\Gamma)$ is uniquely determined by a sequence $\{x_n\}_{n=0}^\infty$ of vertices of Γ such that any two consecutive vertices x_n and x_{n+1} are neighbors in Γ (i.e., $x_n \nearrow x_{n+1}$ or $x_n \searrow x_{n+1}$). In other words, every path in $\Pi(\Gamma)$ is the trajectory of a simple random walk on Γ; i.e., $\Pi(\Gamma)$ is the graph of simple walks on Γ.

For example, the homogeneous tree of degree $2k$ with a marked vertex determining a $\mathbb{Z}_+$-grading is the Cayley graph of the free group with k generators, and the corresponding pascalized graph is the graph of simple random walks on the free group.

Specifying the ideals J for the infinite-dimensional algebras in question in the same manner as in the finite-dimensional case, we obtain

$$Br_\infty / J \cong \mathbb{C}[S_\infty], \quad Br_{\infty,\infty}/J \cong \mathbb{C}[S_\infty \times S_\infty], \quad Part_\infty / J \cong \mathbb{C}[S_\infty].$$

The Bratteli diagrams of the algebras Br_∞, $Br_{\infty,\infty}$, and $Part_\infty$ are the pascalized Bratteli diagrams of the corresponding quotient algebras.

Recall that the Bratteli diagram of the group algebra of the infinite symmetric group is the Young graph $\mathbb{Y}$ (e.g., see [16]). The following three theorems restate known results in terms convenient to us. The Bratteli diagram for the family of Brauer algebras was obtained by Wenzl [19]:

Theorem 1.3. *The Bratteli diagram of the Brauer algebra Br_∞ for a generic parameter is obtained by pascalization of graph $\mathbb{Y}$, $\Gamma(Br_\infty) = \Pi(\mathbb{Y})$.*

The Bratteli diagram of the walled Brauer algebra was independently described in [11] and [14]. Consider the locally semisimple algebra $\mathbb{C}[S_\infty \times S_\infty]$ as the inductive limit of the finite-dimensional algebras

$$\mathbb{C}[S_0 \times S_0] \subset \mathbb{C}[S_1 \times S_0] \subset \mathbb{C}[S_1 \times S_1] \subset \mathbb{C}[S_2 \times S_1] \subset \cdots \subset \mathbb{C}[S_{[n+1/2]} \times S_{[n/2]}] \subset \cdots$$

and denote by $\bar{\mathbb{Y}}$ the corresponding Bratteli diagram, which can readily be obtained from $\mathbb{Y}$.

Theorem 1.4. *The Bratteli diagram of the walled Brauer algebra $Br_{\infty,\infty}$ for a generic parameter is obtained by pascalization of the graph $\bar{\mathbb{Y}}$, $\Gamma(Br_{\infty,\infty}) = \Pi(\bar{\mathbb{Y}})$.*

Martin [12] found the Bratteli diagram of the partition algebra. Let $\bar{\bar{\mathbb{Y}}}$ be the Young graph with each odd level repeated twice, corresponding to the family

$$G_0 \subset G_1 \subset G_2 \subset \cdots, \qquad G_{2i} = G_{2i+1} = \mathbb{C}[S_i], \quad i \in \mathbb{Z}_+.$$

Theorem 1.5. *The Bratteli diagram of the partition algebra $Part_\infty$ for a generic parameter is obtained by pascalization of the graph $\bar{\bar{\mathbb{Y}}}$, $\Gamma(Part_\infty) = \Pi(\bar{\bar{\mathbb{Y}}})$.*

2. A Theorem on Central Measures on Pascalized Graphs

One of the main questions in the theory of locally semisimple algebras is to describe traces (characters) and the K-functor. A *finite trace* (a *character*) on a *-algebra A is a linear functional $f\colon A \to \mathbb{C}$ such that

1. $f(1) = 1$.
2. $f(ab) = f(ba), \quad a, b \in A$.
3. $f(a^*a) \geqslant 0, \qquad a \in A$.

By $\mathrm{Char}(A) = \mathrm{Char}(\Gamma(A))$ we denote the set of all finite traces on A. A trace is said to be *indecomposable* if it is not equal to a nontrivial convex sum of traces.

A *central measure* on the path space Y_Γ of a $\mathbb{Z}_+$-graded infinite graph Γ is a Borel probability measure that induces uniform conditional measures on the finite sets of paths that coincide starting from some level. Denote the set of all central measures on the path space of a graph Γ by $\mathrm{Cent}(\Gamma) = \mathrm{Cent}(Y_\Gamma)$. A central measure is said to be *ergodic* if it is not a nontrivial convex sum of central measures. There is a well-known one-to-one correspondence between the set of finite traces on a locally semisimple algebra and the set of central measures on the path space of the corresponding Bratteli diagram,

$$\mathrm{Char}(A) \longleftrightarrow \mathrm{Cent}(\Gamma(A))$$

(e.g., see [16], [18]). Indecomposable traces correspond to ergodic central measures.

We reduce the problem of finding the traces on a pascalized graph (for some algebras) to the same problem for the original graph and apply this reasoning to the infinite-dimensional Brauer algebras and the partition algebra. We prove that *under certain conditions on the Bratteli diagram, every central measure on the pascalized graph is nonzero only on the original graph* (*treated as a subgraph of the pascalized graph*) *and hence coincides with a central measure on the original graph itself.* In particular, there is a one-to-one correspondence between the traces on the algebras Br_∞ and $Part_\infty$ and on the infinite symmetric group S_∞ as well as between the traces on $Br_{\infty,\infty}$ and on $S_\infty \times S_\infty$.

The central measures for the infinite-dimensional Brauer algebra were described, without proof and without specifying a set of full measure, by Kerov [8]; apparently, the traces on the infinite-dimensional walled Brauer algebra and on the partition algebra have not been considered earlier at all.

For an arbitrary $\mathbb{Z}_+$-graded graph Γ, we denote the number of paths from a vertex d to a vertex d' by $\dim(d; d')$ and the number of paths from the initial vertex to the vertex d by $\dim(d)$. Recall the *ergodic method* for finding central measures

Theorem 2.1 [16]. *Consider an arbitrary $\mathbb{Z}_+$-graded graph and an ergodic central measure μ on it. The set S of paths of the form $s = (s_0 \nearrow s_1 \nearrow \cdots \nearrow s_n \nearrow \dots)$ such that*

$$\mu(C_d) = \lim_{n\to\infty} \frac{\dim(d) \cdot \dim(d; s_n)}{\dim s_n} \tag{3}$$

for every vertex d is of full measure.

Hence to find the central measures it suffices to describe all limits (3). Now we use this method for pascalized graphs.

Lemma 2.2. *If the condition*

$$\lim_{n\to\infty} \max_{\lambda, |\lambda|<n} \frac{\dim(n-2, \lambda)}{\dim(n, \lambda)} = 0 \tag{4}$$

holds for the vertices of $\Pi(\Gamma)$, then $\mu(C_{(n_0,\lambda)}) = 0$ for every ergodic central measure μ on $\Pi(\Gamma)$ and every vertex $(n_0, \lambda) \in \Pi(\Gamma)_{n_0}$ with $|\lambda| < n_0$.

Proof. If μ is an ergodic central measure and s is a path as in Theorem 2.1, then

$$\mu(C_{(n_0,\lambda)}) = \lim_{n\to\infty} \frac{\dim(n_0, \lambda) \cdot \dim((n_0, \lambda); (n, s_n))}{\dim(n, s_n)}.$$

If a vertex (n, s_n) lies on some path issuing from a vertex (n_0, λ) with $|\lambda| < n_0$, then, obviously, $|s_n| < n$. By parity considerations, $|\lambda| \leqslant n_0 - 2$ and $|s_n| \leqslant n - 2$, which proves the existence of the vertices $(n_0 - 2, \lambda)$ and $(n - 2, s_n)$. There is a bijection (which acts by increasing the level number by 2 for each vertex) between the paths from (n_0, λ) to (n, s_n) and the paths from $(n_0 - 2, \lambda)$ to $(n - 2, s_n)$. Thus

$$\mu(C_{(n_0,\lambda)}) = \dim(n_0, \lambda) \cdot \lim_{n\to\infty} \frac{\dim((n_0 - 2, \lambda); (n - 2, s_n))}{\dim(n, s_n)} \leqslant \dim(n_0, \lambda) \cdot \lim_{n\to\infty} \frac{\dim(n - 2, s_n)}{\dim(n, s_n)},$$

which proves the lemma. □

For an arbitrary inductive family of algebras

$$A_0 \cong \mathbb{C} \subset A_1 \subset \cdots \subset A_l \subset \cdots,$$

we set $a_{l+1} = [\dim A_{l+1} / \dim A_l]$, $l \in \mathbb{Z}_+$, where $\dim A_l$ is the dimension of A_l.

Lemma 2.3. *Let Γ be the branching graph of the family*

$$A_0 \cong \mathbb{C} \subset A_1 \subset \cdots \subset A_l \subset \cdots.$$

If

$$\dim(\operatorname{Ind}_{A_l}^{A_{l+1}} \lambda) = a_{l+1} \dim \lambda$$

for every $l \in \mathbb{Z}_+$ and every simple module $\lambda \in \Gamma_l$ (where $\operatorname{Ind}_{A_l}^{A_{l+1}} \lambda$ is induced module), then the ratio $\dim(n, \lambda)/\dim \lambda$ depends only on the level number n in the pascalized graph and the level number $|\lambda|$ in the original graph. In other words, there exist numbers $M(n, l)$, $l \leqslant n$, $n - l = 0$ (mod 2), such that

$$\dim(n, \lambda) = M(n, |\lambda|) \dim \lambda$$

for every simple module $(n, \lambda) \in \Pi(\Gamma)_n$. The numbers $M(n, l)$ can be defined inductively by the relations

$$\begin{aligned} M(n, n) = 1, \quad M(2n+2, 0) &= M(2n+1, 1), \qquad n = 0, 1, 2, \dots, \\ M(n, l) = M(n-1, l-1) + a_{l+1} &M(n-1, l+1), \qquad 0 < l < n, \ n = 0, 1, 2, \dots. \end{aligned} \tag{5}$$

Proof. Using the branching graph $\Pi(\Gamma)$, we can represent $\dim(n,\lambda)$, $0<|\lambda|<n$, as follows:

$$\dim(n,\lambda)=\sum_{\eta\nearrow\lambda}\dim(n-1,\eta)+\sum_{\nu\searrow\lambda}\dim(n-1,\nu).$$

By induction, we obtain

$$\dim(n,\lambda)=M(n-1,|\lambda|-1)\sum_{\eta\nearrow\lambda}\dim\eta+M(n-1,|\lambda|+1)\sum_{\nu\searrow\lambda}\dim\nu.$$

The relation $\sum_{\eta\nearrow\lambda}\dim\eta=\dim\lambda$ holds for every branching graph. By the assumptions of the lemma, $\sum_{\nu\searrow\lambda}\dim\nu=\dim(\mathrm{Ind}_{A_{|\lambda|}}^{A_{|\lambda|+1}}\lambda)=a_{|\lambda|+1}\dim\lambda$, whence it follows that

$$\dim(n,\lambda)=(M(n-1,|\lambda|-1)+a_{|\lambda|+1}M(n-1,|\lambda|+1))\dim\lambda,$$

which proves the desired formula. The case $|\lambda|=0$ can be considered in a similar manner, and the case $|\lambda|=n$ is obvious.

Remark 2.4. The converse of Lemma 2.3 can be proved in a similar way.

Remark 2.5. Let $\{G_n\}$ be an inductive family of finite groups. Then the assumptions of Lemma 2.3 hold for the group algebra $\mathbb{C}[G]=\bigcup\mathbb{C}[G_n]$, and moreover, $a_{l+1}=[G_{l+1}:G_l]$; i.e., the $(l+1)$st dimension ratio for the algebras is equal to the index of the subgroup G_l in the group G_{l+1}. Example 1.2 above shows that the lists of traces on Γ and $\Pi(\Gamma)$ can be very different: there is only one trace on Γ, while the traces on $\Pi(\Gamma)$ are parametrized by the interval $[0,1/2]$.

Lemma 2.6. *If $n\geqslant 2$, then*

$$\frac{M(2n-2,0)}{M(2n,0)}=\frac{M(2n-3,1)}{M(2n-1,1)}>\frac{M(2n-2,2)}{M(2n,2)}>\frac{M(2n-3,3)}{M(2n-1,3)}>\frac{M(2n-2,4)}{M(2n,4)}>\cdots,$$
$$\frac{M(2n-2,0)}{M(2n,0)}>\frac{M(2n-1,1)}{M(2n+1,1)}>\frac{M(2n-2,2)}{M(2n,2)}>\frac{M(2n-1,3)}{M(2n+1,3)}>\frac{M(2n-2,4)}{M(2n,4)}>\cdots.$$

Proof. The relation

$$\frac{M(2n-2,0)}{M(2n,0)}=\frac{M(2n-3,1)}{M(2n-1,1)}$$

follows from (5).

To prove the inequalities, recall the following simple fact:

$$\frac{A}{B}>\frac{C}{D},\quad A,B,C,D,x>0\implies\frac{A}{B}>\frac{A+xC}{B+xD}>\frac{C}{D}.$$

By (5), $M(2n-2,2)=M(2n-3,1)+a_3M(2n-3,3)$ and $M(2n,2)=M(2n-1,1)+a_3M(2n-1,3)$, and we obtain the inequalities

$$\frac{M(2n-3,1)}{M(2n-1,1)}>\frac{M(2n-2,2)}{M(2n,2)}>\frac{M(2n-3,3)}{M(2n-1,3)}.$$

The remaining inequalities can be obtained in a similar manner. □

Corollary 2.7. *Set $\varepsilon(n)=0$ for n even and $\varepsilon(n)=1$ for n odd. Then, under the assumptions of Lemma 2.3,*

$$\max_{\lambda,|\lambda|<n}\frac{\dim(n-2,\lambda)}{\dim(n,\lambda)}=\frac{M(n-2+\varepsilon(n),0)}{M(n+\varepsilon(n),0)}.$$

Corollary 2.8. *The sequence $\left\{\frac{M(2n,0)}{M(2n+2,0)}\right\}_{n=0}^{\infty}$ is decreasing.*

Thus, under the assumptions of Lemma 2.3, the limit (4) is zero if and only if the decreasing sequence $\left\{\frac{M(2n,0)}{M(2n+2,0)}\right\}_{n=0}^{\infty}$ tends to zero.

Lemma 2.9. 1. $\displaystyle\lim_{n\to\infty}\frac{M(2n,0)}{M(2n+2,0)}>0\iff\exists M,C\;\forall n\;M(2n,0)<C\cdot M^n$;

2. $\lim\limits_{n\to\infty}\dfrac{M(2n,0)}{M(2n+2,0)}=0 \iff \forall M\ \exists C\ \forall n\ M(2n,0)>C\cdot M^n.$

Proof. The decreasing sequence $\big\{m_n=\frac{M(2n,0)}{M(2n+2,0)}>0\big\}_{n=0}^{\infty}$ always has a limit $\lim_{n\to\infty} m_n = m\geqslant 0$. If $m>0$, then

$$M(2n,0)=\prod_{k=0}^{n-1}\frac{M(2k+2,0)}{M(2k,0)}<C\Big(\frac{1}{m}\Big)^n.$$

If $m=0$, then for every $M>0$ there exists an N such that $\frac{M(2k,0)}{M(2k+2,0)}<\frac{1}{M}$ for $k>N$, and so

$$M(2n,0)=\prod_{k=0}^{n-1}\frac{M(2k+2,0)}{M(2k,0)}>C_1\cdot M^{n-N}=C_2\cdot M^n. \quad \square$$

Theorem 2.10. *Under the assumptions of Lemma* 2.3,

$$\lim_{n\to\infty}\max_{\lambda,|\lambda|<n}\frac{\dim(n-2,\lambda)}{\dim(n,\lambda)}=0 \iff \sup_l\{a_l=[\dim A_l/\dim A_{l-1}]\}=\infty.$$

Proof. Suppose that $\sup_l\{a_l\}=a$; then, using (5), one can readily verify by induction that $M(n,l)<(a+1)^n$ for every l.

If $\sup_l\{a_l\}=\infty$, then for every M there exists an index L such that $a_L>M$. Suppose that $2n>2L$ and consider the path

$$(0,0)\nearrow(1,1)\nearrow\cdots\nearrow(L,L)\nearrow(L+1,L-1)\nearrow(L+2,L)\nearrow(L+3,L-1)\nearrow\cdots$$
$$\cdots\nearrow(2n-L,L)\nearrow(2n-L+1,L-1)\nearrow(2n-L+2,L-2)\nearrow\cdots\nearrow(2n,0).$$

By induction, we obtain $M(L+2i,L)>M^i$, whence it follows that $M(2n,0)>M^{n-2L}=CM^n$. $\square$

Thus we have obtained a criterion for the central measures to be preserved under pascalization of the graph. Under the assumptions of Theorem 2.10,

1. Every central measure on $\Pi(\Gamma)$ is nonzero only on the subgraph $\Gamma\subset\Pi(\Gamma)$ and hence coincides with a central measure on Γ.

2. Consequently, there exists a bijection $\mathrm{Cent}(\Gamma)\leftrightarrow\mathrm{Cent}(\Pi(\Gamma))$ between the sets of central measures and hence a one-to-one correspondence between the traces on the corresponding algebras.

For the inductive family of Brauer algebras, we have $a_l=\dim\mathbb{C}[S_l]/\dim\mathbb{C}[S_{l-1}]=l$; for the walled Brauer algebras, $a_l=[(l+1)/2]$; and for the partition algebras, $a_{2l}=l$ and $a_{2l+1}=1$. This gives the following theorem.

Theorem 2.11. *The Bratteli diagram for each of the algebras Br_∞, $Br_{\infty,\infty}$, and $Part_\infty$ is the pascalized graph $\Pi(\Gamma)$ of some graph Γ. The central measures on $\Pi(\Gamma)$ are nonzero only on the subgraph $\Gamma\subset\Pi(\Gamma)$ and are in a one-to-one correspondence with the central measures on Γ.*

Corollary 2.12. *Every trace on each of the algebra Br_∞, $Br_{\infty,\infty}$, or $Part_\infty$ is the lift of some trace on the respective quotient algebra $\mathbb{C}[S_\infty]$, $\mathbb{C}[S_\infty\times S_\infty]$, or $\mathbb{C}[S_\infty]$.*

Our results about traces remain true for deformations of the Brauer algebra (the *Birman–Wenzl algebras*, which were studied in [2], [13], and [8]) and for deformations of the walled Brauer algebra (see [1], [10], [5], and [11]) for generic parameter values.

3. Remarks on the K_0-Functor

Recall that the *$K_0(A)$-functor* (the *Grothendieck group*) of a locally semisimple algebra A is the inductive limit of finitely generated infinite cyclic groups whose generators are vertices of the Bratteli diagram of A and whose embeddings are also determined by the same diagram. This description follows from the identification of vertices of the branching graph with the corresponding irreducible representations (simple finite-dimensional modules) of finite-dimensional subalgebras. To each such module, one can assign the induced simple module over the entire A (see [16]). We say that a vertex is *infinitesimal* if the cylinder of paths through this vertex is of zero measure for every finite central measure. A projective module (in particular, a simple module) is said to be *infinitesimal* if the corresponding vertex is infinitesimal, i.e., if all finite traces vanish on it. Hence we have defined the subgroup $I(A) = I$ of infinitesimal modules in the Grothendieck group. One can readily see that simple infinitesimal modules generate I. For the group algebra of the infinite symmetric group, this subgroup is trivial. For the algebras Br_∞, $Br_{\infty,\infty}$, and $Part_\infty$, Theorem 2.11 says that the subgroup I is not trivial: it is generated by all vertices of the difference $\Pi(\Gamma)\backslash\Gamma$ for the corresponding graph Γ. This leads to the following theorem.

Theorem 3.1. *There are isomorphisms*

$$K_0(Br_\infty)/I \cong K_0(\mathbb{C}[S_\infty]), \quad K_0(Br_{\infty,\infty})/I \cong K_0(\mathbb{C}[S_\infty \times S_\infty]), \quad K_0(Part_\infty)/I \cong K_0(\mathbb{C}[S_\infty]).$$

Apparently, the structure of infinitesimal modules over locally semisimple algebras has not been studied earlier. These modules also occur in [17] as modules over the complex group algebra of the group $SL(2, F)$ over a countable field F. Such a module cannot be a submodule of a factor representation admitting a trace (in particular, of the regular representation), because a finite trace corresponds to a finite-dimensional representation or to a II_1-representation. Q2

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Вопросы к авторам

Q0. Не нужно ли всюду по тексту заменить "corepresentation" на "presentation"? вроде бы второй перевод термина «копредставление» здесь больше подходит?

Q1. В русском варианте $k \in \mathbb{Z}$. А может быть, $k \in \mathbb{Z}_- \cup \{1, \dots, n-1\}$?

Q2. В русском тексте факторпредставление. Это существенно?